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REVERSES OF JENSEN'S INTEGRAL INEQUALITY VIA A WEIGHTED OSTROWSKI RESULT WITH APPLICATIONS FOR CONTINUOUS f-DIVERGENCE MEASURES

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ABSTRACT. In this paper we obtain some reverses of Jensen's integral inequality by employing a refinement of the weighted integral inequality of Ostrowski. Applications for continuous f-divergence measures with an example for the Kullback-Leibler divergence are also given.

1. INTRODUCTION

For two *Lebesgue integrable* functions $f,\ g:[a,b]\to\mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g):=\frac{1}{b-a}\int_a^b f(t)g(t)dt-\frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt.$$

In 1935, Grüss [7] showed that

$$|C(f,g)|\le\frac{1}{4}(M-m)(N-n), \tag{1.1}$$

provided that there exists the real numbers m, M, n, N such that

$$m\le f(t)\le M \quad\text{and}\quad n\le g(t)\le N \quad\text{for a.e. } t\in[a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller quantity.

The following inequality was obtained by Ostrowski in 1970, [10]:

$$|C(f,g)|\le\frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{1.3}$$

provided that f is *Lebesgue integrable* and satisfies (1.2) while g is absolutely continuous and $g'\in L_\infty[a,b]$. The constant $\frac{1}{8}$ is best possible in (1.3).

In [5] we obtained the following refinement of Ostrowski's inequality (1.3):

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Theorem 1. *Let f, $g : [a,b] \to \mathbb{R}$ be such that g is absolutely continuous on $[a,b]$ with $g' \in L_\infty [a,b]$ and f is Lebesgue integrable and satisfies (1.2), then*

$$
\begin{aligned}
(1.4)\quad & |C(f,g)| \\
&\leq \frac{1}{2}\|g'\|_\infty \frac{b-a}{M-m}\left(\frac{1}{b-a}\int_a^b f(t)\,dt - m\right)\left(M - \frac{1}{b-a}\int_a^b f(t)\,dt\right) \\
&\leq \frac{1}{8}\|g'\|_\infty (b-a)(M-m).
\end{aligned}
$$

The constants $\frac{1}{2}$ and $\frac{1}{8}$ are best possible.

In this paper we obtain some reverses of Jensen's integral inequality by employing a refinement of the weighted integral inequality of Ostrowski. Applications for continuous f-divergence measures with an example for the Kullback-Leibler divergence are also given.

2. Ostrowski Weighted Inequality

Consider now the *weighted Čebyšev functional*

$$
\begin{aligned}
(2.1)\quad C_w(f,g) := & \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) f(t) g(t)\,dt \\
& - \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) f(t)\,dt \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) g(t)\,dt
\end{aligned}
$$

where f, g, $w : [a,b] \to \mathbb{R}$ and $w(t) \geq 0$ for a.e. $t \in [a,b]$ are measurable functions such that the involved integrals exist and $\int_a^b w(t)\,dt > 0$.

We can also define, as above,

$$
\begin{aligned}
(2.2)\quad C_{h'}(f,g) := & \frac{1}{h(b)-h(a)}\int_a^b f(t) g(t) h'(t)\,dt \\
& - \frac{1}{h(b)-h(a)}\int_a^b f(t) h'(t)\,dt \frac{1}{h(b)-h(a)}\int_a^b g(t) h'(t)\,dt,
\end{aligned}
$$

where h is absolutely continuous and f, g are Lebesgue measurable on $[a,b]$ and such that the above integrals exist.

Lemma 1. *Let $h : [a,b] \to [h(a), h(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). If f is Lebesgue integrable and satisfies the condition $m \leq f(t) \leq M$ for $t \in [a,b]$ and $g : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ and $\frac{g'}{h'}$ is essentially bounded, namely $\frac{g'}{h'} \in L_\infty [a,b]$, then we have*

$$
\begin{aligned}
(2.3)\quad & |C_{h'}(f,g)| \leq \frac{1}{2}\frac{h(b)-h(a)}{M-m}\left\|\frac{g'}{h'}\right\|_\infty \\
& \times \left(\frac{1}{h(b)-h(a)}\int_a^b f(t) h'(t)\,dt - m\right)\left(M - \frac{1}{h(b)-h(a)}\int_a^b f(t) h'(t)\,dt\right) \\
& \leq \frac{1}{8}[h(b)-h(a)](M-m)\left\|\frac{g'}{h'}\right\|_{[a,b],\infty}.
\end{aligned}
$$

The constants $\frac{1}{2}$ and $\frac{1}{8}$ are best possible.

Proof. Since $\frac{g'}{h'} \in L_\infty [c,d]$, hence $\left(g \circ h^{-1}\right)' \in L_\infty [h(c), h(d)]$. Also

$$\left\| \left(g \circ h^{-1}\right)' \right\|_{[h(c),h(d)],\infty} = \left\| \frac{g'}{h'} \right\|_{[c,d],\infty}.$$

Now, if we use the refinement of Ostrowski's inequality (1.4) for the functions $f \circ h^{-1}$ and $g \circ h^{-1}$ on the interval $[h(a), h(b)]$, then we get

$$\begin{aligned} (2.4) \quad & \left| \frac{1}{h(b) - h(a)} \int_{h(a)}^{h(b)} f \circ h^{-1}(u) g \circ h^{-1}(u) du \right. \\ & \left. - \frac{1}{[h(b) - h(a)]^2} \int_{h(a)}^{h(b)} f \circ h^{-1}(u) du \int_{h(a)}^{h(b)} g \circ h^{-1}(u) du \right| \\ & \leq \frac{1}{2} \frac{h(b) - h(a)}{M - m} \left\| \left(g \circ h^{-1}\right)' \right\|_{[h(c),h(d)],\infty} \\ & \times \left(\frac{1}{h(b) - h(a)} \int_{h(a)}^{h(b)} f \circ h^{-1}(t) dt - m \right) \left(M - \frac{1}{h(b) - h(a)} \int_{h(a)}^{h(b)} f(t) dt \right) \\ & \leq \frac{1}{8} [h(b) - h(a)] (M - m) \left\| \left(g \circ h^{-1}\right)' \right\|_{[h(a),h(b)],\infty} \end{aligned}$$

since $m \leq f \circ h^{-1}(u) \leq M$ for all $u \in [h(a), h(b)]$.

Observe also that, by the change of variable $t = h^{-1}(u)$, $u \in [g(a), g(b)]$, we have $u = h(t)$ that gives $du = h'(t) dt$ and

$$\int_{h(a)}^{h(b)} \left(f \circ h^{-1}\right)(u) du = \int_a^b f(t) h'(t) dt,$$

$$\int_{h(a)}^{h(b)} g \circ h^{-1}(u) du = \int_a^b g(t) h'(t) dt,$$

$$\int_{h(a)}^{h(b)} f \circ h^{-1}(u) g \circ h^{-1}(u) du = \int_a^b f(t) g(t) h'(t) dt$$

and

$$\left\| \left(g \circ h^{-1}\right)' \right\|_{[h(a),h(b)],\infty} = \left\| \frac{g'}{h'} \right\|_{[a,b],\infty}.$$

By making use of (2.4) we then get the desired result (2.3).

The best constant follows by the refinement of Ostrowski's inequality (1.4). □

If $w : [a,b] \to \mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W : [a,b] \to [0,\infty)$, $W(x) := \int_a^x w(s) ds$ is strictly increasing and differentiable on (a,b). We have $W'(x) = w(x)$ for any $x \in (a,b)$.

Theorem 2. *Assume that $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$, f is Lebesgue integrable and satisfies the condition $m \leq f(t) \leq M$ for $t \in [a,b]$ and $g : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ with $\frac{g'}{w}$ is essentially bounded, namely $\frac{g'}{w} \in$*

$L_\infty[a,b]$, then we have

$$|C_w(f,g)| \le \frac{1}{2(M-m)}\left\|\frac{g'}{w}\right\|_\infty \times \left(\frac{\int_a^b f(t)w(t)\,dt}{\int_a^b w(s)\,ds}-m\right)\left(M-\frac{\int_a^b f(t)w(t)\,dt}{\int_a^b w(s)\,ds}\right)\int_a^b w(s)\,ds \le \frac{1}{8}(M-m)\left\|\frac{g'}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds. \tag{2.5}$$

The constant $\frac{1}{8}$ is best possible.

The proof follows by Lemma 1 by taking $h(x) := \int_a^x w(t)\,dt$, $x\in[a,b]$.

Remark 1. *Under the assumptions of Theorem 2 and if there exists a constant $K>0$ such that $|g'(t)|\le Kw(t)$ for a.e. $t\in[a,b]$, then by (2.5) we get*

$$|C_w(f,g)| \le \frac{K}{2(M-m)} \times \left(\frac{\int_a^b f(t)w(t)\,dt}{\int_a^b w(s)\,ds}-m\right)\left(M-\frac{\int_a^b f(t)w(t)\,dt}{\int_a^b w(s)\,ds}\right)\int_a^b w(s)\,ds \le \frac{1}{8}(M-m)K\int_a^b w(s)\,ds. \tag{2.6}$$

a). For $w(t)=\frac{1}{\ell(t)}=\ell^{-1}(t)$, $t\in[a,b]\subset(0,\infty)$, where $\ell(t)=t$, define

$$C_{\ell^{-1}}(f,g) := \frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b \frac{f(t)g(t)}{t}dt - \frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b \frac{f(t)}{t}dt\frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b \frac{g(t)}{t}dt. \tag{2.7}$$

If $m\le f(t)\le M$ for $t\in[a,b]$ and $g:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ with $\ell g'$ is essentially bounded, namely $\ell g'\in L_\infty[a,b]$, then we have

$$|C_{\ell^{-1}}(f,g)| \le \frac{1}{2(M-m)}\|\ell g'\|_{[a,b],\infty} \times \left(\frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b \frac{f(t)}{t}dt-m\right)\left(M-\frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b \frac{f(t)}{t}dt\right)\ln\left(\frac{b}{a}\right) \le \frac{1}{8}(M-m)\|\ell g'\|_{[a,b],\infty}\ln\left(\frac{b}{a}\right). \tag{2.8}$$

b). For $w(t)=\exp t$, $t\in[a,b]$, define

$$C_{\exp}(f,g) := \frac{1}{\exp b-\exp a}\int_a^b f(t)g(t)\exp t dt - \frac{1}{\exp b-\exp a}\int_a^b f(t)\exp t dt\frac{1}{\exp b-\exp a}\int_a^b g(t)\exp t dt. \tag{2.9}$$

If $m \leq f(t) \leq M$ for $t \in [a,b]$ and $g : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ with $\frac{g'}{\exp}$ is essentially bounded, namely $\frac{g'}{\exp} \in L_\infty [a,b]$, then we have

$$
\begin{aligned}
(2.10)\quad & \left|C_{\exp}(f,g)\right| \\
& \leq \frac{1}{2(M-m)} \left\| \frac{g'}{\exp} \right\|_{[a,b],\infty} \left(\frac{\int_a^b f(t) \exp t dt}{\exp b - \exp a} - m \right) \left(M - \frac{\int_a^b f(t) \exp t dt}{\exp b - \exp a} \right) \\
& \qquad \times (\exp b - \exp a) \\
& \qquad \leq \frac{1}{8}(M-m) \left\| \frac{g'}{\exp} \right\|_{[a,b],\infty} (\exp b - \exp a).
\end{aligned}
$$

c). For $w(t) = \ell^p(t)$, $t \in [a,b] \subset (0,\infty)$, where $\ell(t) = t$ and $p \neq -1$, define

$$
\begin{aligned}
(2.11)\quad C_{\ell^p}(f,g) := & \frac{p+1}{b^{p+1} - a^{p+1}} \int_a^b t^p f(t) g(t) dt \\
& - \frac{p+1}{b^{p+1} - a^{p+1}} \int_a^b t^p f(t) dt \frac{p+1}{b^{p+1} - a^{p+1}} \int_a^b t^p g(t) dt.
\end{aligned}
$$

If $m \leq f(t) \leq M$ for $t \in [a,b]$ and $g : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ with $g'\ell^{-p}$ is essentially bounded, namely $g'\ell^{-p} \in L_\infty [a,b]$, then we have

$$
\begin{aligned}
(2.12)\quad & \left|C_{\ell^p}(f,g)\right| \leq \frac{b^{p+1} - a^{p+1}}{2(p+1)(M-m)} \left\| g'\ell^{-p} \right\|_\infty \\
& \times \left(\frac{p+1}{b^{p+1} - a^{p+1}} \int_a^b f(t) t^p dt - m \right) \left(M - \frac{p+1}{b^{p+1} - a^{p+1}} \int_a^b f(t) t^p dt \right) \\
& \qquad \leq \frac{b^{p+1} - a^{p+1}}{8(p+1)} (M-m) \left\| g'\ell^{-p} \right\|_{[a,b],\infty}.
\end{aligned}
$$

In probability theory and statistics, the *beta prime distribution* (also known as *inverted beta distribution* or *beta distribution of the second kind*) is an absolutely continuous probability distribution defined for $x > 0$ with two parameters α and β, having the probability density function:

$$
w_{\alpha,\beta}(x) := \frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha,\beta)}
$$

where B is *Beta function*

$$
B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}, \ \alpha, \ \beta > 0.
$$

The cumulative distribution function is

$$
W_{\alpha,\beta}(x) = I_{\frac{x}{1+x}}(\alpha,\beta),
$$

where I is the *regularized incomplete beta function* defined by

$$
I_z(\alpha,\beta) := \frac{B(z;\alpha,\beta)}{B(\alpha,\beta)}.
$$

Here $B(\cdot;\alpha,\beta)$ is the *incomplete beta function* defined by

$$
B(z;\alpha,\beta) := \int_0^z t^{\alpha-1}(1-t)^{\beta-1}, \ \alpha, \ \beta, \ z > 0.
$$

Define

$$C_{w_{\alpha,\beta}}(f,g) := \frac{1}{B(\alpha,\beta)}\int_0^\infty t^{\alpha-1}(1+t)^{-\alpha-\beta} f(t) g(t)\,dt - \frac{1}{B(\alpha,\beta)}\int_0^\infty t^{\alpha-1}(1+t)^{-\alpha-\beta} f(t)\,dt \frac{1}{B(\alpha,\beta)}\int_0^\infty t^{\alpha-1}(1+t)^{-\alpha-\beta} g(t)\,dt, \tag{2.13}$$

provided the integrals exist.

If f is Lebesgue measurable and there exists the constants $m \le f(t) \le M$ for $t \in (0,\infty)$ and $g : (0,\infty) \to \mathbb{R}$ is locally absolutely continuous on $(0,\infty)$ with

$$|g'(t)| \le L t^{\alpha-1}(1+t)^{-\alpha-\beta} \text{ for a.e. } t \in (0,\infty), \tag{2.14}$$

then by (2.6) we get

$$\begin{aligned} \left|C_{w_{\alpha,\beta}}(f,g)\right| &\le \frac{LB(\alpha,\beta)}{2(M-m)} \\ &\times \left(\frac{1}{B(\alpha,\beta)}\int_0^\infty t^{\alpha-1}(1+t)^{-\alpha-\beta} f(t)\,dt - m\right) \\ &\times \left(M - \frac{1}{B(\alpha,\beta)}\int_0^\infty t^{\alpha-1}(1+t)^{-\alpha-\beta} f(t)\,dt\right) \\ &\le \frac{1}{8}(M-m) LB(\alpha,\beta). \end{aligned} \tag{2.15}$$

The probability density of the *normal distribution* on $(-\infty,\infty)$ is

$$w_{\mu,\sigma^2}(x) := \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\ x \in \mathbb{R},$$

where μ is the *mean* or *expectation* of the distribution (and also its *median* and *mode*), σ is the *standard deviation*, and σ^2 is the *variance*.

The cumulative distribution function is

$$W_{\mu,\sigma^2}(x) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right),$$

where the *error function* erf is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-t^2\right) dt.$$

Consider

$$C_{w_{\mu,\sigma^2}}(f,g) := \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^\infty \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) f(t) g(t)\,dt - \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^\infty \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) f(t)\,dt \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^\infty \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) g(t)\,dt, \tag{2.16}$$

provided the integrals exist.

If f is Lebesgue measurable and there exists the constants $m \le f(t) \le M$ for $t \in (-\infty,\infty)$ and $g : (-\infty,\infty) \to \mathbb{R}$ is locally absolutely continuous on $(-\infty,\infty)$ with

$$|g'(t)| \le L\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) \text{ for a.e. } t \in (0,\infty), \tag{2.17}$$

then by (2.6) we get

$$(2.18)\quad |C_w(f,g)| \le \frac{\sqrt{2\pi}\sigma L}{2(M-m)} \times \left(\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)f(t)\,dt - m\right) \times \left(M - \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)f(t)\,dt\right) \le \frac{1}{8}(M-m)\sqrt{2\pi}\sigma L.$$

3. Reverses of Jensen's Inequality

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \ge 0$ for μ-a.e. (almost every) $x \in \Omega$, consider the *Lebesgue space*

$$L_w(\Omega,\mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega w(x)|f(x)|\,d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, S. S. Dragomir obtained in 2002 [4] the following result:

Theorem 3. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L_w(\Omega,\mu)$, where $w \ge 0$ μ-a.e. (almost everywhere) on Ω with $\int_\Omega w d\mu = 1$. Then we have the inequality:*

$$(3.1)\quad 0 \le \int_\Omega w(\Phi\circ f)\,d\mu - \Phi\left(\int_\Omega wf d\mu\right) \le \int_\Omega w(\Phi'\circ f) f d\mu - \int_\Omega w(\Phi'\circ f)\,d\mu \int_\Omega wf d\mu.$$

We have the following reverse of Jensen's inequality:

Theorem 4. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M), $w : [a, b] \to (0,\infty)$ be continuous on $[a, b]$ and $f : [a, b] \to [m, M]$ is absolutely continuous so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L_w[a, b]$.*

(i) *If $\frac{f'}{w} \in L_\infty[a, b]$, then we have the inequality*

$$(3.2)\quad 0\leq\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Phi\circ f)(t)\,dt-\Phi\left(\frac{\int_a^b w(t)f(t)\,dt}{\int_a^b w(s)\,ds}\right)$$
$$\leq\frac{1}{2}\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\frac{1}{\Phi'_-(M)-\Phi'_+(m)}$$
$$\times\left(\frac{\int_a^b(\Phi'\circ f)(t)\,w(t)\,dt}{\int_a^b w(s)\,ds}-\Phi'_+(m)\right)\left(\Phi'_-(M)-\frac{\int_a^b(\Phi'\circ f)(t)\,w(t)\,dt}{\int_a^b w(s)\,ds}\right)$$
$$\times\int_a^b w(s)\,ds$$
$$\leq\frac{1}{8}\left[\Phi'_-(M)-\Phi'_+(m)\right]\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds.$$

(ii) *If* Φ *is twice differentiable on* (m,M) *and* $\frac{(\Phi''\circ f)f'}{w}\in L_\infty[a,b]$, *then*

$$(3.3)\quad 0\leq\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Phi\circ f)(t)\,dt-\Phi\left(\frac{\int_a^b w(t)f(t)\,dt}{\int_a^b w(s)\,ds}\right)$$
$$\leq\frac{1}{2}\left\|\frac{(\Phi''\circ f)f'}{w}\right\|_{[a,b],\infty}\frac{1}{M-m}$$
$$\times\left(\frac{\int_a^b f(t)\,w(t)\,dt}{\int_a^b w(s)\,ds}-m\right)\left(M-\frac{\int_a^b f(t)\,w(t)\,dt}{\int_a^b w(s)\,ds}\right)\int_a^b w(s)\,ds$$
$$\leq\frac{1}{8}(M-m)\left\|\frac{(\Phi''\circ f)f'}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds.$$

Proof. (i) By (3.1) we have

$$(3.4)\qquad 0\leq\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Phi\circ f)(t)\,dt-\Phi\left(\frac{\int_a^b w(t)f(t)\,dt}{\int_a^b w(s)\,ds}\right)$$
$$\leq\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Phi'\circ f)(t)f(t)\,dt$$
$$-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Phi'\circ f)(t)\,dt\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)f(t)\,dt.$$

Since Φ is differentiable convex on (m,M), hence

$$\Phi'_+(m)\leq(\Phi'\circ f)(t)\leq\Phi'_-(M)$$

for $t\in[a,b]$.

If we use the inequality (2.5), then we get

$$\begin{aligned}
&\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Phi\circ f)(t)f(t)\,dt \\
&-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Phi'\circ f)(t)\,dt\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)f(t)\,dt \\
&\leq \frac{1}{2\left(\Phi'_-(M)-\Phi'_+(m)\right)}\left\|\frac{f'}{w}\right\|_{[a,b],\infty} \\
&\times\left(\frac{\int_a^b(\Phi'\circ f)(t)w(t)\,dt}{\int_a^b w(s)\,ds}-\Phi'_+(m)\right)\left(\Phi'_-(M)-\frac{\int_a^b(\Phi'\circ f)(t)w(t)\,dt}{\int_a^b w(s)\,ds}\right) \\
&\times\int_a^b w(s)\,ds \\
&\leq\frac{1}{8}\left[\Phi'_-(M)-\Phi'_+(m)\right]\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds,
\end{aligned}$$

which, together with (3.4), proves the required inequality (3.2).

(ii) If Φ is twice differentiable on (a,b), then

$$(\Phi'\circ f)'(t)=(\Phi''\circ f)(t)f'(t)$$

for $t\in(a,b)$.

Since $m\leq f(t)\leq M$ for $t\in[a,b]$ and

$$\frac{(\Phi''\circ f)f'}{w}\in L_\infty[a,b],$$

then by using the inequality (2.5) we also have

$$\begin{aligned}
&\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Phi\circ f)(t)f(t)\,dt \\
&-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(\Phi'\circ f)(t)\,dt\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)f(t)\,dt \\
&\leq\frac{1}{2(M-m)}\left\|\frac{(\Phi''\circ f)f'}{w}\right\|_{[a,b],\infty} \\
&\times\left(\frac{\int_a^b f(t)w(t)\,dt}{\int_a^b w(s)\,ds}-m\right)\left(M-\frac{\int_a^b f(t)w(t)\,dt}{\int_a^b w(s)\,ds}\right)\int_a^b w(s)\,ds \\
&\leq\frac{1}{8}(M-m)\left\|\frac{(\Phi''\circ f)f'}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds,
\end{aligned}$$

which, together with (3.4), proves (3.3). □

Corollary 1. *Let $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable convex function on (m,M) and $f:[a,b]\to[m,M]$ be absolutely continuous so that $\Phi\circ f$, f, $\Phi'\circ f$, $(\Phi'\circ f)f\in L[a,b]$.*

(i) *If* $f' \in L_\infty[a,b]$, *then we have the inequality*

$$0 \le \frac{1}{b-a}\int_a^b (\Phi\circ f)(t)\,dt - \Phi\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \le \frac{1}{2}\|f'\|_{[a,b],\infty}\frac{b-a}{\Phi'_-(M)-\Phi'_+(m)} \times\left(\frac{1}{b-a}\int_a^b (\Phi'\circ f)(t)\,dt - \Phi'_+(m)\right)\left(\Phi'_-(M) - \frac{1}{b-a}\int_a^b (\Phi'\circ f)(t)\,dt\right) \le \frac{1}{8}(b-a)\left[\Phi'_-(M)-\Phi'_+(m)\right]\|f'\|_{[a,b],\infty}. \tag{3.5}$$

(ii) *If* Φ *is twice differentiable on* (m,M) *and* $(\Phi''\circ f)f' \in L_\infty[a,b]$, *then*

$$0 \le \frac{1}{b-a}\int_a^b (\Phi\circ f)(t)\,dt - \Phi\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \le \frac{1}{2}\|(\Phi''\circ f)f'\|_{[a,b],\infty}\frac{1}{M-m}\left(\frac{1}{b-a}\int_a^b f(t)\,dt - m\right)\left(M-\frac{1}{b-a}\int_a^b f(t)\,dt\right) \le \frac{1}{8}(b-a)(M-m)\|(\Phi''\circ f)f'\|_{[a,b],\infty}. \tag{3.6}$$

Corollary 2. *Let* $\Phi:[a,b]\subset\mathbb{R}\to\mathbb{R}$ *be a differentiable convex function on* (a,b), $w:[a,b]\to(0,\infty)$ *be continuous on* $[a,b]$ *and* $\Phi,\ \Phi'\in L_w[a,b]$.

(i) *If* $\frac{1}{w}\in L_\infty[a,b]$, *then we have the inequality*

$$0 \le \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\Phi(t)\,dt - \Phi\left(\frac{\int_a^b tw(t)\,dt}{\int_a^b w(s)\,ds}\right) \le \frac{1}{2}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\frac{1}{\Phi'_-(M)-\Phi'_+(m)} \times\left(\frac{\int_a^b \Phi'(t)w(t)\,dt}{\int_a^b w(s)\,ds}-\Phi'_+(m)\right)\left(\Phi'_-(M)-\frac{\int_a^b \Phi'(t)w(t)\,dt}{\int_a^b w(s)\,ds}\right) \times\int_a^b w(s)\,ds \le \frac{1}{8}\left[\Phi'_-(b)-\Phi'_+(a)\right]\left\|\frac{1}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds. \tag{3.7}$$

(ii) *If f* Φ *is twice differentiable on* (m,M) *and* $\frac{\Phi''}{w}\in L_{\infty}[a,b]$*, then*

$$
\begin{aligned}
(3.8)\quad 0 &\leq \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\,\Phi(t)\,dt-\Phi\left(\frac{\int_a^b tw(t)\,dt}{\int_a^b w(s)\,ds}\right)\\
&\leq \frac{1}{2}\left\|\frac{(\Phi''\circ f)\,f'}{w}\right\|_{[a,b],\infty}\frac{1}{M-m}\\
&\times\left(\frac{\int_a^b tw(t)\,dt}{\int_a^b w(s)\,ds}-m\right)\left(M-\frac{\int_a^b tw(t)\,dt}{\int_a^b w(s)\,ds}\right)\int_a^b w(s)\,ds\\
&\leq \frac{1}{8}(b-a)\left\|\frac{\Phi''}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds.
\end{aligned}
$$

We observe that, if either in Corollary 1 or 2 we take the weight $w\equiv 1$, then we get the known result

$$
\begin{aligned}
(3.9)\quad 0 &\leq \frac{1}{b-a}\int_a^b \Phi(t)\,dt-\Phi\left(\frac{a+b}{2}\right)\\
&\leq \frac{b-a}{2\left(\Phi'_-(b)-\Phi'_+(a)\right)}\left(\frac{\Phi(b)-\Phi(a)}{b-a}-\Phi'_+(a)\right)\left(\Phi'_-(b)-\frac{\Phi(b)-\Phi(a)}{b-a}\right)\\
&\leq \frac{1}{8}(b-a)\left[\Phi'_-(b)-\Phi'_+(a)\right]
\end{aligned}
$$

with $\frac{1}{8}$ as the best possible constant.

Define the function $\ell(t):=t$, $t\in\mathbb{R}$.

a). Let $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable convex function on (m,M) and $f:[a,b]\subset(0,\infty)\to[m,M]$ be absolutely continuous and so that $\Phi\circ f$, f, $\Phi'\circ f$, $(\Phi'\circ f)\,f\in L_{\ell^{-1}}[a,b]$. If $f'\ell\in L_{\infty}[a,b]$, then by the statement (i) of Theorem 4 we have the inequality

$$
\begin{aligned}
(3.10)\qquad 0 &\leq \frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b \frac{(\Phi\circ f)(t)}{t}dt-\Phi\left(\frac{\int_a^b \frac{f(t)}{t}dt}{\ln\left(\frac{b}{a}\right)}\right)\\
&\leq \frac{1}{2\left(\Phi'_-(M)-\Phi'_+(m)\right)}\|\ell f'\|_{[a,b],\infty}\\
&\times\left(\frac{\int_a^b \frac{(\Phi'\circ f)(t)}{t}dt}{\ln\left(\frac{b}{a}\right)}-\Phi'_+(m)\right)\left(\Phi'_-(M)-\frac{\int_a^b \frac{(\Phi'\circ f)(t)}{t}dt}{\ln\left(\frac{b}{a}\right)}\right)\ln\left(\frac{b}{a}\right)\\
&\leq \frac{1}{8}\left[\Phi'_-(M)-\Phi'_+(m)\right]\ln\left(\frac{b}{a}\right)\|\ell f'\|_{[a,b],\infty}.
\end{aligned}
$$

If Φ is twice differentiable on (m,M) and $(\Phi''\circ f)f'\ell\in L_\infty[a,b]$, then by the statement (ii) of Theorem 4 we have the inequality

$$
\begin{aligned}
(3.11)\qquad & 0\leq\frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b\frac{(\Phi\circ f)(t)}{t}dt-\Phi\left(\frac{\int_a^b\frac{f(t)}{t}dt}{\ln\left(\frac{b}{a}\right)}\right)\\
&\leq\frac{1}{2(M-m)}\left\|(\Phi''\circ f)f'\ell\right\|_{[a,b],\infty}\\
&\times\left(\frac{\int_a^b\frac{f(t)}{t}dt}{\ln\left(\frac{b}{a}\right)}-m\right)\left(M-\frac{\int_a^b\frac{f(t)}{t}wdt}{\ln\left(\frac{b}{a}\right)}\right)\ln\left(\frac{b}{a}\right)\\
&\leq\frac{1}{8}(M-m)\left\|(\Phi''\circ f)f'\ell\right\|_{[a,b],\infty}\ln\left(\frac{b}{a}\right).
\end{aligned}
$$

b). Let $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable convex function on (m,M) and $f:[a,b]\to[m,M]$ be absolutely continuous and so that $\Phi\circ f,\ f,\ \Phi'\circ f,\ (\Phi'\circ f)f\in L_{\exp}[a,b]$. If $\frac{f'}{\exp}\in L_\infty[a,b]$, then by the statement (i) of Theorem 4 we have the inequality

$$
\begin{aligned}
(3.12)\quad & 0\leq\frac{1}{\exp b-\exp a}\int_a^b(\Phi\circ f)(t)\exp tdt-\Phi\left(\frac{\int_a^b f(t)\exp tdt}{\exp b-\exp a}\right)\\
&\leq\frac{1}{2\left(\Phi'_-(M)-\Phi'_+(m)\right)}\left\|\frac{f'}{\exp}\right\|_{[a,b],\infty}\\
&\times\left(\frac{\int_a^b(\Phi'\circ f)(t)\exp tdt}{\exp b-\exp a}-\Phi'_+(m)\right)\left(\Phi'_-(M)-\frac{\int_a^b(\Phi'\circ f)(t)\exp tdt}{\exp b-\exp a}\right)\\
&\times(\exp b-\exp a)\\
&\leq\frac{1}{8}\left[\Phi'_-(M)-\Phi'_+(m)\right]\left\|\frac{f'}{\exp}\right\|_{[a,b],\infty}(\exp b-\exp a).
\end{aligned}
$$

If Φ is twice differentiable on (m,M) and $\frac{(\Phi''\circ f)f'}{\exp}\in L_\infty[a,b]$, then by the statement (ii) of Theorem 4 we have the inequality

$$
\begin{aligned}
(3.13)\qquad & 0\leq\frac{1}{\exp b-\exp a}\int_a^b(\Phi\circ f)(t)\exp tdt-\Phi\left(\frac{\int_a^b f(t)\exp tdt}{\exp b-\exp a}\right)\\
&\leq\frac{1}{2(M-m)}\left\|\frac{(\Phi''\circ f)f'}{\exp}\right\|_{[a,b],\infty}\\
&\times\left(\frac{\int_a^b f(t)\exp tdt}{\exp b-\exp a}-m\right)\left(M-\frac{\int_a^b f(t)\exp tdt}{\exp b-\exp a}\right)(\exp b-\exp a)\\
&\leq\frac{1}{8}(M-m)\left\|\frac{(\Phi''\circ f)f'}{\exp}\right\|_{[a,b],\infty}(\exp b-\exp a).
\end{aligned}
$$

c). Consider the function $\ell^p(t):=t^p,\ t>0,\ p\in\mathbb{R}\setminus\{-1\}$. Let $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable convex function on (m,M) and $f:[a,b]\subset(0,\infty)\to[m,M]$ be absolutely continuous and so that $\Phi\circ f,\ f,\ \Phi'\circ f,\ (\Phi'\circ f)f\in L_{\ell^p}[a,b]$.

If $f'\ell^{-p} \in L_{\infty}[a,b]$, then by the statement (i) of Theorem 4 we have the inequality

$$
\begin{aligned}
(3.14)\quad 0 &\le \frac{p+1}{b^{p+1}-a^{p+1}}\int_a^b t^p (\Phi\circ f)(t)\,dt - \Phi\left(\frac{(p+1)\int_a^b t^p f(t)\,dt}{b^{p+1}-a^{p+1}}\right)\\
&\le \frac{1}{2\left(\Phi'_-(M)-\Phi'_+(m)\right)}\left\|f'\ell^{-p}\right\|_{[a,b],\infty}\\
&\times\left(\frac{(p+1)\int_a^b (\Phi'\circ f)(t)\,t^p dt}{b^{p+1}-a^{p+1}}-\Phi'_+(m)\right)\left(\Phi'_-(M)-\frac{(p+1)\int_a^b (\Phi'\circ f)(t)\,t^p dt}{b^{p+1}-a^{p+1}}\right)\\
&\times\frac{b^{p+1}-a^{p+1}}{p+1}\\
&\le \frac{1}{8(p+1)}\left[\Phi'_-(M)-\Phi'_+(m)\right]\left(b^{p+1}-a^{p+1}\right)\left\|f'\ell^{-p}\right\|_{[a,b],\infty}.
\end{aligned}
$$

If Φ is twice differentiable on (m,M) and $(\Phi''\circ f)f'\ell^{-p} \in L_{\infty}[a,b]$, then by the statement (ii) of Theorem 4 we have the inequality

$$
\begin{aligned}
(3.15)\quad 0 &\le \frac{p+1}{b^{p+1}-a^{p+1}}\int_a^b t^p (\Phi\circ f)(t)\,dt - \Phi\left(\frac{(p+1)\int_a^b t^p f(t)\,dt}{b^{p+1}-a^{p+1}}\right)\\
&\le \frac{1}{2(M-m)}\left\|(\Phi''\circ f)f'\ell^{-p}\right\|_{[a,b],\infty}\\
&\times\left(\frac{(p+1)\int_a^b f(t)\,t^p dt}{b^{p+1}-a^{p+1}}-m\right)\left(M-\frac{(p+1)\int_a^b f(t)\,t^p dt}{b^{p+1}-a^{p+1}}\right)\frac{b^{p+1}-a^{p+1}}{p+1}\\
&\le \frac{1}{8(p+1)}(M-m)\left(b^{p+1}-a^{p+1}\right)\left\|(\Phi''\circ f)f'\ell^{-p}\right\|_{[a,b],\infty}.
\end{aligned}
$$

For $p=-2$, we get from (3.14) that

$$
\begin{aligned}
(3.16)\quad 0 &\le \frac{ab}{b-a}\int_a^b \frac{(\Phi\circ f)(t)}{t^2}dt - \Phi\left(\frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt\right)\\
&\le \frac{1}{2}\left\|f'\ell^{2}\right\|_{[a,b],\infty}\frac{1}{\Phi'_-(M)-\Phi'_+(m)}\\
&\times\left(\frac{ab\int_a^b \frac{(\Phi'\circ f)(t)}{t^2}dt}{b-a}-\Phi'_+(m)\right)\left(\Phi'_-(M)-\frac{ab\int_a^b \frac{(\Phi'\circ f)(t)}{t^2}dt}{b-a}\right)\left(\frac{b-a}{ab}\right)\\
&\le \frac{1}{8}\left[\Phi'_-(M)-\Phi'_+(m)\right]\left(\frac{b-a}{ab}\right)\left\|f'\ell^{2}\right\|_{[a,b],\infty},
\end{aligned}
$$

provided $f'\ell^2 \in L_\infty [a,b]$, while from (3.15) we obtain

$$0 \le \frac{ab}{b-a}\int_a^b \frac{(\Phi\circ f)(t)}{t^2}dt - \Phi\left(\frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt\right) \tag{3.17}$$
$$\le \frac{1}{2}\left\|(\Phi''\circ f) f'\ell^2\right\|_{[a,b],\infty}\frac{1}{M-m}$$
$$\times\left(\frac{ab\int_a^b \frac{f(t)}{t^2}dt}{b-a} - m\right)\left(M - \frac{ab\int_a^b \frac{f(t)}{t^2}dt}{b-a}\right)\left(\frac{b-a}{ab}\right)$$
$$\le \frac{1}{8}(M-m)\left(\frac{b-a}{ab}\right)\left\|(\Phi''\circ f) f'\ell^2\right\|_{[a,b],\infty},$$

provided $(\Phi''\circ f) f'\ell^2 \in L_\infty [a,b]$.

4. Applications for f-Divergence Measure

Assume that I is a finite or an infinite interval of real numbers. Consider the set of all probability densities on I to be $\mathcal{P}(I) := \left\{p \,|p : I \to \mathbb{R},\ p(x) \ge 0,\ \int_I p(x)\,dx = 1\right\}$, where $\int_I$ is the usual Lebesgue integral on the interval I.

The *Kullback-Leibler divergence* [9] is well known among the information divergences. It is defined as:

$$D_{KL}(p,q) := \int_I p(x)\ln\left[\frac{p(x)}{q(x)}\right]dx,\quad p,\ q \in \mathcal{P}(I), \tag{4.1}$$

where ln is to base e.

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. These are the: *variation distance* D_v, *Hellinger distance* D_H [6], χ^2*-divergence* D_{χ^2}, α*-divergence* D_α, *Bhattacharyya distance* D_B [1], *Harmonic distance* D_{Ha}, *Jeffrey's distance* D_J [8], *triangular discrimination* D_Δ [11], etc... They are defined as follows:

$$D_v(p,q) := \int_I |p(x) - q(x)|\,dx,\quad p,\ q \in \mathcal{P}(I); \tag{4.2}$$

$$D_H(p,q) := \int_I \left|\sqrt{p(x)} - \sqrt{q(x)}\right|dx,\quad p,\ q \in \mathcal{P}(I); \tag{4.3}$$

$$D_{\chi^2}(p,q) := \int_I p(x)\left[\left(\frac{q(x)}{p(x)}\right)^2 - 1\right]dx,\quad p,\ q \in \mathcal{P}(I); \tag{4.4}$$

$$D_\alpha(p,q) := \frac{4}{1-\alpha^2}\left[1 - \int_I [p(x)]^{\frac{1-\alpha}{2}}[q(x)]^{\frac{1+\alpha}{2}}dx\right],\quad p,\ q \in \mathcal{P}(I); \tag{4.5}$$

$$D_B(p,q) := \int_I \sqrt{p(x)q(x)}dx,\quad p,\ q \in \mathcal{P}(I); \tag{4.6}$$

$$D_{Ha}(p,q) := \int_I \frac{2p(x)q(x)}{p(x)+q(x)}dx,\quad p,\ q \in \mathcal{P}(I); \tag{4.7}$$

$$D_J(p,q) := \int_I [p(x) - q(x)]\ln\left[\frac{p(x)}{q(x)}\right]dx,\quad p,\ q \in \mathcal{P}(I); \tag{4.8}$$

$$D_{\Delta}(p,q) := \int_I \frac{[p(x)-q(x)]^2}{p(x)+q(x)}dx, \quad p,\ q \in \mathcal{P}(I). \tag{4.9}$$

Csiszár f-divergence is defined as follows [2]

$$I_f(p,q) := \int_I p(x) f\left[\frac{q(x)}{p(x)}\right]dx, \quad p,\ q \in \mathcal{P}(I), \tag{4.10}$$

where f is convex on $(0,\infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u=1$. By appropriately defining this convex function, various divergences are derived. Most of the above distances (4.1)-(4.9), are particular instances of Csiszár f-divergence. There are also many others which are not in this class. For the basic properties of Csiszár f-divergence see [2], [3] and [12].

The following result holds:

Proposition 1. *Assume that* $0<r\leq 1\leq R<\infty$ *and* $p,\ q\in\mathcal{P}(I)$ *with*

$$r \leq \frac{q(x)}{p(x)} \leq R \text{ for a.e. } x\in I.$$

(i) *If* $f:(0,\infty)\to\mathbb{R}$ *is differentiable convex on* $(0,\infty)$ *with* $f(1)=0$ *and if* p, q *are differentiable on the interior of* I, *then we have the inequalities*

$$\begin{aligned} 0 \leq I_f(p,q) &\leq \frac{1}{2}\left\|\frac{q'p-p'q}{p^3}\right\|_{I,\infty} \frac{\left(I_{f'}(p,q)-f'_+(r)\right)\left(f'_-(R)-I_{f'}(p,q)\right)}{f'_-(R)-f'_+(r)} \\ &\leq \frac{1}{8}\left[f'_-(R)-f'_+(r)\right]\left\|\frac{q'p-p'q}{p^3}\right\|_{I,\infty} \end{aligned} \tag{4.11}$$

provided $\frac{q'p-p'q}{q}\in L_\infty(I)$.

(ii) *If* f *is twice differentiable on* $(0,\infty)$ *and* $\frac{f''\left(\frac{q}{p}\right)(q'p-p'q)}{q}\in L_\infty(I)$, *then*

$$\begin{aligned} 0 \leq I_f(p,q) &\leq \frac{1}{2}\left\|\frac{f''\left(\frac{q}{p}\right)(q'p-p'q)}{p^3}\right\|_{I,\infty} \frac{(1-r)(R-1)}{R-r} \\ &\leq \frac{1}{8}(R-r)\left\|\frac{f''\left(\frac{q}{p}\right)(q'p-p'q)}{p^3}\right\|_{I,\infty}. \end{aligned} \tag{4.12}$$

The proof follows by Theorem 4 for the convex function f.

Consider the convex function $f(t)=-\ln t$, $t>0$. We have $I_f(p,q)=D_{KL}(p,q)$,

$$I_{f'}(p,q) = -\int_I p(x)\frac{1}{\frac{q(x)}{p(x)}}dx = -\int_I \frac{p^2(x)}{q(x)}dx$$

and since

$$D_{\chi^2}(p,q) := \int_I p(x)\left[\left(\frac{q(x)}{p(x)}\right)^2-1\right]dx,$$

then

$$I_{f'}(p,q) = -1-D_{\chi^2}(q,p).$$

From (4.11) we get

$$(4.13)\quad 0 \le D_{KL}(p,q) \le \frac{1}{2}\left\|\frac{q'p-p'q}{p^3}\right\|_{I,\infty}\frac{\left(1-r-rD_{\chi^2}(q,p)\right)\left(RD_{\chi^2}(q,p)+R-1\right)}{R-r} \le \frac{1}{8}\frac{R-r}{rR}\left\|\frac{q'p-p'q}{p^3}\right\|_{I,\infty},$$

provided $\frac{q'p-p'q}{p^3} \in L_\infty(I)$, while from (4.12) we get

$$(4.14)\quad 0 \le D_{KL}(p,q) \le \frac{1}{2}\left\|\frac{q'p-p'q}{q^2p}\right\|_{I,\infty}\frac{(1-r)(R-1)}{R-r} \le \frac{1}{8}(R-r)\left\|\frac{q'p-p'q}{q^2p}\right\|_{I,\infty},$$

provided $\frac{q'p-p'q}{q^2p} \in L_\infty(I)$.

Consider the convex function $f(t)=t\ln t$, $t>0$. We have

$$I_f(p,q)=\int_I p(x)\frac{q(x)}{p(x)}\ln\left(\frac{q(x)}{p(x)}\right)dx = D_{KL}(q,p),$$

$$I_{f'}(p,q)=\int_I p(x)\left(\ln\left[\frac{q(x)}{p(x)}\right]+1\right)dx = 1-D_{KL}(p,q).$$

By (4.11) we get

$$(4.15)\quad 0 \le D_{KL}(q,p) \le \frac{1}{2}\left\|\frac{q'p-p'q}{p^3}\right\|_{I,\infty}\frac{\left(\ln r^{-1}-D_{KL}(p,q)\right)\left(D_{KL}(p,q)-\ln R^{-1}\right)}{(\ln R-\ln r)} \le \frac{1}{8}(\ln R-\ln r)\left\|\frac{q'p-p'q}{p^3}\right\|_{I,\infty},$$

provided $\frac{q'p-p'q}{p^3} \in L_\infty(I)$, while by (4.12)

$$(4.16)\quad 0 \le D_{KL}(q,p) \le \frac{1}{2}\left\|\frac{q'p-p'q}{qp^2}\right\|_{I,\infty}\frac{(1-r)(R-1)}{R-r} \le \frac{1}{8}(R-r)\left\|\frac{q'p-p'q}{qp^2}\right\|_{I,\infty}$$

provided $\frac{q'p-p'q}{qp^2} \in L_\infty(I)$.

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