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CAUCHY-SCHWARZ INEQUALITY IMPLIES HÖLDER'S INEQUALITY

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Abstract. The aim of this note is a to give a direct proof that Hölder inequality is directly implied by the Cauchy-Schwarz inequality.

1. Introduction

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space (μ is a positive measure). For all mesurable functions $f, g : \Omega \mapsto \mathbb{C}$ on Ω, we recall the Hölder's inequality:

$$\int_\Omega |fg| d\mu \leq \Big(\int_\Omega |f|^p d\mu \Big)^{\frac{1}{p}} \Big(\int_\Omega |f|^q d\mu \Big)^{\frac{1}{q}}, \ \forall p, q \geq 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1. \qquad (H)$$

If $p = q = 2$ then we obtain the cauchy-Schwarz inequality:

$$\int_\Omega |fg| d\mu \leq \Big(\int_\Omega |f|^2 d\mu \Big)^{\frac{1}{2}} \Big(\int_\Omega |f|^2 d\mu \Big)^{\frac{1}{2}}. \qquad (C.S)$$

Their discrete versions are respectively, given by:

$$\sum_{i=1}^n |x_i y_i| \leq \left[\sum_{i=1}^n |x_i|^p \right]^{\frac{1}{p}} \left[\sum_{i=1}^n |y_i|^q \right]^{\frac{1}{q}} := \|x\|_p \|y\|_q, \qquad (H)_d$$

and

$$\sum_{i=1}^n |x_i y_i| \leq \left[\sum_{i=1}^n |x_i|^2 \right]^{\frac{1}{2}} \left[\sum_{i=1}^n |y_i|^2 \right]^{\frac{1}{2}} := \|x\|_2 \|y\|_2, \qquad (C.S)_d$$

for all positive integer n and all vectors $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{K}^n$, where the field $\mathbb{K}$ is real or complex.

Easily, we have $(H) \Longrightarrow (C.S)$.
It is natural to raise the question: is the converse true ?.

There is a positive answer to this question but, in general, not showed by a direct proof. Indeed, the converse was already known in the literature but through indirect implications. See for instance, [3], [5], [6], [4], and [2].

Many connections between classical discrete inequalities were studied in the book [6], where in particular the equivalence $(H)_d \Longleftrightarrow (C.S)_d$ was deducted through several intermidiate results.

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A. W. Marshall and I. Olkin pointed out in their book [5] that the Cauchy-Schwarz inequality implies Lyapunov's inequality which itself implies the arithmetic-geometric mean inequality. The conclusions are that, in a sense, the arithmetic-geometric mean inequality, Hölder's inequality, the Cauchy-Schwarz inequality, and Lyapunov's inequality are all equivalent [[5], p. 457].

In 2006, Y-C Li and S-Y Shaw [4] gave a proof of Hölder's inequality by using the Cauchy-Schwarz inequality. Their method lies on the fact that the convexity of a function on an open and finite interval is equivalent to continuity and midconvexity.

In 2007, the equivalence between the integral inequalities (H) and $(C-S)$ was studied by C. Finol and M. Wójtowicz in [2]. They gave a proof $(C-S)$ implies (H) by using density arguments, induction and the conclusions were obtained after three steps of proof.

The aim of this paper is to provide a direct proof that $(C-S)$ implies (H). Our method is quite different from those made in [4] and [2].

Our method of proof is based on a direct consequence of Young's inequality.

Let a, b be two positive numbers and let $\alpha \in [0,1]$. We denote by $Y(\alpha)$ the Young's inequality:

$$a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b. \tag{$Y(\alpha)$}$$

2. Proof of the implication: $(C-S) \Longrightarrow (H)$

We avoid the trivial cases, so we suppose that $1 < p, q$ with $1/p + 1/q = 1$. We suppose also that $||f||_p \neq 0$ and $||g||_q \neq 0$.

By using Young's inequality $(Y(\frac{1}{p}))$, for all positive numbers a and b, we have:

$$ab = \left[(\sqrt{a}^p)^{\frac{1}{p}} (\sqrt{b}^q)^{\frac{1}{q}}\right]^2 \leq \left[\frac{1}{p}\sqrt{a}^p + \frac{1}{q}\sqrt{b}^q\right]^2 = \frac{1}{p^2}a^p + \frac{1}{q^2}b^q + \frac{2}{pq}a^{\frac{p}{2}} b^{\frac{q}{2}}. \tag{2.1}$$

In the inequalty (2.1), we set $a = \frac{|f(x)|}{||f||_p}$ and $b = \frac{|g(x)|}{||g||_q}$ then

$$\frac{|f(x)g(x)|}{||f||_p ||g||_q} \leq \frac{|f(x)|^p}{p^2 ||f||_p^p} + \frac{|g(x)|^q}{q^2 ||g||_q^q} + \frac{2}{pq} \frac{|f(x)|^{p/2}}{||f||_p^{p/2}} \frac{|g(x)|^{q/2}}{||g||_q^{q/2}}. \tag{2.2}$$

By integrating both sides of (2.2), we get

$$\int_{\Omega} \frac{|f(x)g(x)|}{||f||_p ||g||_q} d\mu(x) \leq \frac{1}{p^2} + \frac{1}{q^2} + \frac{2}{pq ||f||_p^{p/2} ||g||_q^{q/2}} \int_{\Omega} |f|^{p/2} |g|^{q/2} \, d\mu.$$

Therefore, we have

$$\int_{\Omega} |fg| d\mu \leq \Big(\frac{1}{p^2} + \frac{1}{q^2}\Big) ||f||_p ||g||_q + \frac{2}{pq} ||f||_p^{1-\frac{p}{2}} ||g||_q^{1-\frac{q}{2}} \int_{\Omega} |f|^{p/2} |g|^{q/2} d\mu. \tag{2.3}$$

Now, by using the Cauchy-Schwarz, we obtain the following

$$\int_{\Omega} |f|^{p/2} |g|^{q/2} d\mu \leq \left[\int_{\Omega} |f|^p d\mu\right]^{\frac{1}{2}} \left[\int_{\Omega} |g|^q d\mu\right]^{\frac{1}{2}} = ||f||_p^{\frac{p}{2}} ||g||_q^{\frac{q}{2}}. \tag{2.4}$$

From (2.3) and (2.4), we deduce that

$$\int_\Omega |fg|d\mu \leq \left(\frac{1}{p^2}+\frac{1}{q^2}+\frac{2}{pq}\right)||f||_p||g||_q = \left(\frac{1}{p}+\frac{1}{q}\right)^2 ||f||_p||g||_q = ||f||_p||g||_q.$$

This end the proof.

Remark. The inequality (2.3) implies the following improvement to Hölder's inequality.

$$\int_\Omega |fg|\,d\mu \;\leq\; ||f||_p||g||_q\left(1-\frac{1}{pq}\left\|\frac{|f|^{\frac{p}{2}}}{||f||_p^{\frac{p}{2}}}-\frac{|g|^{\frac{q}{2}}}{||g||_q^{\frac{q}{2}}}\right\|_2^2\right), \tag{2.5}$$

for all $f \in L_p \setminus \{0\}$ and all $g \in L_q \setminus \{0\}$.

The inequality (2.5) above was obtained by J. M. Aldaz [1] in a different manner.

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