

SOME IYENGAR TYPE WEIGHTED INTEGRAL INEQUALITIES

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ABSTRACT. In this paper we obtain some weighted integral inequalities related to the celebrated results of Iyengar and Ostrowski. Particular cases for some weights of interest are given as well.

1. INTRODUCTION

In 1938, Iyengar proved the following theorem obtaining bounds for a trapezoidal quadrature rule for functions whose derivative are bounded (see for example [16, p. 471]).

Theorem 1. *Let f be a differentiable function on (a, b) and assume that there is a constant $M > 0$ such that $|f'(x)| \leq M$, for any $x \in (a, b)$. Then we have*

$$(1.1) \quad \left| \int_a^b f(x) dx - (b-a) \frac{f(a) + f(b)}{2} \right| \leq \frac{M(b-a)^2}{4} - \frac{1}{4M} (f(a) - f(b))^2.$$

Using a classical inequality due to Hayashi (see for example, [15, pp. 311-312]), Agarwal and Dragomir proved in [1] the following generalization of Theorem 1.

Theorem 2. *Let $f : I \subseteq \mathbb{R} \mapsto \mathbb{R}$ be a differentiable mapping in $\overset{\circ}{I}$, the interior of I , and let $a, b \in \overset{\circ}{I}$ with $a < b$. Let $M = \sup_{x \in [a, b]} f'(x) < \infty$ and $m = \inf_{x \in [a, b]} f'(x) > -\infty$. If $m < M$, then we have*

$$(1.2) \quad \begin{aligned} & \left| \int_a^b f(t) dt - (b-a) \frac{f(a) + f(b)}{2} \right| \\ & \leq \frac{[f(b) - f(a) - m(b-a)][M(b-a) - f(b) + f(a)]}{2(M-m)} \\ & \leq \frac{1}{8} (M-m)(b-a)^2. \end{aligned}$$

Thus, by placing $m = -M$ in (1.2) the Iyengar's result (1.1) is recovered.

As pointed out in [12], it should be noted that Theorem 1 and Theorem 2 are equivalent, in the sense that we can also obtain Theorem 2 from Theorem 1. Indeed, we can write the condition $m \leq f'(x) \leq M$ for $x \in [a, b]$ as $|f'(x) - \frac{m+M}{2}| \leq \frac{1}{2}(M-m)$ for $x \in [a, b]$. Let $g(x) := f(x) - \frac{m+M}{2}x$ and $M_1 := \frac{1}{2}(M-m)$ and if we apply Theorem 1 for g and M_1 , then we get Theorem 2.

For some Iyengar type inequalities see [2]-[4], [7]-[10], [12]-[14] and [18]-[21].

Motivated by the above results, in this paper we obtain some weighted integral inequalities related to (1.1), (1.2) and Ostrowski's result from 1970 regarding an

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upper bound for the absolute value of Čebyšev's functional. Particular cases for some weights of interest are given as well.

2. WEIGHTED IYENGAR TYPE INEQUALITIES

In [5] we obtained the following inequality:

Lemma 1. *Let $h : [a, b] \rightarrow \mathbb{R}$ be an integrable function on $[a, b]$ such that*

$$-\infty < \gamma \leq h(x) \leq \Gamma < \infty \text{ for a.e. } x \text{ on } [a, b].$$

Then we have the inequality

$$(2.1) \quad \begin{aligned} & \frac{1}{b-a} \int_a^b \left| \int_a^x h(t) dt - \frac{x-a}{b-a} \int_a^b h(u) du \right| dx \\ & \leq \frac{1}{2} \left(\frac{1}{b-a} \int_a^b h(u) du - \gamma \right) \left(\Gamma - \frac{1}{b-a} \int_a^b h(u) du \right) \frac{b-a}{\Gamma - \gamma} \\ & \leq \frac{1}{8} (\Gamma - \gamma) (b-a), \end{aligned}$$

with the constants $\frac{1}{2}$ and $\frac{1}{8}$ best possible.

Using this fact, we can improve the Agarwal-Dragomir inequality (1.2) and Iyengar inequality (1.1) as follows:

Theorem 3. *If $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$ and there exist the real numbers m, M with $m \leq f'(x) \leq M$ for almost every (a.e.) $x \in [a, b]$, then*

$$(2.2) \quad \begin{aligned} & \left| \int_a^b f(t) dt - (b-a) \frac{f(a) + f(b)}{2} \right| \\ & \leq \int_a^b \left| f(x) - \frac{(x-a)f(b) + (b-x)f(a)}{b-a} \right| dx \\ & \leq \frac{[f(b) - f(a) - m(b-a)][M(b-a) - f(b) + f(a)]}{2(M-m)} \\ & \leq \frac{1}{8} (M-m)(b-a)^2. \end{aligned}$$

If $|f'(x)| \leq M$ for a.e. $x \in [a, b]$ with $M > 0$, then

$$(2.3) \quad \begin{aligned} & \left| \int_a^b f(t) dt - (b-a) \frac{f(a) + f(b)}{2} \right| \\ & \leq \int_a^b \left| f(x) - \frac{(x-a)f(b) + (b-x)f(a)}{b-a} \right| dx \\ & \leq \frac{M(b-a)^2}{4} - \frac{1}{4M} (f(a) - f(b))^2 \leq \frac{M(b-a)^2}{4}. \end{aligned}$$

Proof. We observe that, if we take $h(t) = f'(t)$, then

$$\begin{aligned} \int_a^x h(t) dt - \frac{x-a}{b-a} \int_a^b h(u) du &= \int_a^x f'(t) dt - \frac{x-a}{b-a} \int_a^b f'(u) du \\ &= f(x) - f(a) - \frac{x-a}{b-a} [f(b) - f(a)] \\ &= f(x) - \frac{(x-a)f(b) + (b-x)f(a)}{b-a} \end{aligned}$$

and by (2.1) we get

$$\begin{aligned} (2.4) \quad & \frac{1}{b-a} \int_a^b \left| f(x) - \frac{(x-a)f(b) + (b-x)f(a)}{b-a} \right| dx \\ & \leq \frac{1}{2} \left(\frac{f(b) - f(a)}{b-a} - m \right) \left(M - \frac{f(b) - f(a)}{b-a} \right) \frac{b-a}{M-m} \\ & \leq \frac{1}{8} (M-m)(b-a), \end{aligned}$$

provided $m \leq f'(x) \leq M$ for a.e. $x \in [a, b]$.

By the properties of modulus and integral we also have

$$\begin{aligned} (2.5) \quad & \int_a^b \left| f(x) - \frac{(x-a)f(b) + (b-x)f(a)}{b-a} \right| dx \\ & \geq \left| \int_a^b f(x) dx - \int_a^b \frac{(x-a)f(b) + (b-x)f(a)}{b-a} dx \right| \\ & = \left| \int_a^b f(x) dx - (b-a) \frac{f(a) + f(b)}{2} \right|, \end{aligned}$$

which completes the proof of (2.2). \square

In order to extend the Iyengar type inequalities presented above for weighted integrals, we need the following result as well:

Lemma 2. *Let $h : [a, b] \rightarrow [h(a), h(b)]$ be a continuous strictly increasing function that is differentiable on (a, b) . If $g : [a, b] \rightarrow \mathbb{R}$ are absolutely continuous on $[a, b]$ and there exist the real numbers $k < K$ with*

$$(2.6) \quad k \leq \frac{g'(x)}{h'(x)} \leq K \text{ for a.e. } x \in [a, b],$$

then we have

$$\begin{aligned} (2.7) \quad & \left| \int_a^b g(t) h'(t) dt - (h(b) - h(a)) \frac{g(a) + g(b)}{2} \right| \\ & \leq \int_a^b \left| g(t) - \frac{(h(t) - h(a))g(b) + (h(b) - h(t))g(a)}{h(b) - h(a)} \right| h'(t) dt \\ & \leq \frac{1}{2(K-k)} [g(b) - g(a) - k(h(b) - h(a))] [K(h(b) - h(a)) - g(b) + g(a)] \\ & \leq \frac{1}{8} (K-k)(h(b) - h(a))^2. \end{aligned}$$

If $\left| \frac{g'(x)}{h'(x)} \right| \leq K$ for a.e. $x \in [a, b]$ with $K > 0$, then we have

$$(2.8) \quad \left| \int_a^b g(t) h'(t) dt - (h(b) - h(a)) \frac{g(a) + g(b)}{2} \right| \\ \leq \int_a^b \left| g(t) - \frac{(h(t) - h(a))g(b) + (h(b) - h(t))g(a)}{h(b) - h(a)} \right| h'(t) dt \\ \leq \frac{1}{4}K(h(b) - h(a))^2 - \frac{1}{4K}(g(b) - g(a))^2 \leq \frac{1}{4}K(h(b) - h(a))^2.$$

Proof. We observe that the function $f := g \circ h^{-1}$ is absolutely continuous on $[h(a), h(b)]$ and using the chain rule and the derivative of inverse functions we have

$$(2.9) \quad (g \circ h^{-1})'(z) = (g' \circ h^{-1})(z) (h^{-1})'(z) = \frac{(g' \circ h^{-1})(z)}{(h' \circ h^{-1})(z)}$$

for almost every (a.e.) $z \in [h(c), h(d)]$.

By the condition (2.6) we have

$$k \leq (g \circ h^{-1})'(z) \leq K \text{ for a.e. } z \in [h(c), h(d)].$$

If we use the inequality (2.2) for the function $f := g \circ h^{-1}$ on the interval $[h(a), h(b)]$, then we get

$$\left| \int_{h(a)}^{h(b)} (g \circ h^{-1})(z) dz - (h(b) - h(a)) \frac{(g \circ h^{-1})(h(a)) + (g \circ h^{-1})(h(b))}{2} \right| \\ \leq \int_{h(a)}^{h(b)} \left| (g \circ h^{-1})(z) - \frac{(z - h(a))(g \circ h^{-1})(h(b)) + (h(b) - z)(g \circ h^{-1})(h(a))}{h(b) - h(a)} \right| dz \\ \leq \frac{1}{2(K - k)} [(g \circ h^{-1})(h(b)) - (g \circ h^{-1})(h(a)) - k(h(b) - h(a))] \\ \times [K(h(b) - h(a)) - (g \circ h^{-1})(h(b)) + (g \circ h^{-1})(h(a))] \\ \leq \frac{1}{8}(K - k)(h(b) - h(a))^2,$$

which is equivalent to

$$(2.10) \quad \left| \int_{h(a)}^{h(b)} (g \circ h^{-1})(z) dz - (h(b) - h(a)) \frac{g(a) + g(b)}{2} \right| \\ \leq \int_{h(a)}^{h(b)} \left| (g \circ h^{-1})(z) - \frac{(z - h(a))g(b) + (h(b) - z)g(a)}{h(b) - h(a)} \right| dz \\ \leq \frac{1}{2(K - k)} [g(b) - g(a) - k(h(b) - h(a))] [K(h(b) - h(a)) - g(b) + g(a)] \\ \leq \frac{1}{8}(K - k)(h(b) - h(a))^2.$$

If we change the variable $t = h^{-1}(z)$, $z \in [h(c), h(d)]$, then we have $z = h(t)$, which gives $dz = h'(t) dt$,

$$\int_{h(a)}^{h(b)} (g \circ h^{-1})(z) dz = \int_a^b g(t) h'(t) dt,$$

$$\begin{aligned} & \int_{h(a)}^{h(b)} \left| (g \circ h^{-1})(z) - \frac{(z - h(a))g(b) + (h(b) - z)g(a)}{h(b) - h(a)} \right| dz \\ &= \int_a^b \left| g(t) - \frac{(h(t) - h(a))g(b) + (h(b) - h(t))g(a)}{h(b) - h(a)} \right| h'(t) dt \end{aligned}$$

and by (2.10) we get the desired result (2.7). \square

The following weighted integral inequality holds:

Theorem 4. *Assume that $w : [a, b] \rightarrow (0, \infty)$ is continuous on $[a, b]$. If $g : [a, b] \rightarrow \mathbb{R}$ are absolutely continuous on $[a, b]$ and there exist the real numbers $k < K$ with*

$$(2.11) \quad kw(x) \leq g'(x) \leq w(x)K \text{ for a.e. } x \in [a, b],$$

then we have

$$\begin{aligned} (2.12) \quad & \left| \int_a^b g(t)w(t)dt - \frac{g(a) + g(b)}{2} \int_a^b w(t)dt \right| \\ & \leq \int_a^b \left| g(t) - \frac{g(b) \int_a^t w(s)ds + g(a) \int_t^b w(s)ds}{\int_a^b w(t)dt} \right| w(t)dt \\ & \leq \frac{1}{2(K-k)} \left[g(b) - g(a) - k \int_a^b w(t)dt \right] \left[K \int_a^b w(t)dt - g(b) + g(a) \right] \\ & \leq \frac{1}{8} (K-k) \left(\int_a^b w(t)dt \right)^2. \end{aligned}$$

If $|g'(x)| \leq Kw(x)$ for a.e. $x \in [a, b]$ with $K > 0$, then we have

$$\begin{aligned} (2.13) \quad & \left| \int_a^b g(t)w(t)dt - \frac{g(a) + g(b)}{2} \int_a^b w(t)dt \right| \\ & \leq \int_a^b \left| g(t) - \frac{g(b) \int_a^t w(s)ds + g(a) \int_t^b w(s)ds}{\int_a^b w(t)dt} \right| w(t)dt \\ & \leq \frac{1}{4}K \left(\int_a^b w(t)dt \right)^2 - \frac{1}{4K} (g(b) - g(a))^2 \leq \frac{1}{4}K \left(\int_a^b w(t)dt \right)^2. \end{aligned}$$

The proof follows by Lemma 2 for $h(x) := \int_a^x w(s)ds$.

a). If $g : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ are absolutely continuous on $[a, b]$ and there exist the real numbers $k < K$ with

$$(2.14) \quad \frac{k}{x} \leq g'(x) \leq \frac{K}{x} \text{ for a.e. } x \in [a, b],$$

then

$$\begin{aligned}
(2.15) \quad & \left| \int_a^b \frac{g(t)}{t} dt - \frac{g(a) + g(b)}{2} \ln \left(\frac{b}{a} \right) \right| \\
& \leq \int_a^b \left| g(t) - \frac{g(b) \ln \left(\frac{t}{a} \right) + g(a) \ln \left(\frac{b}{t} \right)}{\ln \left(\frac{b}{a} \right)} \right| \frac{1}{t} dt \\
& \leq \frac{1}{2(K-k)} \left[g(b) - g(a) - k \ln \left(\frac{b}{a} \right) \right] \left[K \ln \left(\frac{b}{a} \right) - g(b) + g(a) \right] \\
& \leq \frac{1}{8} (K-k) \left(\ln \left(\frac{b}{a} \right) \right)^2.
\end{aligned}$$

If $|g'(x)| \leq \frac{K}{x}$ for a.e. $x \in [a, b] \subset (0, \infty)$ with $K > 0$, then we have

$$\begin{aligned}
(2.16) \quad & \left| \int_a^b \frac{g(t)}{t} dt - \frac{g(a) + g(b)}{2} \ln \left(\frac{b}{a} \right) \right| \\
& \leq \int_a^b \left| g(t) - \frac{g(b) \ln \left(\frac{t}{a} \right) + g(a) \ln \left(\frac{b}{t} \right)}{\ln \left(\frac{b}{a} \right)} \right| \frac{1}{t} dt \\
& \leq \frac{1}{4} K \left(\ln \left(\frac{b}{a} \right) \right)^2 - \frac{1}{4K} (g(b) - g(a))^2 \leq \frac{1}{4} K \left(\ln \left(\frac{b}{a} \right) \right)^2.
\end{aligned}$$

b). If $g : [a, b] \rightarrow \mathbb{R}$ are absolutely continuous on $[a, b]$ and there exist the real numbers $k < K$ with

$$(2.17) \quad k \exp x \leq g'(x) \leq K \exp x \text{ for a.e. } x \in [a, b],$$

$$\begin{aligned}
(2.18) \quad & \left| \int_a^b g(t) \exp t dt - \frac{g(a) + g(b)}{2} (\exp b - \exp a) \right| \\
& \leq \int_a^b \left| g(t) - \frac{g(b) (\exp t - \exp a) + g(a) (\exp b - \exp t)}{(\exp b - \exp a)} \right| \exp t dt \\
& \leq \frac{1}{2(K-k)} [g(b) - g(a) - k(\exp b - \exp a)] [K(\exp b - \exp a) - g(b) + g(a)] \\
& \leq \frac{1}{8} (K-k) (\exp b - \exp a)^2.
\end{aligned}$$

If $|g'(x)| \leq K \exp(x)$ for a.e. $x \in [a, b]$ with $K > 0$, then we have

$$\begin{aligned}
(2.19) \quad & \left| \int_a^b g(t) \exp t dt - \frac{g(a) + g(b)}{2} (\exp b - \exp a) \right| \\
& \leq \int_a^b \left| g(t) - \frac{g(b) (\exp t - \exp a) + g(a) (\exp b - \exp t)}{(\exp b - \exp a)} \right| \exp t dt \\
& \leq \frac{1}{4} K (\exp b - \exp a)^2 - \frac{1}{4K} (g(b) - g(a))^2 \leq \frac{1}{4} K (\exp b - \exp a)^2.
\end{aligned}$$

c). If $g : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ are absolutely continuous on $[a, b]$ and there exist the real numbers $k < K$ and $p \neq -1$ with

$$(2.20) \quad kx^p \leq g'(x) \leq Kx^p \text{ for a.e. } x \in [a, b],$$

then

$$\begin{aligned}
(2.21) \quad & \left| \int_a^b g(t) t^p dt - \frac{b^{p+1} - a^{p+1}}{p+1} \frac{g(a) + g(b)}{2} \right| \\
& \leq \int_a^b \left| g(t) - \frac{g(b)(t^{p+1} - a^{p+1}) + g(a)(b^{p+1} - t^{p+1})}{b^{p+1} - a^{p+1}} \right| t^p dt \\
& \leq \frac{1}{2(K-k)} \left[g(b) - g(a) - k \frac{b^{p+1} - a^{p+1}}{p+1} \right] \left[K \frac{b^{p+1} - a^{p+1}}{p+1} - g(b) + g(a) \right] \\
& \leq \frac{1}{8(p+1)^2} (K-k) (b^{p+1} - a^{p+1})^2.
\end{aligned}$$

If $|g'(x)| \leq Kx^p$ for a.e. $x \in [a, b] \subset (0, \infty)$ with $K > 0$, then we have

$$\begin{aligned}
(2.22) \quad & \left| \int_a^b g(t) t^p dt - \frac{b^{p+1} - a^{p+1}}{p+1} \frac{g(a) + g(b)}{2} \right| \\
& \leq \int_a^b \left| g(t) - \frac{g(b)(t^{p+1} - a^{p+1}) + g(a)(b^{p+1} - t^{p+1})}{b^{p+1} - a^{p+1}} \right| t^p dt \\
& \leq \frac{1}{4(p+1)^2} K (b^{p+1} - a^{p+1})^2 - \frac{1}{4K} (g(b) - g(a))^2 \leq \frac{1}{4(p+1)^2} K (b^{p+1} - a^{p+1})^2.
\end{aligned}$$

3. SOME RELATED RESULTS

For two *Lebesgue integrable* functions $f, g : [a, b] \rightarrow \mathbb{R}$, consider the *Čebyšev functional*:

$$C(f, g) := \frac{1}{b-a} \int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2} \int_a^b f(t)dt \int_a^b g(t)dt.$$

In 1935, Grüss [11] showed that

$$(3.1) \quad |C(f, g)| \leq \frac{1}{4} (M - m)(N - n),$$

provided that there exists the real numbers m, M, n, N such that

$$(3.2) \quad m \leq f(t) \leq M \quad \text{and} \quad n \leq g(t) \leq N \quad \text{for a.e. } t \in [a, b].$$

The constant $\frac{1}{4}$ is best possible in (3.1) in the sense that it cannot be replaced by a smaller quantity.

The following inequality was obtained by Ostrowski in 1970, [17]:

$$(3.3) \quad |C(f, g)| \leq \frac{1}{8} (b-a) (M - m) \|g'\|_\infty,$$

provided that f is *Lebesgue integrable* and satisfies (3.2) while g is absolutely continuous and $g' \in L_\infty[a, b]$. The constant $\frac{1}{8}$ is best possible in (3.3).

In [5] we obtained the following refinement of Ostrowski's inequality:

Lemma 3. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be such that g is absolutely continuous on $[a, b]$ with $g' \in L_\infty[a, b]$ and f is Lebesgue integrable and satisfies (3.2), then*

$$(3.4) \quad |C(f, g)| \leq \frac{1}{2} (b-a) \|g'\|_\infty \frac{\left(\frac{1}{b-a} \int_a^b f(t) dt - m\right) \left(M - \frac{1}{b-a} \int_a^b f(t) dt\right)}{M - m} \\ \leq \frac{1}{8} \|g'\|_\infty (b-a) (M - m).$$

As a particular case of this inequality that may be seen as a perturbed Iyengar type inequality, we have:

Corollary 1. *If $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$ and there exist the real numbers γ, Γ with $\gamma \leq f''(x) \leq \Gamma$ for almost every (a.e.) $x \in [a, b]$, then*

$$(3.5) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{12} (b-a) [f'(b) - f'(a)] - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ \leq \frac{1}{4} (b-a)^2 \frac{\left(\frac{f'(b) - f'(a)}{b-a} - \gamma\right) \left(\Gamma - \frac{f'(b) - f'(a)}{b-a}\right)}{\Gamma - \gamma} \\ \leq \frac{1}{16} (b-a)^2 (\Gamma - \gamma).$$

In particular, if $|f''(x)| \leq \Gamma$ for almost every (a.e.) $x \in [a, b]$, then

$$(3.6) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{12} (b-a) [f'(b) - f'(a)] - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ \leq \frac{1}{8} \Gamma (b-a)^2 - \frac{1}{8\Gamma} [f'(b) - f'(a)]^2 \\ \leq \frac{1}{8} (b-a)^2 \Gamma.$$

Proof. Using (3.4) for f'' and $g(t) := \frac{1}{2}(t-a)(b-t)$ with $x \in [a, b]$, then we get

$$(3.7) \quad \left| \frac{1}{2(b-a)} \int_a^b (t-a)(b-t) f''(t) dt \right. \\ \left. - \frac{1}{2(b-a)} \int_a^b (t-a)(b-t) dt \frac{1}{b-a} \int_a^b f''(t) dt \right| \\ \leq \frac{1}{2} \|g'\|_{[a,b], \infty} (b-a) \frac{\left(\frac{1}{b-a} \int_a^b f''(t) dt - \gamma\right) \left(\Gamma - \frac{1}{b-a} \int_a^b f''(t) dt\right)}{\Gamma - \gamma} \\ \leq \frac{1}{8} \|g'\|_{[a,b], \infty} (b-a) (\Gamma - \gamma).$$

We have

$$\frac{1}{2} \int_a^b (t-a)(b-t) f''(t) dt \\ = \frac{1}{2} (t-a)(b-t) f'(t) \Big|_a^b - \int_a^b \left(\frac{a+b}{2} - t\right) f'(t) dt \\ = \int_a^b \left(t - \frac{a+b}{2}\right) f'(t) dt = \frac{f(a) + f(b)}{2} (b-a) - \int_a^b f(t) dt$$

and

$$\begin{aligned} \int_a^b (t-a)(b-t) dt &= \frac{a^2+b^2}{2} (b-a) - \int_a^b t^2 dt \\ &= \frac{a^2+b^2}{2} (b-a) - \frac{b^3-a^3}{3} \\ &= (b-a) \left[\frac{a^2+b^2}{2} - \frac{b^2+ab+a^2}{3} \right] = \frac{1}{6} (b-a)^3 \end{aligned}$$

then

$$\begin{aligned} &\frac{1}{2(b-a)} \int_a^b (t-a)(b-t) f''(t) dt \\ &- \frac{1}{2(b-a)} \int_a^b (t-a)(b-t) dt \frac{1}{b-a} \int_a^b f''(t) dt \\ &= \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{12} (b-a) [f'(b) - f'(a)]. \end{aligned}$$

Also $g'(t) := \frac{a+b}{2} - t$, which gives $\|g'\|_{[a,b],\infty} = \frac{1}{2} (b-a)$ and by (3.7) we get (3.5). \square

Consider now the *weighted Čebyšev functional*

$$(3.8) \quad C_w(f, g) := \frac{1}{\int_a^b w(t) dt} \int_a^b w(t) f(t) g(t) dt \\ - \frac{1}{\int_a^b w(t) dt} \int_a^b w(t) f(t) dt \frac{1}{\int_a^b w(t) dt} \int_a^b w(t) g(t) dt$$

where $f, g, w : [a, b] \rightarrow \mathbb{R}$ and $w(t) \geq 0$ for a.e. $t \in [a, b]$ are measurable functions such that the involved integrals exist and $\int_a^b w(t) dt > 0$.

We can also define, as above,

$$(3.9) \quad C_{h'}(f, g) := \frac{1}{h(b) - h(a)} \int_a^b f(t) g(t) h'(t) dt \\ - \frac{1}{h(b) - h(a)} \int_a^b f(t) h'(t) dt \frac{1}{h(b) - h(a)} \int_a^b g(t) h'(t) dt,$$

where h is absolutely continuous and f, g are Lebesgue measurable on $[a, b]$ and such that the above integrals exist.

Lemma 4. *Let $h : [a, b] \rightarrow [h(a), h(b)]$ be a continuous strictly increasing function that is differentiable on (a, b) . If f is Lebesgue integrable and satisfies the condition $m \leq f(t) \leq M$ for $t \in [a, b]$ and $g : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$ and*

$\frac{g'}{h'}$ is essentially bounded, namely $\frac{g'}{h'} \in L_\infty [a, b]$, then we have

$$(3.10) \quad |C_{h'}(f, g)| \leq \frac{1}{2} [h(b) - h(a)] \left\| \frac{g'}{h'} \right\|_\infty \\ \times \frac{\left(\frac{1}{h(b)-h(a)} \int_a^b f(t) h'(t) dt - m \right) \left(M - \frac{1}{h(b)-h(a)} \int_a^b f(t) h'(t) dt \right)}{M - m} \\ \leq \frac{1}{8} [h(b) - h(a)] (M - m) \left\| \frac{g'}{h'} \right\|_{[a, b], \infty}.$$

The constants $\frac{1}{2}$ and $\frac{1}{8}$ are best possible.

Proof. Since $\frac{g'}{h'} \in L_\infty [c, d]$, hence $(g \circ h^{-1})' \in L_\infty [h(c), h(d)]$. Also

$$\left\| (g \circ h^{-1})' \right\|_{[h(c), h(d)], \infty} = \left\| \frac{g'}{h'} \right\|_{[c, d], \infty}.$$

Now, if we use the refinement of Ostrowski's inequality (3.4) for the functions $f \circ h^{-1}$ and $g \circ h^{-1}$ on the interval $[h(a), h(b)]$, then we get

$$(3.11) \quad \left| \frac{1}{h(b) - h(a)} \int_{h(a)}^{h(b)} f \circ h^{-1}(u) g \circ h^{-1}(u) du \right. \\ \left. - \frac{1}{[h(b) - h(a)]^2} \int_{h(a)}^{h(b)} f \circ h^{-1}(u) du \int_{h(a)}^{h(b)} g \circ h^{-1}(u) du \right| \\ \leq \frac{1}{2} (h(b) - h(a)) \left\| (g \circ h^{-1})' \right\|_{[h(c), h(d)], \infty} \\ \times \frac{\left(\frac{1}{h(b)-h(a)} \int_{h(a)}^{h(b)} f \circ h^{-1}(t) dt - m \right) \left(M - \frac{1}{h(b)-h(a)} \int_{h(a)}^{h(b)} f(t) dt \right)}{M - m} \\ \leq \frac{1}{8} [h(b) - h(a)] (M - m) \left\| (g \circ h^{-1})' \right\|_{[h(a), h(b)], \infty}$$

since $m \leq f \circ h^{-1}(u) \leq M$ for all $u \in [h(a), h(b)]$.

Observe also that, by the change of variable $t = h^{-1}(u)$, $u \in [g(a), g(b)]$, we have $u = h(t)$ that gives $du = h'(t) dt$ and

$$\int_{h(a)}^{h(b)} (f \circ h^{-1})(u) du = \int_a^b f(t) h'(t) dt, \\ \int_{h(a)}^{h(b)} g \circ h^{-1}(u) du = \int_a^b g(t) h'(t) dt, \\ \int_{h(a)}^{h(b)} f \circ h^{-1}(u) g \circ h^{-1}(u) du = \int_a^b f(t) g(t) h'(t) dt$$

and

$$\left\| (g \circ h^{-1})' \right\|_{[h(a), h(b)], \infty} = \left\| \frac{g'}{h'} \right\|_{[a, b], \infty}.$$

By making use of (3.11) we then get the desired result (3.10).

The best constant follows by the refinement of Ostrowski's inequality (3.4). \square

If $w : [a, b] \rightarrow \mathbb{R}$ is continuous and positive on the interval $[a, b]$, then the function $W : [a, b] \rightarrow [0, \infty)$, $W(x) := \int_a^x w(s) ds$ is strictly increasing and differentiable on (a, b) . We have $W'(x) = w(x)$ for any $x \in (a, b)$.

Theorem 5. Assume that $w : [a, b] \rightarrow (0, \infty)$ is continuous on $[a, b]$, f is Lebesgue integrable and satisfies the condition $m \leq f(t) \leq M$ for $t \in [a, b]$ and $g : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$ with $\frac{g'}{w}$ is essentially bounded, namely $\frac{g'}{w} \in L_\infty[a, b]$, then we have

$$(3.12) \quad |C_w(f, g)| \leq \frac{1}{2(M-m)} \left\| \frac{g'}{w} \right\|_\infty \\ \times \left(\frac{\int_a^b f(t) w(t) dt}{\int_a^b w(s) ds} - m \right) \left(M - \frac{\int_a^b f(t) w(t) dt}{\int_a^b w(s) ds} \right) \int_a^b w(s) ds \\ \leq \frac{1}{8} (M-m) \left\| \frac{g'}{w} \right\|_{[a,b],\infty} \int_a^b w(s) ds.$$

The constant $\frac{1}{8}$ is best possible.

Remark 1. Under the assumptions of Theorem 5 and if there exists a constant $K > 0$ such that $|g'(t)| \leq Kw(t)$ for a.e. $t \in [a, b]$, then by (3.12) we get

$$(3.13) \quad |C_w(f, g)| \leq \frac{1}{2(M-m)} K \\ \times \left(\frac{\int_a^b f(t) w(t) dt}{\int_a^b w(s) ds} - m \right) \left(M - \frac{\int_a^b f(t) w(t) dt}{\int_a^b w(s) ds} \right) \int_a^b w(s) ds \\ \leq \frac{1}{8} (M-m) K \int_a^b w(s) ds.$$

a). For $w(t) = \frac{1}{\ell(t)} = \ell^{-1}(t)$, $t \in [a, b] \subset (0, \infty)$, where $\ell(t) = t$, define

$$(3.14) \quad C_{\ell^{-1}}(f, g) := \frac{1}{\ln\left(\frac{b}{a}\right)} \int_a^b \frac{f(t)g(t)}{t} dt - \frac{1}{\ln\left(\frac{b}{a}\right)} \int_a^b \frac{f(t)}{t} dt \frac{1}{\ln\left(\frac{b}{a}\right)} \int_a^b \frac{g(t)}{t} dt.$$

If $m \leq f(t) \leq M$ for $t \in [a, b]$ and $g : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$ with $\ell g'$ is essentially bounded, namely $\ell g' \in L_\infty[a, b]$, then we have

$$(3.15) \quad |C_{\ell^{-1}}(f, g)| \\ \leq \frac{1}{2} \|\ell g'\|_{[a,b],\infty} \frac{\left(\frac{1}{\ln\left(\frac{b}{a}\right)} \int_a^b \frac{f(t)}{t} dt - m \right) \left(M - \frac{1}{\ln\left(\frac{b}{a}\right)} \int_a^b \frac{f(t)}{t} dt \right)}{M-m} \ln\left(\frac{b}{a}\right) \\ \leq \frac{1}{8} (M-m) \|\ell g'\|_{[a,b],\infty} \ln\left(\frac{b}{a}\right).$$

b). For $w(t) = \exp t$, $t \in [a, b]$, define

$$(3.16) \quad C_{\exp}(f, g) := \frac{1}{\exp b - \exp a} \int_a^b f(t) g(t) \exp t dt \\ - \frac{1}{\exp b - \exp a} \int_a^b f(t) \exp t dt \frac{1}{\exp b - \exp a} \int_a^b g(t) \exp t dt.$$

If $m \leq f(t) \leq M$ for $t \in [a, b]$ and $g : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$ with $\frac{g'}{\exp}$ is essentially bounded, namely $\frac{g'}{\exp} \in L_\infty [a, b]$, then we have

$$(3.17) \quad |C_{\exp}(f, g)| \leq \frac{1}{2(M-m)} \left\| \frac{g'}{\exp} \right\|_{[a,b],\infty} \left(\frac{\int_a^b f(t) \exp t dt}{\exp b - \exp a} - m \right) \left(M - \frac{\int_a^b f(t) \exp t dt}{\exp b - \exp a} \right) \times (\exp b - \exp a) \leq \frac{1}{8} (M-m) \left\| \frac{g'}{\exp} \right\|_{[a,b],\infty} (\exp b - \exp a).$$

c). For $w(t) = \ell^p(t)$, $t \in [a, b] \subset (0, \infty)$, where $\ell(t) = t$ and $p \neq -1$, define

$$(3.18) \quad C_{\ell^p}(f, g) := \frac{p+1}{b^{p+1} - a^{p+1}} \int_a^b t^p f(t) g(t) dt - \frac{p+1}{b^{p+1} - a^{p+1}} \int_a^b t^p f(t) dt \frac{p+1}{b^{p+1} - a^{p+1}} \int_a^b t^p g(t) dt.$$

If $m \leq f(t) \leq M$ for $t \in [a, b]$ and $g : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$ with $g' \ell^{-p}$ is essentially bounded, namely $g' \ell^{-p} \in L_\infty [a, b]$, then we have

$$(3.19) \quad |C_{\ell^p}(f, g)| \leq \frac{b^{p+1} - a^{p+1}}{2(p+1)(M-m)} \|g' \ell^{-p}\|_\infty \times \left(\frac{p+1}{b^{p+1} - a^{p+1}} \int_a^b f(t) t^p dt - m \right) \left(M - \frac{p+1}{b^{p+1} - a^{p+1}} \int_a^b f(t) t^p dt \right) \leq \frac{b^{p+1} - a^{p+1}}{8(p+1)} (M-m) \|g' \ell^{-p}\|_{[a,b],\infty}.$$

We have:

Lemma 5. *Let $h : [a, b] \rightarrow [h(a), h(b)]$ be a continuous strictly increasing function that is twice differentiable on (a, b) . If $g : [a, b] \rightarrow \mathbb{R}$ has an absolutely continuous derivative on $[a, b]$ and there exist the real numbers $n < N$ with*

$$(3.20) \quad \phi \leq \frac{g''(x)h'(x) - g'(x)h''(x)}{[h'(x)]^3} \leq \Phi \text{ for a.e. } x \in [a, b],$$

then we have

$$(3.21) \quad \left| \frac{g(a) + g(b)}{2} - \frac{1}{12} (h(b) - h(a)) \left[\frac{g'(b)}{h'(b)} - \frac{g'(a)}{h'(a)} \right] - \frac{1}{h(b) - h(a)} \int_a^b g(t) h^{-1}(t) dt \right| \leq \frac{1}{4} \frac{[h(b) - h(a)]^2}{\Phi - \phi} \left(\frac{\frac{g'(b)}{h'(b)} - \frac{g'(a)}{h'(a)}}{h(b) - h(a)} - \phi \right) \left(\Phi - \frac{\frac{g'(b)}{h'(b)} - \frac{g'(a)}{h'(a)}}{h(b) - h(a)} \right) \leq \frac{1}{16} [h(b) - h(a)]^2 (\Phi - \phi).$$

In particular, if

$$(3.22) \quad \left| \frac{g''(x)h'(x) - g'(x)h''(x)}{[h'(x)]^3} \right| \leq \Phi \text{ for a.e. } x \in [a, b]$$

then

$$(3.23) \quad \left| \frac{g(a) + g(b)}{2} - \frac{1}{12} (h(b) - h(a)) \left[\frac{g'(b)}{h'(b)} - \frac{g'(a)}{h'(a)} \right] - \frac{1}{h(b) - h(a)} \int_a^b g(t) h^{-1}(t) dt \right| \\ \leq \frac{1}{8} \Phi [h(b) - h(a)]^2 - \frac{1}{8\Phi} \left[\frac{g'(b)}{h'(b)} - \frac{g'(a)}{h'(a)} \right]^2 \\ \leq \frac{1}{8} [h(b) - h(a)]^2 \Phi.$$

Proof. We observe that the function $f := g \circ h^{-1}$ has an absolutely continuous derivative on $[h(a), h(b)]$ and using the chain rule and the derivative of inverse functions we have

$$\begin{aligned} (g \circ h^{-1})''(z) &= \left(\frac{(g' \circ h^{-1})(z)}{(h' \circ h^{-1})(z)} \right)' \\ &= \frac{(g' \circ h^{-1})'(z) (h' \circ h^{-1})(z) - (g' \circ h^{-1})(z) (h' \circ h^{-1})'(z)}{[(h' \circ h^{-1})(z)]^2} \\ &= \frac{\frac{(g'' \circ h^{-1})(z)}{(h'' \circ h^{-1})(z)} (h' \circ h^{-1})(z) - (g' \circ h^{-1})(z) \frac{(h'' \circ h^{-1})(z)}{(h'' \circ h^{-1})(z)}}{[(h' \circ h^{-1})(z)]^2}} \\ &= \frac{(g'' \circ h^{-1})(z) (h' \circ h^{-1})(z) - (g' \circ h^{-1})(z) (h'' \circ h^{-1})(z)}{[(h' \circ h^{-1})(z)]^3} \end{aligned}$$

for almost every (a.e.) $z \in [h(a), h(b)]$.

If $x \in [a, b]$ and we put $z = h(x)$, then

$$\begin{aligned} &(g \circ h^{-1})''(h(x)) \\ &= \frac{(g'' \circ h^{-1})(h(x)) (h' \circ h^{-1})(h(x)) - (g' \circ h^{-1})(h(x)) (h'' \circ h^{-1})(h(x))}{[(h' \circ h^{-1})(h(x))]^3} \\ &= \frac{g''(x)h'(x) - g'(x)h''(x)}{[h'(x)]^3} \in [\phi, \Phi]. \end{aligned}$$

If we use the inequality (3.5) for $f = g \circ h^{-1}$ and the interval $[h(a), h(b)]$, then we get

$$\begin{aligned}
& \left| \frac{g \circ h^{-1}(h(a)) + g \circ h^{-1}(h(b))}{2} \right. \\
& \quad - \frac{1}{12} (h(b) - h(a)) \left[\frac{(g' \circ h^{-1})(h(b))}{(h' \circ h^{-1})(h(b))} - \frac{(g' \circ h^{-1})(h(a))}{(h' \circ h^{-1})(h(a))} \right] \\
& \quad \left. - \frac{1}{h(b) - h(a)} \int_{h(a)}^{h(b)} (g \circ h^{-1})(z) dz \right| \\
& \leq \frac{1}{4} \frac{(h(b) - h(a))^2}{\Phi - \phi} \\
& \times \left(\frac{\frac{(g' \circ h^{-1})(h(b))}{(h' \circ h^{-1})(h(b))} - \frac{(g' \circ h^{-1})(h(a))}{(h' \circ h^{-1})(h(a))}}{h(b) - h(a)} - \phi \right) \left(\Phi - \frac{\frac{(g' \circ h^{-1})(h(b))}{(h' \circ h^{-1})(h(b))} - \frac{(g' \circ h^{-1})(h(a))}{(h' \circ h^{-1})(h(a))}}{h(b) - h(a)} \right) \\
& \leq \frac{1}{16} [h(b) - h(a)]^2 (\Phi - \phi),
\end{aligned}$$

which is equivalent to (3.21). \square

Theorem 6. *Assume that $w : [a, b] \rightarrow (0, \infty)$ is absolutely continuous on $[a, b]$, $g : [a, b] \rightarrow \mathbb{R}$ has an absolutely continuous derivative on $[a, b]$ and there exist the real numbers $\phi < \Phi$ with*

$$(3.24) \quad \phi \leq \frac{g''(x)w(x) - g'(x)w'(x)}{w^3(x)} \leq \Phi \text{ for a.e. } x \in [a, b],$$

then we have

$$\begin{aligned}
(3.25) \quad & \left| \frac{g(a) + g(b)}{2} \right. \\
& \quad - \frac{1}{12} \left[\frac{g'(b)}{w(b)} - \frac{g'(a)}{w(a)} \right] \int_a^b w(s) ds - \frac{1}{\int_a^b w(s) ds} \int_a^b g(t) w(t) dt \left| \right. \\
& \leq \frac{1}{4} \frac{\left(\int_a^b w(s) ds \right)^2}{\Phi - \phi} \left(\frac{\frac{g'(b)}{w(b)} - \frac{g'(a)}{w(a)}}{\int_a^b w(s) ds} - \phi \right) \left(\Phi - \frac{\frac{g'(b)}{w(b)} - \frac{g'(a)}{w(a)}}{\int_a^b w(s) ds} \right) \\
& \leq \frac{1}{16} (\Phi - \phi) \left(\int_a^b w(s) ds \right)^2.
\end{aligned}$$

In particular, if

$$(3.26) \quad \left| \frac{g''(x)w(x) - g'(x)w'(x)}{w^3(x)} \right| \leq \Phi \text{ for a.e. } x \in [a, b],$$

then

$$(3.27) \quad \left| \frac{g(a) + g(b)}{2} - \frac{1}{12} \left[\frac{g'(b)}{w(b)} - \frac{g'(a)}{w(a)} \right] \int_a^b w(s) ds - \frac{1}{\int_a^b w(s) ds} \int_a^b g(t) w(t) dt \right| \leq \frac{1}{8} \Phi \left(\int_a^b w(s) ds \right)^2 - \frac{1}{8\Phi} \left[\frac{g'(b)}{w(b)} - \frac{g'(a)}{w(a)} \right]^2 \leq \frac{1}{8} \Phi \left(\int_a^b w(s) ds \right)^2.$$

a). Assume that $g : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ has an absolutely continuous derivative on $[a, b]$ and there exist the real numbers $\phi < \Phi$ with

$$(3.28) \quad \phi \leq g''(x)x^2 + g'(x)x \leq \Phi \text{ for a.e. } x \in [a, b],$$

then by (3.25) for $w(t) = \frac{1}{t}$ we get

$$(3.29) \quad \left| \frac{g(a) + g(b)}{2} - \frac{1}{12} [g'(b)b - g'(a)a] \ln \left(\frac{b}{a} \right) - \frac{1}{\ln \left(\frac{b}{a} \right)} \int_a^b \frac{g(t)}{t} dt \right| \leq \frac{1}{4} \frac{\left(\ln \left(\frac{b}{a} \right) \right)^2}{\Phi - \phi} \left(\frac{g'(b)b - g'(a)a}{\ln \left(\frac{b}{a} \right)} - \phi \right) \left(\Phi - \frac{g'(b)b - g'(a)a}{\ln \left(\frac{b}{a} \right)} \right) \leq \frac{1}{16} (\Phi - \phi) \left(\ln \left(\frac{b}{a} \right) \right)^2.$$

b). Assume that $g : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ has an absolutely continuous derivative on $[a, b]$ and there exist the real numbers $\phi < \Phi$ with

$$(3.30) \quad \phi \leq \frac{g''(x) - g'(x)}{\exp(2x)} \leq \Phi \text{ for a.e. } x \in [a, b],$$

then by (3.25) for $w(t) = \exp t$ we get

$$(3.31) \quad \left| \frac{g(a) + g(b)}{2} - \frac{1}{12} [g'(b) \exp(-b) - g'(a) \exp(-a)] (\exp b - \exp a) - \frac{1}{(\exp b - \exp a)} \int_a^b g(t) \exp t dt \right| \leq \frac{1}{4} \frac{(\exp b - \exp a)^2}{\Phi - \phi} \times \left(\frac{g'(b) \exp(-b) - g'(a) \exp(-a)}{\exp b - \exp a} - \phi \right) \left(\Phi - \frac{g'(b) \exp(-b) - g'(a) \exp(-a)}{\exp b - \exp a} \right) \leq \frac{1}{16} (\Phi - \phi) (\exp b - \exp a)^2.$$

c). Assume that $g : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ has an absolutely continuous derivative on $[a, b]$ and there exist the real numbers $\phi < \Phi$ with

$$(3.32) \quad \phi \leq \frac{g''(x)x - pg'(x)}{x^{2p+1}} \leq \Phi \text{ for a.e. } x \in [a, b], \quad p \neq -1,$$

then by (3.25) for $w(t) = t^p$ we get

$$\begin{aligned}
 (3.33) \quad & \left| \frac{g(a) + g(b)}{2} \right. \\
 & \left. - \frac{b^{p+1} - a^{p+1}}{12(p+1)} [g'(b)b^{-p} - g'(a)a^{-p}] - \frac{p+1}{b^{p+1} - a^{p+1}} \int_a^b g(t)t^p dt \right| \\
 & \leq \frac{1}{4} \frac{(b^{p+1} - a^{p+1})^2}{(p+1)^2 (\Phi - \phi)} \\
 & \times \left(\frac{(p+1)(g'(b)b^{-p} - g'(a)a^{-p})}{b^{p+1} - a^{p+1}} - \phi \right) \left(\Phi - \frac{(p+1)(g'(b)b^{-p} - g'(a)a^{-p})}{b^{p+1} - a^{p+1}} \right) \\
 & \leq \frac{1}{16} (\Phi - \phi) \frac{(b^{p+1} - a^{p+1})^2}{(p+1)^2}.
 \end{aligned}$$

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