Received 09/05/18

Asymptotic series related to Ramanujan's expansion for the harmonic number

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Abstract In this paper, we present various asymptotic series for the harmonic number $H_n = \sum_{k=1}^{n} \frac{1}{k}$. More precisely, we give a recursive relation for determining the coefficients $\mu_j(h)$ such that

$$H_n \sim \frac{1}{2}\psi\left(2m+h\right) + \gamma + \sum_{j=1}^{\infty} \frac{\mu_j(h)}{(2m+h)^j}$$

as $n \to \infty$, where $h \in \mathbb{R}$, $m = \frac{1}{2}n(n+1)$, ψ denotes the digamma function and γ is the Euler–Mascheroni constant. We also give recursive relations for determining the constants a_ℓ, b_ℓ, α_ℓ, and β_ℓ such that

$$H_n \sim \frac{1}{2}\ln\left(2m+\frac{1}{3}\right)+\gamma+\sum_{\ell=1}^{\infty}\frac{a_\ell}{(2m+b_\ell)^{2\ell}} \text{ and } H_n \sim \frac{1}{2}\psi\left(2m+\frac{5}{6}\right)+\gamma+\sum_{\ell=1}^{\infty}\frac{\alpha_\ell}{(2m+\beta_\ell)^{2\ell}}$$

as $n \to \infty$.

Keywords Harmonic number; Euler–Mascheroni constant; Asymptotic expansion
Mathematics Subject Classification (2010) Primary 41A60; Secondary 40A05

1 Introduction

Ramanujan (see [2, p. 531] and [14, p. 276]) proposed, without a proof and without a formula for the general term, the following asymptotic expansion for the nth harmonic number:

$$H_n := \sum_{k=1}^{n} \frac{1}{k} \sim \frac{1}{2}\ln(2m) + \gamma + \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \frac{1}{2310m^5} - \frac{191}{360360m^6} + \frac{29}{30030m^7} - \frac{2833}{1166880m^8} + \frac{140051}{17459442m^9} - \cdots \tag{1.1}$$

as $n \to \infty$, where $m = \frac{1}{2}n(n+1)$ ($n \in \mathbb{N} := \{1, 2, \ldots\}$) is the nth triangular number and γ is the Euler–Mascheroni constant.

Berndt [2, pp. 531–532] simply verified that Ramanujan's expansion coincides with the following Euler expansion:

$$H_n \sim \ln n + \gamma - \sum_{j=1}^{\infty} \frac{B_j}{jn^j}, \tag{1.2}$$

where B_j $(j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ are the Bernoulli numbers defined by the following generating function:

$$\frac{z}{e^z - 1} = \sum_{j=0}^{\infty} B_j \frac{z^j}{j!}, \qquad |z| < 2\pi.$$

Hirschhorn [8] presented a natural derivation for Ramanujan's expansion. However, Berndt and Hirschhorn did not give the general formula for the coefficients of $\frac{1}{m^j}$ $(j \in \mathbb{N})$ in Ramanujan's expansion. The complete proof of expansion (1.1) was given by Villarino [15, Theorem 1.1] who proved that for every integer $r \geq 1$, there exists a $\Theta_r, 0 < \Theta_r < 1$, for which the following equation is true:

$$H_n = \frac{1}{2}\ln(2m) + \gamma + \sum_{j=1}^{r} \frac{R_j}{m^j} + \Theta_r \cdot \frac{R_{r+1}}{m^{r+1}}, \tag{1.3}$$

with

$$R_j = \frac{(-1)^{j-1}}{2j \cdot 8^j} \left\{ 1 + \sum_{k=1}^{j} \binom{j}{k} (-4)^k B_{2k}(\tfrac{1}{2}) \right\}, \tag{1.4}$$

where $B_n(t)$ denotes the Bernoulli polynomials defined by the following generating function:

$$\frac{ze^{tz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(t) \frac{z^n}{n!}, \qquad |z| < 2\pi. \tag{1.5}$$

By using the relation

$$B_n(\tfrac{1}{2}) = -(1 - 2^{1-n}) B_n \quad \text{for} \quad n \in \mathbb{N}_0$$

(see [1, p. 805]), it follows from (1.3) and (1.4) that

$$H_n \sim \frac{1}{2}\ln(2m) + \gamma + \sum_{j=1}^{\infty} \frac{R_j}{m^j} \tag{1.6}$$

with

$$R_j = \frac{(-1)^{j-1}}{2j \cdot 8^j} \left\{ 1 - \sum_{k=1}^{j} \binom{j}{k} (-4)^k (1 - 2^{1-2k}) B_{2k} \right\}. \tag{1.7}$$

Ramanujan's expansion (1.1) was also researched in [4, 5, 6, 7, 9].

Also in [15], Villarino remarked that there might exist a series expansion for the logarithm of the factorial in terms of $\frac{1}{m}$. Villarino's remark has been considered by Nemes [13] and Chen [3].

Mortici and Chen [11, Theorem 2] presented the following approximation formula:

$$\begin{aligned} H_n = \frac{1}{2}\ln\left(n^2 + n + \frac{1}{3}\right) + \gamma - \frac{1}{180(n^2 + n + \frac{1}{3})^2} + \frac{8}{2835(n^2 + n + \frac{1}{3})^3} \\ - \frac{5}{1512(n^2 + n + \frac{1}{3})^4} + \frac{592}{93555(n^2 + n + \frac{1}{3})^5} + O\left(\frac{1}{(n^2 + n + \frac{1}{3})^6}\right). \end{aligned} \tag{1.8}$$

Very recently, Mortici and Villarino [12, Theorem 2] and Chen [4, Theorem 3.3] developed the approximation formula (1.8) to produce a complete asymptotic expansion:

$$H_n \sim \frac{1}{2} \ln \left(2m + \frac{1}{3} \right) + \gamma + \sum_{j=2}^{\infty} \frac{\rho_j}{(2m + \frac{1}{3})^j}. \tag{1.9}$$

Moreover, the authors gave a formula for determining the coefficients ρ_j in (1.9).

Euler's gamma function:

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} \mathrm{d}t, \qquad x > 0$$

is one of the most important functions in mathematical analysis and has applications in many diverse areas. The logarithmic derivative of the gamma function:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

is known as the psi (or digamma) function. $\psi(x)$ is connected to the Euler–Mascheroni constant and harmonic numbers through the well known relation (see [1, p. 258, Eq. (6.3.2)])

$$\psi(n+1) = -\gamma + H_n, \qquad n \in \mathbb{N}. \tag{1.10}$$

Hence, various approximations of the psi function are used in this relation and interpreted as approximation for the harmonic number H_n or as approximation of the constant γ.

The psi function has the following asymptotic expansion (see [10, p. 33]):

$$\psi(x+a) \sim \ln x + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} B_k(a)}{k x^k}, \quad x \to \infty, \quad a \in \mathbb{R}, \tag{1.11}$$

where $B_n(t)$ is the Bernoulli polynomials defined by (1.5).

In view of (1.6), (1.9) and (1.11), we can let

$$H_n \sim \frac{1}{2} \psi\left(2m + h\right) + \gamma + \sum_{j=1}^{\infty} \frac{\mu_j}{(2m+h)^j}, \qquad n \to \infty, \tag{1.12}$$

where $h \in \mathbb{R}$ and $m = \frac{1}{2} n(n+1)$. The first aim of present paper is to determine the coefficients $\mu_j \equiv \mu_j(h)$ in (1.12). The second aim of present paper is to determine the constants a_ℓ, b_ℓ, α_ℓ, and β_ℓ such that

$$H_n \sim \frac{1}{2} \ln \left(2m + \frac{1}{3} \right) + \gamma + \sum_{\ell=1}^{\infty} \frac{a_\ell}{(2m + b_\ell)^{2\ell}}, \qquad n \to \infty$$

and

$$H_n \sim \frac{1}{2} \psi \left(2m + \frac{5}{6} \right) + \gamma + \sum_{\ell=1}^{\infty} \frac{\alpha_\ell}{(2m + \beta_\ell)^{2\ell}}, \qquad n \to \infty.$$

2 Main results

Theorem 2.1. *Let $h \in \mathbb{R}$ and $m = \frac{1}{2}n(n+1)$. The harmonic number has the following asymptotic expansion:*

$$H_n \sim \frac{1}{2}\psi\left(2m+h\right) + \gamma + \sum_{j=1}^{\infty} \frac{\mu_j}{(2m+h)^j}, \qquad n \to \infty, \tag{2.1}$$

with the coefficients $\mu_j \equiv \mu_j(h)$ $(j \in \mathbb{N})$ given by the recurrence relation

$$\mu_1 = \frac{1}{6} - \frac{B_1(h)}{2}, \quad \mu_j = 2^j R_j - \frac{(-1)^{j-1}B_j(h)}{2j} - \sum_{k=1}^{j-1} \mu_k \left(-h\right)^{j-k} \binom{j-1}{j-k}, \qquad j \geq 2, \tag{2.2}$$

where R_j are given in (1.7) *and $B_n(t)$ is the Bernoulli polynomials.*

Proof. Write (2.1) as

$$H_n \sim \frac{1}{2}\psi\left(2m+h\right) + \gamma + \sum_{j=1}^{\infty} \frac{\mu_j}{(2m)^j} \left(1 + \frac{h}{2m}\right)^{-j}. \tag{2.3}$$

The choice $x = 2m$ and $a = h$ in (1.11) yields

$$\psi(2m+h) \sim \ln(2m) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}B_k(h)}{k \cdot 2^k m^k}. \tag{2.4}$$

Direct computation yields

$$\begin{aligned}
\sum_{j=1}^{\infty} \frac{\mu_j}{(2m)^j} \left(1 + \frac{h}{2m}\right)^{-j} &= \sum_{j=1}^{\infty} \frac{\mu_j}{(2m)^j} \sum_{k=0}^{\infty} \binom{-j}{k} \frac{h^m}{(2m)^k} \\
&= \sum_{j=1}^{\infty} \frac{\mu_j}{2^j} \sum_{k=0}^{\infty} (-1)^k \binom{k+j-1}{k} \frac{h^k}{2^k} \frac{1}{m^{j+k}} \\
&= \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{j} \frac{\mu_k}{2^j} \left(-h\right)^{j-k} \binom{j-1}{j-k} \right\} \frac{1}{m^j}.
\end{aligned} \tag{2.5}$$

Substituting (2.4) and (2.5) into (2.3) we have

$$H_n \sim \frac{1}{2}\ln(2m) + \gamma + \sum_{j=1}^{\infty} \left\{ \frac{(-1)^{j-1}B_j(h)}{j \cdot 2^{j+1}} + \sum_{k=1}^{j} \frac{\mu_k}{2^j} \left(-h\right)^{j-k} \binom{j-1}{j-k} \right\} \frac{1}{m^j}. \tag{2.6}$$

Equating coefficients of the term m^{-j} on the right sides of (1.6) and (2.6), we obtain

$$\frac{(-1)^{j-1}B_j(h)}{j \cdot 2^{j+1}} + \sum_{k=1}^{j} \frac{\mu_k}{2^j} \left(-h\right)^{j-k} \binom{j-1}{j-k} = R_j, \qquad j \in \mathbb{N}. \tag{2.7}$$

For $j = 1$ we obtain $\mu_1 = \frac{1}{6} - \frac{B_1(h)}{2}$, and for $j \geq 2$ we have

$$\frac{(-1)^{j-1}B_j(h)}{j \cdot 2^{j+1}} + \sum_{k=1}^{j-1} \frac{\mu_k}{2^j} \left(-h\right)^{j-k} \binom{j-1}{j-k} + \frac{\mu_j}{2^j} = R_j, \qquad j \geq 2,$$

which yields the recursive formula (2.2). The proof of Theorem 2.1 is complete. □

The first few coefficients $\mu_j \equiv \mu_j(h)$ are:

$$\mu_1 = -\frac{1}{2}h + \frac{5}{12},$$
$$\mu_2 = -\frac{1}{4}h^2 + \frac{1}{6}h + \frac{1}{120},$$
$$\mu_3 = -\frac{1}{6}h^3 + \frac{1}{6}h^2 - \frac{1}{15}h + \frac{4}{315},$$
$$\mu_4 = -\frac{1}{8}h^4 + \frac{1}{6}h^3 - \frac{1}{10}h^2 + \frac{4}{105}h - \frac{23}{1680}.$$

Setting $h = 0$ in (2.1), we obtain the following explicit asymptotic expansion:

$$H_n \sim \gamma + \frac{1}{2}\psi(2m) + \frac{5}{24m} + \frac{1}{480m^2} + \frac{1}{630m^3} - \frac{23}{26880m^4} + \cdots, \qquad n \to \infty. \tag{2.8}$$

Setting $h = \frac{5}{6}$ in (2.1) yields

$$\begin{aligned} H_n \sim{}& \frac{1}{2}\psi\left(2m + \frac{5}{6}\right) + \gamma - \frac{19}{720(2m+\frac{5}{6})^2} - \frac{1069}{45360(2m+\frac{5}{6})^3} \\ &- \frac{263}{17280(2m+\frac{5}{6})^4} - \cdots, \qquad n \to \infty. \end{aligned} \tag{2.9}$$

Theorem 2.2. *The harmonic number has the following asymptotic series:*

$$H_n \sim \frac{1}{2}\ln\left(2m + \frac{1}{3}\right) + \gamma + \sum_{\ell=1}^{\infty} \frac{a_\ell}{(2m+b_\ell)^{2\ell}}, \qquad n \to \infty, \tag{2.10}$$

where a_ℓ and b_ℓ are given by a pair of recurrence relations

$$a_\ell = 2^{2\ell}\left\{R_{2\ell} + \frac{1}{4\ell 6^{2\ell}} - \sum_{k=1}^{\ell-1} \frac{a_k}{2^{2k}}\left(-\frac{b_k}{2}\right)^{2\ell-2k}\binom{2\ell-1}{2\ell-2k}\right\}, \qquad \ell \geq 2 \tag{2.11}$$

and

$$b_\ell = \frac{2^{2\ell}}{\ell a_\ell}\left\{\frac{1}{(4\ell+2)6^{2\ell+1}} + \sum_{k=1}^{\ell-1} \frac{a_k}{2^{2k}}\left(-\frac{b_k}{2}\right)^{2\ell-2k+1}\binom{2\ell}{2\ell-2k+1} - R_{2\ell+1}\right\}, \qquad \ell \geq 2, \tag{2.12}$$

with $a_1 = -\frac{1}{180}$ and $b_1 = \frac{37}{63}$. Here R_j are given in (1.7).

Proof. Write (2.10) as

$$H_n \sim \frac{1}{2}\ln(2m) + \frac{1}{2}\ln\left(1 + \frac{1}{6m}\right) + \gamma + \sum_{j=1}^{\infty} \frac{a_j}{2^{2j}m^{2j}}\left(1 + \frac{b_j}{2m}\right)^{-2j}. \tag{2.13}$$

The Maclaurin expansion of $\ln(1+x)$ with $x = \frac{1}{6m}$ gives

$$\frac{1}{2}\ln\left(1 + \frac{1}{6m}\right) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{2j6^j}\frac{1}{m^j}. \tag{2.14}$$

Direct computation yields

$$\begin{aligned}\sum_{j=1}^{\infty}\frac{a_j}{2^{2j}m^{2j}}\left(1+\frac{b_j}{2m}\right)^{-2j} &= \sum_{j=1}^{\infty}\frac{a_j}{2^{2j}m^{2j}}\sum_{k=0}^{\infty}\binom{-2j}{k}\left(\frac{b_j}{2}\right)^{k}\frac{1}{m^k}\\ &= \sum_{j=1}^{\infty}\frac{a_j}{2^{2j}m^{2j}}\sum_{k=0}^{\infty}(-1)^k\binom{k+2j-1}{k}\left(\frac{b_j}{2}\right)^{k}\frac{1}{m^k}\\ &= \sum_{j=2}^{\infty}\sum_{k=0}^{j-2}\frac{a_{k+1}}{2^{2k+2}}(-1)^{j-k}\binom{j+k-1}{j-k-2}\left(\frac{b_{k+1}}{2}\right)^{j-k-2}\frac{1}{m^{j+k}},\end{aligned}$$

which can be written as

$$\sum_{j=1}^{\infty}\frac{a_j}{2^{2j}m^{2j}}\left(1+\frac{b_j}{2m}\right)^{-2j} \sim \sum_{j=2}^{\infty}\left\{\sum_{k=1}^{\lfloor j/2\rfloor}\frac{a_k}{2^{2k}}\left(-\frac{b_k}{2}\right)^{j-2k}\binom{j-1}{j-2k}\right\}\frac{1}{m^j}. \tag{2.15}$$

Substituting (2.14) and (2.15) into (2.13) we have

$$H_n \sim \frac{1}{2}\ln(2m)+\gamma+\frac{1}{12m}+\sum_{j=2}^{\infty}\left\{\frac{(-1)^{j-1}}{2j6^j}+\sum_{k=1}^{\lfloor j/2\rfloor}\frac{a_k}{2^{2k}}\left(-\frac{b_k}{2}\right)^{j-2k}\binom{j-1}{j-2k}\right\}\frac{1}{m^j}. \tag{2.16}$$

Equating coefficients of the term m^{-j} on the right sides of (1.6) and (2.16), we obtain

$$\frac{(-1)^{j-1}}{2j6^j}+\sum_{k=1}^{\lfloor j/2\rfloor}\frac{a_k}{2^{2k}}\left(-\frac{b_k}{2}\right)^{j-2k}\binom{j-1}{j-2k} = R_j, \qquad j\geq 2. \tag{2.17}$$

Setting $j=2\ell$ and $j=2\ell+1$ in (2.17), respectively, yields

$$-\frac{1}{4\ell 6^{2\ell}}+\sum_{k=1}^{\ell}\frac{a_k}{2^{2k}}\left(-\frac{b_k}{2}\right)^{2\ell-2k}\binom{2\ell-1}{2\ell-2k} = R_{2\ell} \tag{2.18}$$

and

$$\frac{1}{(4\ell+2)6^{2\ell+1}}+\sum_{k=1}^{\ell}\frac{a_k}{2^{2k}}\left(-\frac{b_k}{2}\right)^{2\ell-2k+1}\binom{2\ell}{2\ell-2k+1} = R_{2\ell+1}. \tag{2.19}$$

For $\ell=1$, from (2.18) and (2.19) we obtain

$$a_1 = -\frac{1}{180} \quad \text{and} \quad b_1 = \frac{37}{63},$$

and for $\ell\geq 2$ we have

$$-\frac{1}{4\ell 6^{2\ell}}+\sum_{k=1}^{\ell-1}\frac{a_k}{2^{2k}}\left(-\frac{b_k}{2}\right)^{2\ell-2k}\binom{2\ell-1}{2\ell-2k}+\frac{a_\ell}{2^{2\ell}} = R_{2\ell}$$

and

$$\frac{1}{(4\ell+2)6^{2\ell+1}}+\sum_{k=1}^{\ell-1}\frac{a_k}{2^{2k}}\left(-\frac{b_k}{2}\right)^{2\ell-2k+1}\binom{2\ell}{2\ell-2k+1}-\frac{\ell a_\ell}{2^{2\ell}}b_\ell = R_{2\ell+1}.$$

We then obtain the recurrence relations (2.11) and (2.12). The proof of Theorem 2.2 is complete. □

Here we give explicit numerical values of some first terms of a_ℓ and b_ℓ by using the formula (2.11) and (2.12). This shows how easily we can determine the constants a_ℓ and b_ℓ in (2.10).

$$\begin{aligned}
a_1 &= -\frac{1}{180}, \quad b_1 = \frac{37}{63},\\
a_2 &= -\frac{181}{22680} - 3a_1 b_1^2 = -\frac{1063}{476280},\\
b_2 &= \frac{17605}{11693} + \frac{476280}{1063} a_1 b_1^3 = \frac{2212979}{2209977},\\
a_3 &= -\frac{1480211}{43783740} - 5a_1 b_1^4 - 10 a_2 b_2^2 = -\frac{115541458428859}{14223875580975060},\\
b_3 &= \frac{292957461659709}{115541458428859} + \frac{14223875580975060}{115541458428859} a_1 b_1^5 + \frac{47412918603250200}{115541458428859} a_2 b_2^3\\
&= \frac{120123908928332403877 1}{766031897022703578729}.
\end{aligned}$$

We then obtain, as $n \to \infty$,

$$H_n \sim \frac{1}{2}\ln\left(2m + \frac{1}{3}\right) + \gamma + \frac{-\frac{1}{180}}{(2m + \frac{37}{63})^2} + \frac{-\frac{1063}{476280}}{(2m + \frac{2212979}{2209977})^4} + \frac{-\frac{115541458428859}{14223875580975060}}{(2m + \frac{1201239089283324038771}{766031897022703578729})^6} + \cdots. \tag{2.20}$$

Theorem 2.3. *The harmonic number has the following asymptotic series:*

$$H_n \sim \frac{1}{2}\psi\left(2m + \frac{5}{6}\right) + \gamma + \sum_{\ell=1}^{\infty} \frac{\alpha_\ell}{(2m + \beta_\ell)^{2\ell}}, \qquad n \to \infty, \tag{2.21}$$

where α_ℓ and β_ℓ are given by a pair of recurrence relations

$$\alpha_\ell = 2^{2\ell}\left\{R_{2\ell} + \frac{B_{2\ell}(\frac{5}{6})}{2\ell \cdot 2^{2\ell+1}} - \sum_{k=1}^{\ell-1} \frac{\alpha_k}{2^{2k}}\left(-\frac{\beta_k}{2}\right)^{2\ell-2k}\binom{2\ell-1}{2\ell-2k}\right\}, \qquad j \geq 2 \tag{2.22}$$

and

$$\beta_\ell = \frac{2^{2\ell}}{\ell\alpha_\ell}\left\{\frac{B_{2\ell+1}(\frac{5}{6})}{(2\ell+1)\cdot 2^{2\ell+2}} + \sum_{k=1}^{\ell-1}\frac{\alpha_k}{2^{2k}}\left(-\frac{\beta_k}{2}\right)^{2\ell-2k+1}\binom{2\ell}{2\ell-2k+1} - R_{2\ell+1}\right\}, \qquad j \geq 2 \tag{2.23}$$

with $\alpha_1 = -\frac{19}{720}$ and $\beta_1 = \frac{463}{1197}$. Here R_j are given in (1.7) and $B_n(t)$ is the Bernoulli polynomials.

Proof. By (2.15), we can write (2.21) as

$$H_n \sim \frac{1}{2}\psi\left(2m + \frac{5}{6}\right) + \gamma + \sum_{j=2}^{\infty}\left\{\sum_{k=1}^{\lfloor j/2\rfloor}\frac{\alpha_k}{2^{2k}}\left(-\frac{\beta_k}{2}\right)^{j-2k}\binom{j-1}{j-2k}\right\}\frac{1}{m^j}. \tag{2.24}$$

The choice $x = 2m$ and $a = \frac{5}{6}$ in (1.11) yields

$$\psi\left(2m + \frac{5}{6}\right) \sim \ln(2m) + \sum_{j=1}^{\infty}\frac{(-1)^{j-1}B_j(\frac{5}{6})}{j\cdot 2^j m^j}. \tag{2.25}$$

Substituting (2.25) into (2.24) yields

$$H_n \sim \frac{1}{2}\ln(2m) + \gamma + \frac{1}{12m} + \sum_{j=2}^{\infty} \left\{ \frac{(-1)^{j-1}B_j(\frac{5}{6})}{j \cdot 2^{j+1}} + \sum_{k=1}^{\lfloor j/2 \rfloor} \frac{\alpha_k}{2^{2k}} \left(-\frac{\beta_k}{2}\right)^{j-2k} \binom{j-1}{j-2k} \right\} \frac{1}{m^j}. \tag{2.26}$$

Equating coefficients of the term m^{-j} on the right sides of (1.6) and (2.26), we obtain

$$\frac{(-1)^{j-1}B_j(\frac{5}{6})}{j \cdot 2^{j+1}} + \sum_{k=1}^{\lfloor j/2 \rfloor} \frac{\alpha_k}{2^{2k}} \left(-\frac{\beta_k}{2}\right)^{j-2k} \binom{j-1}{j-2k} = R_j, \qquad j \geq 2. \tag{2.27}$$

Setting $j = 2\ell$ and $j = 2\ell + 1$ in (2.27), respectively, yields

$$-\frac{B_{2\ell}(\frac{5}{6})}{2\ell \cdot 2^{2\ell+1}} + \sum_{k=1}^{\ell} \frac{\alpha_k}{2^{2k}} \left(-\frac{\beta_k}{2}\right)^{2\ell-2k} \binom{2\ell-1}{2\ell-2k} = R_{2\ell} \tag{2.28}$$

and

$$\frac{B_{2\ell+1}(\frac{5}{6})}{(2\ell+1) \cdot 2^{2\ell+2}} + \sum_{k=1}^{\ell} \frac{\alpha_k}{2^{2k}} \left(-\frac{\beta_k}{2}\right)^{2\ell-2k+1} \binom{2\ell}{2\ell-2k+1} = R_{2\ell+1}. \tag{2.29}$$

For $\ell = 1$, from (2.28) and (2.29) we obtain

$$\alpha_1 = -\frac{19}{720} \quad \text{and} \quad \beta_1 = \frac{463}{1197},$$

and for $\ell \geq 2$ we have

$$-\frac{B_{2\ell}(\frac{5}{6})}{2\ell \cdot 2^{2\ell+1}} + \sum_{k=1}^{\ell-1} \frac{\alpha_k}{2^{2k}} \left(-\frac{\beta_k}{2}\right)^{2\ell-2k} \binom{2\ell-1}{2\ell-2k} + \frac{\alpha_\ell}{2^{2\ell}} = R_{2\ell}$$

and

$$\frac{B_{2\ell+1}(\frac{5}{6})}{(2\ell+1) \cdot 2^{2\ell+2}} + \sum_{k=1}^{\ell-1} \frac{\alpha_k}{2^{2k}} \left(-\frac{\beta_k}{2}\right)^{2\ell-2k+1} \binom{2\ell}{2\ell-2k+1} - \frac{\ell\alpha_\ell}{2^{2\ell}} b_\ell = R_{2\ell+1}.$$

We then obtain the recurrence relations (2.22) and (2.23). The proof of Theorem 2.3 is complete. □

Here we give explicit numerical values of some first terms of α_ℓ and β_ℓ by using the formula (2.22) and (2.23). This shows how easily we can determine the constants α_ℓ and

β_ℓ in (2.21).

$$\begin{aligned}
\alpha_1 &= -\frac{19}{720}, \quad \beta_1 = \frac{463}{1197}, \\
\alpha_2 &= -\frac{4093}{362880} - 3\alpha_1\beta_1^2 = \frac{16369}{28957824}, \\
\beta_2 &= -\frac{4645291}{900295} - \frac{28957824}{16369}\alpha_1\beta_1^3 = -\frac{1589397889}{646591869}, \\
\alpha_3 &= -\frac{92371859}{2802159360} - 5\alpha_1\beta_1^4 - 10\alpha_2\beta_2^2 = -\frac{616958946986009430417 7}{9614962744604074585728 0}, \\
\beta_3 &= \frac{2006884623211057871127}{6169589469860094304177} + \frac{96149627446040745857280}{6169589469860094304177}\alpha_1\beta_1^5 \\
&\quad + \frac{320498758153469152857600}{6169589469860094304177}\alpha_2\beta_2^3 \\
&= -\frac{13693567486511666914983651936 19}{11967619158838672633962182810439}.
\end{aligned}$$

We then obtain, as $n \to \infty$,

$$H_n \sim \frac{1}{2}\psi\left(2m + \frac{5}{6}\right) + \gamma + \frac{-\frac{19}{720}}{(2m + \frac{463}{1197})^2} + \frac{\frac{16369}{28957824}}{(2m - \frac{1589397889}{646591869})^4} + \frac{-\frac{6169589469860094304177}{96149627446040745857280}}{(2m - \frac{1369356748651166691498365193619}{11967619158838672633962182810439})^6} + \cdots . \tag{2.30}$$

From a computational viewpoint, the formulas (2.20) and (2.30) are better than the formulas (1.1), (1.8), (2.8) and (2.9),

It follows from (2.20) and (2.30) that

$$H_n \sim \frac{1}{2}\ln\left(2m + \frac{1}{3}\right) + \gamma + \frac{-\frac{1}{180}}{(2m + \frac{37}{63})^2} := u_n \tag{2.31}$$

and

$$H_n \sim \frac{1}{2}\psi\left(2m + \frac{5}{6}\right) + \gamma + \frac{-\frac{19}{720}}{(2m + \frac{463}{1197})^2} := v_n. \tag{2.32}$$

Moreover, we have, as $n \to \infty$,

$$H_n = u_n + O\big(n^{-8}\big) \quad \text{and} \quad H_n = v_n + O\big(n^{-8}\big).$$

It is observed from Table 1 that, between approximation formulas (2.31) and (2.32), for $n \geq 2$, the formula (2.32) is better than the formula (2.31).

Table 1. Comparison between approximation formulas (2.31) and (2.32).

n	$u_n - H_n$	$H_n - v_n$
2	9.799×10^{-7}	7.620×10^{-7}
10	1.470×10^{-11}	4.189×10^{-12}
100	2.143×10^{-19}	5.437×10^{-20}
1000	2.222×10^{-27}	5.630×10^{-28}
10000	2.230×10^{-35}	5.650×10^{-36}

References

[1] Abramowitz, M., Stegun, I.A. (eds.): Handbook of mathematical functions with formulas, graphs, and mathematical tables. National Bureau of Standards, Applied Mathematics Series, vol. 55. Dover, New York (1972)

[2] Berndt, B.C.: Ramanujan's Notebooks Part V. Springer, Berlin (1998)

[3] Chen, C.P.: Stirling expansions into negative powers of a triangular number. Ramanujan J. 39 (2016) 107–116.

[4] Chen, C.P.: On the coefficients of asymptotic expansion for the harmonic number by Ramanujan. Ramanujan J. 40 (2016) 279–290.

[5] Chen, C.P.: Ramanujan's formula for the harmonic number, Appl. Math. Comput. 317 (2018), 121–128.

[6] Chen, C.P., Cheng, J.X.: Ramanujan's asymptotic expansion for the harmonic numbers. Ramanujan J. **38**, 123–128 (2015)

[7] L. Feng, W. Wang, Riordan array approach to the coefficients of Ramanujans harmonic number expansion. Results Math. 71 (2017), 1413–1419.

[8] Hirschhorn, M.D.: Ramanujan's enigmatic formula for the harmonic series. Ramanujan J. **27**, 343–347 (2012)

[9] Issaka, A.: An asymptotic series related to Ramanujan's expansion for the nth Harmonic number. Ramanujan J. 39 (2016) 303–313.

[10] Luke, Y.L.: The Special Functions and their Approximations, vol. I. Academic Press, New York (1969)

[11] Mortici, C., Chen, C.P.: On the harmonic number expansion by Ramanujan. J. Inequal. Appl. 2013 (2013) 222.

[12] Mortici, C., Villarino, M.B.: On the Ramanujan-Lodge harmonic number expansion. Appl. Math. Comput. **251**, 423–430 (2015)

[13] Nemes, G.: Asymptotic expansion for $\log n!$ in terms of the reciprocal of a triangular number. Acta Math. Hung. **129**, 254–262 (2010)

[14] Ramanujan, S.: Notebook II, Narosa, New Delhi (1988)

[15] Villarino, M.B.: Ramanujan's harmonic number expansion into negative powers of a triangular number. J. Inequal. Pure Appl. Math. **9**(3) (2008) Article 89. `http://www.emis.de/journals/JIPAM/images/245_07_JIPAM/245_07.pdf`.