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SOME INEQUALITIES FOR THE ČEBYŠEV FUNCTIONAL

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we establish some inequalities for the *Čebyšev functional*

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2}\int_a^b f(t)dt \int_a^b g(t)dt,$$

of two *Lebesgue integrable* functions $f,\ g : [a,b] \to \mathbb{R}$.

1. Introduction

For two *Lebesgue integrable* functions $f,\ g : [a,b] \to \mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2}\int_a^b f(t)dt \int_a^b g(t)dt. \tag{1.1}$$

In 1935, Grüss [17] showed that

$$|C(f,g)| \le \frac{1}{4}(M-m)(N-n), \tag{1.2}$$

provided that there exists the real numbers m, M, n, N such that

$$m \le f(t) \le M \quad \text{and} \quad n \le g(t) \le N \quad \text{for a.e. } t \in [a,b]. \tag{1.3}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known result, even though it was obtained by Čebyšev in 1882, [4], states that

$$|C(f,g)| \le \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2, \tag{1.4}$$

provided that f', g' exist and are continuous on $[a,b]$ and $\|f'\|_\infty = \sup_{t\in[a,b]} |f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (1.4) also holds if $f,\ g : [a,b] \to \mathbb{R}$ are assumed to be *absolutely continuous* and $f',\ g' \in L_\infty[a,b]$ while $\|f'\|_\infty = \operatorname{essup}_{t\in[a,b]} |f'(t)|$.

A mixture between Grüss' result (1.2) and Čebyšev's one (1.4) is the following inequality obtained by Ostrowski in 1970, [24]:

$$|C(f,g)| \le \frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{1.5}$$

provided that f is *Lebesgue integrable* and satisfies (1.3) while g is absolutely continuous and $g' \in L_\infty[a,b]$. The constant $\frac{1}{8}$ is best possible in (1.5).

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The case of *euclidean norms* of the derivative was considered by A. Lupaş in [21] in which he proved that

$$|C(f,g)| \leq \frac{1}{\pi^2}\|f'\|_2\|g'\|_2(b-a), \tag{1.6}$$

provided that f, g are absolutely continuous and $f', g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Consider now the *weighted Čebyšev functional*

$$\begin{aligned} C_w(f,g) &:= \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) f(t) g(t)\,dt \\ &\quad - \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) f(t)\,dt \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) g(t)\,dt \end{aligned} \tag{1.7}$$

where $f, g, w : [a,b] \to \mathbb{R}$ and $w(t) \geq 0$ for a.e. $t \in [a,b]$ are measurable functions such that the involved integrals exist and $\int_a^b w(t)\,dt > 0$.

In [6], Cerone and Dragomir obtained, among others, the following inequalities:

$$\begin{aligned} &|C_w(f,g)| \\ &\leq \frac{1}{2}(M-m)\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t) - \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s) g(s)\,ds\right| dt \\ &\leq \frac{1}{2}(M-m)\left[\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t) - \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s) g(s)\,ds\right|^p dt\right]^{\frac{1}{p}} \\ &\leq \frac{1}{2}(M-m)\operatorname*{essup}_{t\in[a,b]}\left|g(t) - \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s) g(s)\,ds\right| \end{aligned} \tag{1.8}$$

for $p > 1$, provided $-\infty < m \leq f(t) \leq M < \infty$ for a.e. $t \in [a,b]$ and the corresponding integrals are finite. The constant $\frac{1}{2}$ is sharp in all the inequalities in (1.8) in the sense that it cannot be replaced by a smaller constant.

In addition, if $-\infty < n \leq g(t) \leq N < \infty$ for a.e. $t \in [a,b]$, then the following refinement of the celebrated Grüss inequality is obtained:

$$\begin{aligned} &|C_w(f,g)| \\ &\leq \frac{1}{2}(M-m)\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t) - \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s) g(s)\,ds\right| dt \\ &\leq \frac{1}{2}(M-m)\left[\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t) - \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s) g(s)\,ds\right|^2 dt\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}(M-m)(N-n). \end{aligned} \tag{1.9}$$

Here, the constants $\frac{1}{2}$ and $\frac{1}{4}$ are also sharp in the sense mentioned above.

For other inequality of Grüss' type see [1]-[5], [7]-[16], [18]-[23] and [25]-[28].

In this paper we establish some new inequalities for the *Čebyšev functional* $C(f,g)$ under several conditions for the integrable functions f, $g : [a,b] \to \mathbb{R}$.

2. The Results

We have:

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be a Lebesgue integrable function and so that $F(b)=0$, where $F(x):=\int_a^x f(t)\,dt$. Then we have*

$$\int_a^b F^2(x)\,dx=\left|\int_a^b f(x)\left(\int_a^x F(s)\,ds\right)dx\right| \tag{2.1}$$
$$\leq\begin{cases}\int_a^b|f(x)|\,dx\int_a^b|F(s)|\,ds,\\[2ex] \left(\int_a^b|f(x)|^p\,dx\right)^{1/p}\left[\int_a^b\left(\int_a^x|F(s)|\,ds\right)^q dx\right]^{1/q},\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \operatorname{essup}_{x\in[a,b]}|f(x)|\int_a^b\left(\int_a^x|F(s)|\,ds\right)dx.\end{cases}$$

Proof. Using integration by parts we have

$$\begin{aligned}\int_a^b F^2(x)\,dx&=\int_a^b F(x)F(x)\,dx=\int_a^b F(x)\,d\left(\int_a^x F(s)\,ds\right)\\&=F(x)\int_a^x F(s)\,ds\Big|_a^b-\int_a^b F'(x)\left(\int_a^x F(s)\,ds\right)dx\\&=F(b)\int_a^b F(s)\,ds-\int_a^b f(x)\left(\int_a^x F(s)\,ds\right)dx\\&=-\int_a^b f(x)\left(\int_a^x F(s)\,ds\right)dx=\left|\int_a^b f(x)\left(\int_a^x F(s)\,ds\right)dx\right|.\end{aligned}$$

We also have

$$\begin{aligned}&\left|\int_a^b f(x)\left(\int_a^x F(s)\,ds\right)dx\right|\\&\leq\int_a^b|f(x)|\left|\int_a^x F(s)\,ds\right|dx\leq\int_a^b|f(x)|\left(\int_a^x|F(s)|\,ds\right)dx\\&\leq\max_{x\in[a,b]}\left(\int_a^x|F(s)|\,ds\right)\int_a^b|f(x)|\,dx=\int_a^b|F(s)|\,ds\int_a^b|f(x)|\,dx.\end{aligned}$$

Using Hölder's integral inequality we also have

$$\int_a^b|f(x)|\left(\int_a^x|F(s)|\,ds\right)dx\leq\left(\int_a^b|f(x)|^p\,dx\right)^{1/p}\left[\int_a^b\left(\int_a^x|F(s)|\,ds\right)^q dx\right]^{1/q}$$

for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, which proves the second branch of (2.1)

Finally,

$$\int_a^b|f(x)|\left(\int_a^x|F(s)|\,ds\right)dx\leq\operatorname*{essup}_{x\in[a,b]}|f(x)|\int_a^b\left(\int_a^x|F(s)|\,ds\right)dx,$$

and the lemma is proved. □

The following result is due to Ostrowski [24]:

Lemma 2. *Let $f:[a,b]\to\mathbb{R}$ be a Lebesgue integrable function so that $F(b)=0$ and there exists $\alpha\in[0,1]$ such that*

$$-\alpha\leq f(x)\leq 1-\alpha \text{ for a.e. } x\in[a,b].$$

Then we have

$$\int_a^b |F(x)|\,dx\leq \frac{1}{2}\alpha(1-\alpha)(b-a)^2\leq\frac{1}{8}(b-a)^2. \tag{2.2}$$

Corollary 1. *With the assumptions of Lemma 2 we have*

$$\int_a^b F^2(x)\,dx\leq\frac{1}{2}\alpha(1-\alpha)(b-a)^2\int_a^b|f(x)|\,dx\leq\frac{1}{8}(b-a)^2\int_a^b|f(x)|\,dx. \tag{2.3}$$

The proof follows by the first branch of (2.1) and by (2.2), namely

$$\int_a^b F^2(x)\,dx\leq\int_a^b|f(x)|\,dx\int_a^b|F(s)|\,ds\leq\frac{1}{2}\alpha(1-\alpha)(b-a)^2\int_a^b|f(x)|\,dx.$$

Lemma 3. *Let $h:[a,b]\to\mathbb{R}$ be a Lebesgue integrable function on $[a,b]$ such that*

$$-\infty<\gamma\leq h(x)\leq\Gamma<\infty \text{ for a.e. on } [a,b], \tag{2.4}$$

then we have the inequality

$$\begin{aligned}
&\int_a^b\left|\int_a^x h(t)\,dt-\frac{x-a}{b-a}\int_a^b h(s)\,ds\right|^2 dx \\
&\leq\frac{1}{2}(b-a)^2\left(\frac{\frac{1}{b-a}\int_a^b h(s)\,ds-\gamma}{\Gamma-\gamma}\right)\left(\frac{\Gamma-\frac{1}{b-a}\int_a^b h(s)\,ds}{\Gamma-\gamma}\right)\\
&\times(\Gamma-\gamma)\int_a^b\left|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right|dt\\
&\leq\frac{1}{8}(b-a)^2(\Gamma-\gamma)\int_a^b\left|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right|dt.
\end{aligned} \tag{2.5}$$

Proof. Let

$$f(t):=\frac{1}{\Gamma-\gamma}\left[h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right],\ t\in[a,b].$$

Then

$$\begin{aligned}
F(x)&=\frac{1}{\Gamma-\gamma}\int_a^x\left(h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right)dt\\
&=\frac{1}{\Gamma-\gamma}\left(\int_a^x h(t)\,dt-\frac{x-a}{b-a}\int_a^b h(s)\,ds\right)
\end{aligned}$$

for $x\in[a,b]$ and

$$F(b)=F(a)=0.$$

Since $\gamma\leq h(x)\leq\Gamma<\infty$ for a.e. on $[a,b]$, hence

$$f(t)\leq\frac{1}{\Gamma-\gamma}\left(\Gamma-\frac{1}{b-a}\int_a^b h(s)\,ds\right)=1-\frac{\frac{1}{b-a}\int_a^b h(s)\,ds-\gamma}{\Gamma-\gamma} \tag{2.6}$$

and

$$-\frac{\frac{1}{b-a}\int_a^b h(s)\,ds-\gamma}{\Gamma-\gamma}=\frac{\gamma-\frac{1}{b-a}\int_a^b h(s)\,ds}{\Gamma-\gamma}\leq f(t) \tag{2.7}$$

for a.e. $t\in[a,b]$.

By denoting

$$\alpha:=\frac{\frac{1}{b-a}\int_a^b h(s)\,ds-\gamma}{\Gamma-\gamma},$$

we have $\alpha\in[0,1]$ and $-\alpha\leq f(t)\leq 1-\alpha$ for a.e. $t\in[a,b]$.

By employing the inequality (2.3) we get

$$\begin{aligned}
&\frac{1}{(\Gamma-\gamma)^2}\int_a^b\left|\int_a^x h(t)\,dt-\frac{x-a}{b-a}\int_a^b h(s)\,ds\right|^2 dx\\
&\leq\frac{1}{2}\left(\frac{\frac{1}{b-a}\int_a^b h(s)\,ds-\gamma}{\Gamma-\gamma}\right)\left(\frac{\Gamma-\frac{1}{b-a}\int_a^b h(s)\,ds}{\Gamma-\gamma}\right)(b-a)^2\\
&\times\frac{1}{\Gamma-\gamma}\int_a^b\left|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right|dt\\
&\leq\frac{1}{8}(b-a)^2\frac{1}{\Gamma-\gamma}\int_a^b\left|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right|dt,
\end{aligned}$$

which is equivalent to (2.5). □

We have:

Corollary 2. *Let $g:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ such that*

$$-\infty<\gamma\leq g'(x)\leq\Gamma<\infty \text{ for a.e. on } [a,b], \tag{2.8}$$

then we have the inequality

$$\begin{aligned}
&\int_a^b\left|g(x)-\frac{(x-a)\,g(b)+(b-x)\,g(a)}{b-a}\right|^2 dx\\
&\leq\frac{1}{2}(b-a)^2\left(\frac{\frac{g(b)-g(a)}{b-a}-\gamma}{\Gamma-\gamma}\right)\left(\frac{\Gamma-\frac{g(b)-g(a)}{b-a}}{\Gamma-\gamma}\right)\\
&\times(\Gamma-\gamma)\int_a^b\left|g'(t)-\frac{g(b)-g(a)}{b-a}\right|dt\\
&\leq\frac{1}{8}(b-a)^2(\Gamma-\gamma)\int_a^b\left|g'(t)-\frac{g(b)-g(a)}{b-a}\right|dt.
\end{aligned} \tag{2.9}$$

Remark 1. *Using the Cauchy-Bunyakovsky-Schwarz (CBS) inequality we get*

$$\begin{aligned}
(2.10)\qquad & \left|\frac{1}{b-a}\int_a^b g(x)\,dx-\frac{g(a)+g(b)}{2}\right|^2 \\
&\le \frac{1}{b-a}\int_a^b\left|g(x)-\frac{(x-a)\,g(b)+(b-x)\,g(a)}{b-a}\right|^2 dx \\
&\le \frac{1}{2}(b-a)\left(\frac{\frac{g(b)-g(a)}{b-a}-\gamma}{\Gamma-\gamma}\right)\left(\frac{\Gamma-\frac{g(b)-g(a)}{b-a}}{\Gamma-\gamma}\right) \\
&\times(\Gamma-\gamma)\int_a^b\left|g'(t)-\frac{g(b)-g(a)}{b-a}\right|dt \\
&\le \frac{1}{8}(b-a)(\Gamma-\gamma)\int_a^b\left|g'(t)-\frac{g(b)-g(a)}{b-a}\right|dt.
\end{aligned}$$

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a Lebesgue integrable function on $[a,b]$ such that there exists the real numbers $m<M$ with the property*

$$(2.11)\qquad -\infty<m\le f(x)\le M<\infty \text{ for a.e. on } [a,b],$$

and $g:[a,b]\to\mathbb{R}$ is absolutely continuous with $g'\in L_2[a,b]$, then we have the inequality

$$\begin{aligned}
(2.12)\qquad |C(f,g)|^2 &\le \frac{1}{2}\|g'\|_{[a,b],2}^2\left(\frac{\frac{1}{b-a}\int_a^b f(s)\,ds-m}{M-m}\right)\left(\frac{M-\frac{1}{b-a}\int_a^b f(s)\,ds}{M-m}\right) \\
&\quad\times(M-m)\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt \\
&\le \frac{1}{8}(M-m)\|g'\|_{[a,b],2}^2\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt \\
&\le \frac{1}{16}(M-m)^2(b-a)\|g'\|_{[a,b],2}^2.
\end{aligned}$$

Proof. Integrating by parts, we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b\left(\int_a^x f(t)\,dt-\frac{x-a}{b-a}\int_a^b f(s)\,ds\right)g\prime(x)\,dx \\
&=\frac{1}{b-a}\left[\left(\int_a^x f(t)\,dt-\frac{x-a}{b-a}\int_a^b f(s)\,ds\right)g(x)\bigg|_a^b\right. \\
&\left.-\int_a^b g(x)\left(f(x)-\frac{1}{b-a}\int_a^b f(s)\,ds\right)dx\right] \\
&=-\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx+\frac{1}{b-a}\int_a^b f(s)\,ds\frac{1}{b-a}\int_a^b g(x)\,dx,
\end{aligned}$$

which gives that

$$(2.13)\qquad C(f,g)=\frac{1}{b-a}\int_a^b\left(\frac{x-a}{b-a}\int_a^b f(s)\,ds-\int_a^x f(t)\,dt\right)g\prime(x)\,dx.$$

Using (CBS) integral inequality and the inequality (2.5), we have

$$\begin{aligned}
|C(f,g)|^2 &= \frac{1}{(b-a)^2}\left|\int_a^b \left(\frac{x-a}{b-a}\int_a^b f(s)\,ds - \int_a^x f(t)\,dt\right) g\prime(x)\,dx\right|^2 \\
&\le \frac{1}{(b-a)^2}\int_a^b \left|\frac{x-a}{b-a}\int_a^b f(s)\,ds - \int_a^x f(t)\,dt\right|^2 \|g'\|_{[a,b],2}^2 \\
&\le \frac{1}{2}\|g'\|_{[a,b],2}^2 \left(\frac{\frac{1}{b-a}\int_a^b f(s)\,ds - m}{M-m}\right)\left(\frac{M-\frac{1}{b-a}\int_a^b f(s)\,ds}{M-m}\right) \\
&\times (M-m)\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt \\
&\le \frac{1}{8}(M-m)\|g'\|_{[a,b],2}^2 \int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt,
\end{aligned}$$

which proves the first and second inequality in (2.12).

By (CBS) integral inequality we also have

$$\begin{aligned}
\left(\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt\right)^2 &\le (b-a)\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right|^2 dt \\
&= (b-a)\int_a^b f^2(t)\,dt - \left(\int_a^b f(s)\,ds\right)^2 \\
&\le \frac{1}{4}(b-a)^2 (M-m)^2,
\end{aligned}$$

where for the last inequality we used Grüss' inequality (1.2) for $g = f$.

This proves the last part of (2.12). □

3. Related Results

If we use the unweighted version of (1.7), we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b \phi(t)\psi(t)\,dt - \frac{1}{b-a}\int_a^b \phi(t)\,dt \frac{1}{b-a}\int_a^b \psi(t)\,dt\right| \qquad (3.1) \\
&\le \frac{1}{2}(M-m)\frac{1}{b-a}\int_a^b \left|\psi(t) - \frac{1}{b-a}\int_a^b \psi(s)\,ds\right| dt
\end{aligned}$$

where ϕ is integrable and $m \le \phi(t) \le M$ for a.e. $t \in [a,b]$ and ψ integrable.

Lemma 4. *Let $h : [a,b] \to \mathbb{R}$ be a Lebesgue integrable function and so that $H(b) = 0$, where $H(x) := \int_a^x h(t)\,dt$. If there exists the constants $k < K$ such that*

$$k \le \int_a^x H(t)\,dt \le K \text{ for a.e. } x \in [a,b], \qquad (3.2)$$

then we have

$$\int_a^b H^2(x)\,dx \le \frac{1}{2}(K-k)\int_a^b |h(t)|\,dt. \qquad (3.3)$$

Proof. If we use the equality in (2.1) and (3.1), then we get

$$\frac{1}{b-a}\int_a^b F^2(x)\,dx$$
$$=\left|\frac{1}{b-a}\int_a^b f(x)\left(\int_a^x F(s)\,ds\right)dx\right|$$
$$=\left|\frac{1}{b-a}\int_a^b f(x)\left(\int_a^x F(s)\,ds\right)dx-\frac{1}{b-a}\int_a^b f(x)\,\frac{1}{b-a}\int_a^b\left(\int_a^x F(s)\,ds\right)dx\right|$$
$$\leq\frac{1}{2}(K-k)\,\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt$$
$$=\frac{1}{2}(K-k)\,\frac{1}{b-a}\int_a^b|f(t)|\,dt,$$

which is equivalent to (3.3). □

Lemma 5. *Let $f:[a,b]\to\mathbb{R}$ be a Lebesgue integrable function such that there exists the constants n, N with the property*

$$n(b-a)\leq\int_a^x F(t)\,dt-\frac{1}{2}(b-a)\int_a^b f(s)\,ds\leq N(b-a), \tag{3.4}$$

for a.e. $x\in[a,b]$, then

$$\int_a^b\left|\int_a^x f(t)\,dt-\frac{x-a}{b-a}\int_a^b f(s)\,ds\right|^2 dx \leq\frac{1}{2}(N-n)(b-a)\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt. \tag{3.5}$$

The proof follows by Lemma 4 for the function $h(t)=f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds$, $t\in[a,b]$.

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a Lebesgue integrable function on $[a,b]$ such that there exists the real numbers $n<N$ with the property (3.4) and $g:[a,b]\to\mathbb{R}$ is absolutely continuous with $g'\in L_2[a,b]$, then we have the inequality*

$$|C(f,g)|^2\leq\frac{1}{2}(N-n)\,\|g'\|_{[a,b],2}^2\,\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt. \tag{3.6}$$

Proof. As in the proof of Theorem 1 we have

$$|C(f,g)|^2=\frac{1}{(b-a)^2}\left|\int_a^b\left(\frac{x-a}{b-a}\int_a^b f(s)\,ds-\int_a^x f(t)\,dt\right)g'(x)\,dx\right|^2$$
$$\leq\frac{1}{(b-a)^2}\int_a^b\left|\frac{x-a}{b-a}\int_a^b f(s)\,ds-\int_a^x f(t)\,dt\right|^2\|g'\|_{[a,b],2}^2.$$

By making use of the inequality (3.5) we deduce the desired result (3.6). □

Corollary 3. *Let $f:[a,b]\to\mathbb{R}$ be a Lebesgue integrable function on $[a,b]$ such that*

$$\left\|\frac{1}{b-a}\int_a^x F(t)\,dt-\frac{1}{2}\int_a^b f(s)\,ds\right\|_{[a,b],\infty}<\infty \tag{3.7}$$

and $g:[a,b]\to\mathbb{R}$ is absolutely continuous with $g'\in L_2[a,b]$, then we have the inequality

$$\begin{aligned}|C(f,g)|^2\le\|g'\|_{[a,b],2}^2&\left\|\frac{1}{b-a}\int_a^x F(t)\,dt-\frac{1}{2}\int_a^b f(s)\,ds\right\|_{[a,b],\infty}\\ &\times\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt.\end{aligned} \tag{3.8}$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.org/dragomir`

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa