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WEIGHTED VERSIONS OF TRAPEZOID AND MIDPOINT INEQUALITIES FOR TWICE DIFFERENTIABLE FUNCTIONS AND APPLICATIONS

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Abstract. In this paper we establish amongst other some upper bounds for the quantities

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{h(b)-h(a)}\int_a^b f(t)\,h'(t)\,dt\right|$$

and

$$\left|f\circ h^{-1}\left(\frac{h(a)+h(b)}{2}\right)-\frac{1}{h(b)-h(a)}\int_a^b f(t)\,h'(t)\,dt\right|$$

under the assumptions that $h:[a,b]\to[h(a),h(b)]$ is a *continuous strictly increasing function* that is *differentiable* on (a,b) and $f:[a,b]\to\mathbb{C}$ is an absolutely continuous function on $[a,b]$. When h is an integral, namely $h(x)=\int_a^x w(s)\,ds$, where $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$, then some weighted inequalities are provided. Applications for some particular functions of interest are also given.

1. Introduction

Let $u:[a,b]\to\mathbb{C}$ be a function such that its derivative is absolutely continuous on $[a,b]$. Then one has the trapezoid inequalities [11] with error bounds in terms of the *Lebesgue norms* of the second derivative:

$$\left|\frac{u(a)+u(b)}{2}(b-a)-\int_a^b u(s)\,ds\right| \tag{1.1}$$

$$\leq\begin{cases}\dfrac{(b-a)^3}{12}\|u''\|_{[a,b],\infty} & \text{if } u''\in L_\infty[a,b];\\[2ex] \dfrac{1}{2}\left[B(q+1,q+1)\right]^{\frac{1}{q}}(b-a)^{2+\frac{1}{q}}\|u''\|_{[a,b],p} & \text{if } u''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{(b-a)^2}{8}\|u''\|_{[a,b],1}. & \end{cases}$$

where $B(\cdot,\cdot)$ is the *Beta function*

$$B(\alpha,\beta):=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt,\quad \alpha,\ \beta>0.$$

A simple proof of the fact can be done by the use of the following identity:

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$$\frac{u(a)+u(b)}{2}(b-a)-\int_a^b u(s)\,ds=\frac{1}{2}\int_a^b (b-s)(s-a)u''(s)\,ds, \tag{1.2}$$

and using the *Hölder integral inequality*.

Let $u(t)=g(t)-\frac{1}{2}\lambda t^2$ with g having an absolutely continuous derivative on $[a,b]$ and $\lambda\in\mathbb{C}$, then

$$\begin{aligned}&\frac{u(a)+u(b)}{2}(b-a)-\int_a^b u(s)\,ds\\&=\frac{g(a)+g(b)}{2}(b-a)-\int_a^b g(s)\,ds-\frac{1}{12}\lambda(b-a)^3\end{aligned}$$

and by (1.1) we get the *perturbed version of trapezoid inequality*

$$\begin{aligned}&\left|\frac{g(a)+g(b)}{2}(b-a)-\frac{1}{12}\lambda(b-a)^3-\int_a^b g(s)\,ds\right|\\&\leq\begin{cases}\dfrac{(b-a)^3}{12}\|g''-\lambda\|_{[a,b],\infty} & \text{if } g''\in L_\infty[a,b];\\[2ex] \dfrac{1}{2}[B(q+1,q+1)]^{\frac{1}{q}}(b-a)^{2+\frac{1}{q}}\|g''-\lambda\|_{[a,b],p} & \text{if } g''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{(b-a)^2}{8}\|g''-\lambda\|_{[a,b],1}. \end{cases}\end{aligned} \tag{1.3}$$

The midpoint version is as follows.

Let $u:[a,b]\to\mathbb{R}$ be a function such that its derivative is absolutely continuous on $[a,b]$. Then one has the inequalities [6], [7] and [8]

$$\begin{aligned}&\left|\int_a^b u(s)\,ds-u\left(\frac{a+b}{2}\right)(b-a)\right|\\&\leq\begin{cases}\dfrac{(b-a)^3}{24}\|u''\|_{[a,b],\infty} & \text{if } u''\in L_\infty[a,b];\\[2ex] \dfrac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}}\|u''\|_{[a,b],p} & \text{if } u''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{(b-a)^2}{8}\|u''\|_{[a,b],1}. \end{cases}\end{aligned} \tag{1.4}$$

A simple proof of this inequality may be done by using the Hölder integral inequality and the identity:

$$\begin{aligned}&\int_a^b u(s)\,ds-u\left(\frac{a+b}{2}\right)(b-a)\\&=\frac{1}{2}\int_a^{\frac{a+b}{2}}(s-a)^2u''(s)\,ds+\frac{1}{2}\int_{\frac{a+b}{2}}^b(b-s)^2u''(s)\,ds.\end{aligned} \tag{1.5}$$

Let $u(t) = g(t) - \frac{1}{2}\lambda t^2$ with g having an absolutely continuous derivative on $[a,b]$ and $\lambda \in \mathbb{C}$, then

$$\int_a^b u(s)\,ds - u\left(\frac{a+b}{2}\right)(b-a) = \int_a^b g(s)\,ds - g\left(\frac{a+b}{2}\right)(b-a) - \frac{1}{24}\lambda(b-a)^3$$

and by (1.4) we get the *perturbed version of mid-point inequality*

$$\left|\int_a^b g(s)\,ds - g\left(\frac{a+b}{2}\right)(b-a) - \frac{1}{24}\lambda(b-a)^3\right| \tag{1.6}$$
$$\leq \begin{cases} \dfrac{(b-a)^3}{24}\left\|g'' - \lambda\right\|_{[a,b],\infty} & \text{if } u'' \in L_\infty[a,b]; \\ \dfrac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}}\left\|g'' - \lambda\right\|_{[a,b],p} & \text{if } u'' \in L_p[a,b], \\ & p > 1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{(b-a)^2}{8}\left\|g'' - \lambda\right\|_{[a,b],1}. \end{cases}$$

For related results, see [1]-[5], [9]-[10] and [13]-[38].

In this paper we establish amongst other some upper bounds for the quantities

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{h(b)-h(a)}\int_a^b f(t)\,h'(t)\,dt\right|$$

and

$$\left|f \circ h^{-1}\left(\frac{h(a)+h(b)}{2}\right) - \frac{1}{h(b)-h(a)}\int_a^b f(t)\,h'(t)\,dt\right|$$

under the assumptions that $h : [a,b] \to [h(a), h(b)]$ is a *continuous strictly increasing function* that is *differentiable* on (a,b) and $f : [a,b] \to \mathbb{C}$ is an absolutely continuous function on $[a,b]$. When h is an integral, namely $h(x) = \int_a^x w(s)\,ds$, where $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$, then some weighted inequalities are provided. Applications for some particular functions of interest are also given.

2. Some Perturbed Inequalities of Interest

Now, for $\gamma,\ \Gamma \in \mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of *complex-valued functions*

$$\bar{U}_{[a,b]}(\gamma,\Gamma) := \left\{f : [a,b] \to \mathbb{C} \mid \operatorname{Re}\left[(\Gamma - f(t))\left(\overline{f(t)} - \overline{\gamma}\right)\right] \geq 0 \text{ for a.e. } t \in [a,b]\right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma) := \left\{f : [a,b] \to \mathbb{C} \mid \left|f(t) - \frac{\gamma+\Gamma}{2}\right| \leq \frac{1}{2}|\Gamma - \gamma| \text{ for a.e. } t \in [a,b]\right\}.$$

This family of functions is a particular case of the class introduced in [12]

$$\bar{\Delta}_{[a,b],g}(\gamma,\Gamma) := \left\{ f:[a,b]\to\mathbb{C} \mid \left| f(t) - \frac{\gamma+\Gamma}{2} g(t) \right| \le \frac{1}{2}|\Gamma-\gamma| |g(t)| \text{ for a.e. } t\in[a,b] \right\},$$

where $g:[a,b]\to\mathbb{C}$.

The following representation result may be stated.

Proposition 1. *For any* $\gamma,\Gamma\in\mathbb{C}$, $\gamma\ne\Gamma$, *we have that* $\bar{U}_{[a,b]}(\gamma,\Gamma)$ *and* $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ *are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{2.1}$$

Proof. We observe that for any $z\in\mathbb{C}$ we have the equivalence

$$\left| z - \frac{\gamma+\Gamma}{2}\right| \le \frac{1}{2}|\Gamma-\gamma|$$

if and only if

$$\operatorname{Re}\left[(\Gamma - z)(\bar{z}-\bar{\gamma})\right] \ge 0.$$

This follows by the equality

$$\frac{1}{4}|\Gamma-\gamma|^2 - \left| z - \frac{\gamma+\Gamma}{2}\right|^2 = \operatorname{Re}\left[(\Gamma - z)(\bar{z}-\bar{\gamma})\right]$$

that holds for any $z\in\mathbb{C}$.

The equality (2.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 1. *For any* $\gamma,\Gamma\in\mathbb{C}$, $\gamma\ne\Gamma$,*we have that*

$$\bar{U}_{[a,b]}(\gamma,\Gamma) = \{f:[a,b]\to\mathbb{C} \mid (\operatorname{Re}\Gamma - \operatorname{Re} f(t))(\operatorname{Re} f(t) - \operatorname{Re}\gamma) + (\operatorname{Im}\Gamma - \operatorname{Im} f(t))(\operatorname{Im} f(t) - \operatorname{Im}\gamma) \ge 0 \text{ a.e. } t\in[a,b]\}. \tag{2.2}$$

Now, if we assume that $\operatorname{Re}(\Gamma)\ge\operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma)\ge\operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\bar{S}_{[a,b]}(\gamma,\Gamma) := \{f:[a,b]\to\mathbb{C} \mid \operatorname{Re}(\Gamma)\ge\operatorname{Re} f(t)\ge\operatorname{Re}(\gamma) \text{ and } \operatorname{Im}(\Gamma)\ge\operatorname{Im} f(t)\ge\operatorname{Im}(\gamma) \text{ for a.e. } t\in[a,b]\}. \tag{2.3}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset \ne \bar{S}_{[a,b]}(\gamma,\Gamma) \subseteq \bar{U}_{[a,b]}(\gamma,\Gamma). \tag{2.4}$$

Theorem 1. *Assume that* $f:[a,b]\subset I\to\mathbb{C}$ *is differentiable on* $\mathring{I}$ *and the derivative is absolutely continuous on* $[a,b]$ *and such that there exists the complex numbers* γ, $\Gamma\in\mathbb{C}$, $\gamma\ne\Gamma$ *with* $f''\in\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$, *then we have*

$$\left| \frac{f(a)+f(b)}{2}(b-a) - \frac{1}{24}(\gamma+\Gamma)(b-a)^3 - \int_a^b f(s)\,ds \right| \le \frac{(b-a)^3}{24}|\Gamma-\gamma| \tag{2.5}$$

and

$$\left| \int_a^b f(s)\,ds - f\left(\frac{a+b}{2}\right)(b-a) - \frac{1}{48}(\gamma+\Gamma)(b-a)^3 \right| \le \frac{(b-a)^3}{48}|\Gamma-\gamma|. \tag{2.6}$$

Since $f'' \in \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$, hence

$$\left\| f'' - \frac{\gamma+\Gamma}{2} \right\|_{[a,b],\infty} \leq \frac{1}{2}\left|\Gamma-\gamma\right|.$$

The proof follows then by the inequalities (1.3) and (1.6) by taking $\lambda = \frac{\gamma+\Gamma}{2}$.

We remark that if $\gamma, \Gamma \in \mathbb{R}$ and

$$\gamma \leq f''(t) \leq \Gamma \text{ for a.e. } t \in [a,b]$$

then the inequalities (2.5) and (2.6) also hold.

We have:

Theorem 2. *Assume that $f : [a,b] \subset I \to \mathbb{C}$ is twice differentiable on $\mathring{I}$ and the second derivative is of bounded variation on $[a,b]$. Then*

$$\left| \frac{f(a)+f(b)}{2}(b-a) - \frac{1}{12}(b-a)^3 \frac{f''(a)+f''(b)}{2} - \int_a^b f(s)\,ds \right| \tag{2.7}$$
$$\leq \frac{(b-a)^3}{12}\left\| f'' - \frac{f''(a)+f''(b)}{2} \right\|_{[a,b],\infty} \leq \frac{(b-a)^3}{24}\bigvee_a^b (f'')$$

and

$$\left| \int_a^b f(s)\,ds - f\left(\frac{a+b}{2}\right)(b-a) - \frac{1}{24}(b-a)^3 \frac{f''(a)+f''(b)}{2} \right| \tag{2.8}$$
$$\leq \frac{(b-a)^3}{24}\left\| f'' - \frac{f''(a)+f''(b)}{2} \right\|_{[a,b],\infty} \leq \frac{(b-a)^3}{48}\bigvee_a^b (f'').$$

Proof. For $t \in (a,b)$ we have

$$\left| f''(t) - \frac{f''(a)+f''(b)}{2} \right| = \left| \frac{f''(t)-f''(a)+f''(t)-f''(b)}{2} \right|$$
$$\leq \frac{1}{2}\left[\left|f''(t)-f''(a)\right| + \left|f''(b)-f''(t)\right| \right] \leq \frac{1}{2}\bigvee_a^b (f''),$$

which implies that

$$\left\| f'' - \frac{f''(a)+f''(b)}{2} \right\|_{[a,b],\infty} \leq \frac{1}{2}\bigvee_a^b (f'').$$

Using the inequalities (1.3) and (1.6) for $\lambda = \frac{f''(a)+f''(b)}{2}$, we deduce the desired results (2.7) and (2.8). □

We also have:

Theorem 3. *Assume that $f : [a,b] \subset I \to \mathbb{C}$ is twice differentiable on $\mathring{I}$ and the second derivative is Lipschitzian with the constant $K > 0$ on $[a,b]$. Then*

$$\left| \frac{f(a)+f(b)}{2}(b-a) - \frac{1}{12}(b-a)^3 f''\left(\frac{a+b}{2}\right) - \int_a^b f(s)\,ds \right| \tag{2.9}$$
$$\leq \frac{(b-a)^3}{12}\left\| f'' - f''\left(\frac{a+b}{2}\right) \right\|_{[a,b],\infty} \leq \frac{(b-a)^3}{24}K$$

and

$$\left|\int_a^b f(s)\,ds - f\left(\frac{a+b}{2}\right)(b-a) - \frac{1}{24}(b-a)^3 f''\left(\frac{a+b}{2}\right)\right| \tag{2.10}$$
$$\le \frac{(b-a)^3}{24}\left\|f'' - f''\left(\frac{a+b}{2}\right)\right\|_{[a,b],\infty} \le \frac{(b-a)^3}{48}K.$$

The proof follows then by the inequalities (1.3) and (1.6) on taking $\lambda = f''\left(\frac{a+b}{2}\right)$ and observing that

$$\left\|f'' - f''\left(\frac{a+b}{2}\right)\right\|_{[a,b],\infty} = \operatorname*{essup}_{t\in[a,b]}\left|f''(t) - f''\left(\frac{a+b}{2}\right)\right|$$
$$\le \operatorname*{essup}_{t\in[a,b]}\left|t - \frac{a+b}{2}\right| = \frac{1}{2}(b-a).$$

By using the inequalities (1.3) and (1.6) and taking $\lambda = \frac{f'(b)-f'(a)}{b-a}$, then we also have:

Theorem 4. *Assume that $f : [a,b] \subset I \to \mathbb{C}$ is twice differentiable on $\mathring{I}$ and the second derivative is Lipschitzian with the constant $K > 0$ on $[a,b]$. Then*

$$\left|\frac{f(a)+f(b)}{2}(b-a) - \frac{1}{12}(b-a)^2[f'(b)-f'(a)] - \int_a^b f(s)\,ds\right| \tag{2.11}$$

$$\le \begin{cases} \dfrac{(b-a)^3}{12}\left\|f'' - \frac{f'(b)-f'(a)}{b-a}\right\|_{[a,b],\infty} & \text{if } g'' \in L_\infty[a,b]; \\ \dfrac{1}{2}[B(q+1,q+1)]^{\frac{1}{q}}(b-a)^{2+\frac{1}{q}}\left\|f'' - \frac{f'(b)-f'(a)}{b-a}\right\|_{[a,b],p} & \text{if } g'' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{(b-a)^2}{8}\left\|f'' - \frac{f'(b)-f'(a)}{b-a}\right\|_{[a,b],1} \end{cases}$$

and

$$\left|\int_a^b f(s)\,ds - f\left(\frac{a+b}{2}\right)(b-a) - \frac{1}{24}(b-a)^2[f'(b)-f'(a)]\right| \tag{2.12}$$

$$\le \begin{cases} \dfrac{(b-a)^3}{24}\left\|f'' - \frac{f'(b)-f'(a)}{b-a}\right\|_{[a,b],\infty} & \text{if } u'' \in L_\infty[a,b]; \\ \dfrac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}}\left\|f'' - \frac{f'(b)-f'(a)}{b-a}\right\|_{[a,b],p} & \text{if } u'' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{(b-a)^2}{8}\left\|f'' - \frac{f'(b)-f'(a)}{b-a}\right\|_{[a,b],1}. \end{cases}$$

3. Composite Inequalities

We have:

Theorem 5. *Let $h:[a,b]\to[h(a),h(b)]$ be a continuous strictly increasing function that is twice differentiable on (a,b). Assume that $f:[a,b]\subset I\to\mathbb{C}$ is differentiable on $\mathring{I}$ and the derivative is absolutely continuous on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{h(b)-h(a)}\int_a^b f(t)\,h'(t)\,dt\right| \tag{3.1}$$

$$\leq\begin{cases}\dfrac{(h(b)-h(a))^2}{12}\left\|\frac{f''h'-f'h''}{[h']^3}\right\|_{[a,b],\infty} & \text{if } \frac{f''h'-f'h''}{[h']^3}\in L_\infty[a,b];\\[2ex] \dfrac{1}{2}[B(q+1,q+1)]^{\frac{1}{q}}(h(b)-h(a))^{1+\frac{1}{q}} & \text{if } \frac{f''h'-f'h''}{[h']^{3-1/p}}\in L_p[a,b],\\ \times\left\|\frac{f''h'-f'h''}{[h']^{3-1/p}}\right\|_{[a,b],p} & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{h(b)-h(a)}{8}\left\|\frac{f''h'-f'h''}{[h']^2}\right\|_{[a,b],1}. & \text{if } \frac{f''h'-f'h''}{[h']^2}\in L_1[a,b].\end{cases}$$

Proof. We write the inequality (1.1) for the function $u=f\circ h^{-1}$ on the interval $[h(a),h(b)]$ we have

$$\left|\frac{f\circ h^{-1}(h(a))+f\circ h^{-1}(h(b))}{2}(h(b)-h(a))-\int_{h(a)}^{h(b)}f\circ h^{-1}(z)\,dz\right| \tag{3.2}$$

$$\leq\begin{cases}\dfrac{(h(b)-h(a))^3}{12}\operatorname{essup}_{z\in[h(a),h(b)]}\left|\left(f\circ h^{-1}\right)''(z)\right| & \text{if } \left(f\circ h^{-1}\right)''\in L_\infty[a,b];\\[2ex] \dfrac{1}{2}[B(q+1,q+1)]^{\frac{1}{q}}(h(b)-h(a))^{2+\frac{1}{q}} & \text{if } \left(f\circ h^{-1}\right)''\in L_p[a,b],\\ \times\left(\int_{h(a)}^{h(b)}\left|\left(f\circ h^{-1}\right)''(z)\right|^p dz\right)^{1/p} & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{(h(b)-h(a))^2}{8}\int_{h(a)}^{h(b)}\left|\left(f\circ h^{-1}\right)''(z)\right|dz. & \end{cases}$$

Using the chain rule and the derivative of inverse functions we have

$$\left(f\circ h^{-1}\right)'(z)=\left(f'\circ h^{-1}\right)(z)\left(h^{-1}\right)'(z)=\frac{\left(f'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)} \tag{3.3}$$

for every $z\in(h(m),h(M))$.

We have by (3.3) that

$$\begin{aligned}\left(f\circ h^{-1}\right)''(z)&=\left(\frac{\left(f'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\right)'\\&=\frac{\left(f'\circ h^{-1}\right)'(z)\left(h'\circ h^{-1}\right)(z)-\left(f'\circ h^{-1}\right)(z)\left(h'\circ h^{-1}\right)'(z)}{\left[\left(h'\circ h^{-1}\right)(z)\right]^2}\\&=\frac{\frac{\left(f''\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\left(h'\circ h^{-1}\right)(z)-\left(f'\circ h^{-1}\right)(z)\frac{\left(h''\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}}{\left[\left(h'\circ h^{-1}\right)(z)\right]^2}\\&=\frac{\left(f''\circ h^{-1}\right)(z)\left(h'\circ h^{-1}\right)(z)-\left(f'\circ h^{-1}\right)(z)\left(h''\circ h^{-1}\right)(z)}{\left[\left(h'\circ h^{-1}\right)(z)\right]^3}\end{aligned}$$

for a.e. $z\in(h(m),h(M))$.

Using the change of variable $z=h(t)$, $t\in[a,b]$, we have $dz=h'(t)\,dt$ and

$$\int_{h(a)}^{h(b)}f\circ h^{-1}(z)\,dz=\int_a^b f(t)\,h'(t)\,dt.$$

If $\frac{f''h'-f'h''}{[h']^3}\in L_\infty[a,b]$, then

$$\begin{aligned}(3.4)\qquad &\operatorname*{essup}_{z\in[h(a),h(b)]}\left|\left(f\circ h^{-1}\right)''(z)\right|\\&=\operatorname*{essup}_{z\in[h(a),h(b)]}\left|\frac{\left(f''\circ h^{-1}\right)(z)\left(h'\circ h^{-1}\right)(z)-\left(f'\circ h^{-1}\right)(z)\left(h''\circ h^{-1}\right)(z)}{\left[\left(h'\circ h^{-1}\right)(z)\right]^3}\right|\\&=\operatorname*{essup}_{x\in[a,b]}\left|\frac{f''(x)\,h'(x)-f'(x)\,h''(x)}{[h'(x)]^3}\right|=\left\|\frac{f''h'-f'h''}{[h']^3}\right\|_{[a,b],\infty}.\end{aligned}$$

Using the change of variable $z=h(t)$, $t\in[a,b]$, we have $dz=h'(t)\,dt$ and

$$\begin{aligned}&\int_{h(a)}^{h(b)}\left|\left(f\circ h^{-1}\right)''(z)\right|^p dz\\&=\int_{h(a)}^{h(b)}\left|\frac{\left(f''\circ h^{-1}\right)(z)\left(h'\circ h^{-1}\right)(z)-\left(f'\circ h^{-1}\right)(z)\left(h''\circ h^{-1}\right)(z)}{\left[\left(h'\circ h^{-1}\right)(z)\right]^3}\right|^p dz\\&=\int_a^b\left|\frac{f''(t)\,h'(t)-f'(t)\,h''(t)}{[h'(t)]^3}\right|^p h'(t)\,dt=\int_a^b\left|\frac{f''(t)\,h'(t)-f'(t)\,h''(t)}{[h'(t)]^{3-1/p}}\right|^p dt.\end{aligned}$$

Therefore

$$\left(\int_{h(a)}^{h(b)}\left|\left(f\circ h^{-1}\right)''(z)\right|^p dz\right)^{1/p}=\left\|\frac{f''h'-f'h''}{[h']^{3-1/p}}\right\|_{[a,b],p}$$

for $p>1$ and

$$\int_{h(a)}^{h(b)}\left|\left(f\circ h^{-1}\right)''(z)\right|dz=\left\|\frac{f''h'-f'h''}{[h']^2}\right\|_{[a,b],1}.$$

Now, if we employ (3.2) we get the desired inequality (3.1). □

a) If we consider the function $h:[a,b]\subset(0,\infty)\to\mathbb{R}$, $h(t)=\ln t$, $t>0$, then by (3.1) for $\ell(t)=t$, we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b\frac{f(t)}{t}dt\right| \tag{3.5}$$
$$\leq\begin{cases}\dfrac{\left(\ln\left(\frac{b}{a}\right)\right)^2}{12}\left\|f''\ell^2+f'\ell\right\|_{[a,b],\infty} & \text{if } f''\ell^2+f'\ell\in L_\infty[a,b];\\[2ex] \dfrac{1}{2}\left[B(q+1,q+1)\right]^{\frac{1}{q}}\left(\ln\left(\frac{b}{a}\right)\right)^{1+\frac{1}{q}} & \text{if } f''\ell^{2-1/p}+f'\ell^{1-1/p}\in L_p[a,b],\\ \times\left\|f''\ell^{2-1/p}+f'\ell^{1-1/p}\right\|_{[a,b],p} & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{\ln\left(\frac{b}{a}\right)}{8}\left\|f''\ell+f'\right\|_{[a,b],1}. & \text{if } f''\ell+f'\in L_1[a,b].\end{cases}$$

b). If we consider the function $h:[a,b]\subset(0,\infty)\to\mathbb{R}$, $h(t)=\exp t$, $t\in\mathbb{R}$, then by (3.1) for $\ell(t)=t$, we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\exp b-\exp a}\int_a^b f(t)\exp t\,dt\right| \tag{3.6}$$
$$\leq\begin{cases}\dfrac{(\exp b-\exp a)^2}{12}\left\|\frac{f''-f'}{\exp(2\ell)}\right\|_{[a,b],\infty} & \text{if } \frac{f''-f'}{\exp(2\ell)}\in L_\infty[a,b];\\[2ex] \dfrac{1}{2}\left[B(q+1,q+1)\right]^{\frac{1}{q}}(\exp b-\exp a)^{1+\frac{1}{q}} & \text{if } \frac{f''-f'}{\exp[(2-1/p)\ell]}\in L_p[a,b],\\ \times\left\|\frac{f''-f'}{\exp[(2-1/p)\ell]}\right\|_{[a,b],p} & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{\exp b-\exp a}{8}\left\|\frac{f''-f'}{\exp(\ell)}\right\|_{[a,b],1}. & \text{if } \frac{f''-f'}{\exp(\ell)}\in L_1[a,b].\end{cases}$$

c). If we consider the function $h:[a,b]\subset(0,\infty)\to\mathbb{R}$, $h(t)=t^r$, $t>0$, $r>0$ then by (3.1) for $\ell(t)=t$, we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{r}{b^r-a^r}\int_a^b f(t)\,t^{r-1}dt\right| \tag{3.7}$$
$$\leq\begin{cases}\dfrac{(b^r-a^r)^2}{12}\left\|\frac{f''-(r-1)\ell^{-1}f'}{r^2\ell^{2(r-1)}}\right\|_{[a,b],\infty} & \text{if } \frac{f''-(r-1)\ell^{-1}f'}{r^2\ell^{2(r-1)}}\in L_\infty[a,b];\\[2ex] \dfrac{1}{2}\left[B(q+1,q+1)\right]^{\frac{1}{q}}(b^r-a^r)^{1+\frac{1}{q}} & \text{if } \frac{f''-f'(r-1)\ell^{-1}}{r^{2-1/p}\ell^{(r-1)(2-1/p)}}\in L_p[a,b],\\ \times\left\|\frac{f''-f'(r-1)\ell^{-1}}{r^{2-1/p}\ell^{(r-1)(2-1/p)}}\right\|_{[a,b],p} & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{b^r-a^r}{8}\left\|\frac{f''-f'(r-1)\ell^{-1}}{r\ell^{r-1}}\right\|_{[a,b],1}. & \text{if } \frac{f''-f'(r-1)\ell^{-1}}{r\ell^{r-1}}\in L_1[a,b].\end{cases}$$

If $w:[a,b]\to\mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W:[a,b]\to[0,\infty)$, $W(x):=\int_a^x w(s)\,ds$ is strictly increasing and differentiable on (a,b). We have $W'(x)=w(x)$ for any $x\in(a,b)$.

Corollary 2. *Assume that $w : [a,b] \to (0,\infty)$ is absolutely continuous on $[a,b]$ and $f : [a,b] \subset I \to \mathbb{C}$ is differentiable on $\mathring{I}$ and the derivative is absolutely continuous on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{\int_a^b w(s)\,ds}\int_a^b f(t)\,w(t)\,dt\right| \tag{3.8}$$
$$\leq \begin{cases} \frac{1}{12}\left(\int_a^b w(s)\,ds\right)^2 \left\|\frac{f''w-f'w'}{w^3}\right\|_{[a,b],\infty} & \text{if } \frac{f''w-f'w'}{w^3} \in L_\infty[a,b]; \\ \frac{1}{2}\left[B(q+1,q+1)\right]^{\frac{1}{q}}\left(\int_a^b w(s)\,ds\right)^{1+\frac{1}{q}} & \text{if } \frac{f''w-f'w'}{w^{3-1/p}} \in L_p[a,b], \\ \times\left\|\frac{f''w-f'w'}{w^{3-1/p}}\right\|_{[a,b],p} & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{8}\int_a^b w(s)\,ds \left\|\frac{f''w-f'w'}{w^2}\right\|_{[a,b],1}. & \text{if } \frac{f''w-f'w'}{w^2} \in L_1[a,b]. \end{cases}$$

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* $a,\ b \in I$ as

$$M_g(a,b) := g^{-1}\left(\frac{g(a)+g(b)}{2}\right). \tag{3.9}$$

If $I=\mathbb{R}$ and $g(t)=t$ is the *identity function*, then $M_g(a,b)=A(a,b):=\frac{a+b}{2}$, the *arithmetic mean*. If $I=(0,\infty)$ and $g(t)=\ln t$, then $M_g(a,b)=G(a,b):=\sqrt{ab}$, the *geometric mean*. If $I=(0,\infty)$ and $g(t)=-\frac{1}{t}$, then $M_g(a,b)=H(a,b):=\frac{2ab}{a+b}$, the *harmonic mean*. If $I=(0,\infty)$ and $g(t)=t^p$, $p\neq 0$, then $M_g(a,b)=M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent p*. Finally, if $I=\mathbb{R}$ and $g(t)=\exp t$, then

$$M_g(a,b)=LME(a,b):=\ln\left(\frac{\exp a+\exp b}{2}\right), \tag{3.10}$$

the *LogMeanExp function*.

By employing the inequality (1.4) and using a similar argument to the one in the proof of Theorem 5 we can state the following result as well:

Theorem 6. *Let $h:[a,b]\to[h(a),h(b)]$ be a continuous strictly increasing function that is twice differentiable on (a,b). Assume that $f:[a,b]\subset I\to\mathbb{C}$ is differentiable on $\mathring{I}$ and the derivative is absolutely continuous on $[a,b]$, then*

$$\left|f\left(M_h(a,b)\right)-\frac{1}{h(b)-h(a)}\int_a^b f(t)\,h'(t)\,dt\right| \tag{3.11}$$

$$\leq\begin{cases}\dfrac{(h(b)-h(a))^2}{24}\left\|\frac{f''h'-f'h''}{[h']^3}\right\|_{[a,b],\infty} & \text{if } \frac{f''h'-f'h''}{[h']^3}\in L_\infty[a,b];\\[2ex] \dfrac{1}{8(2q+1)^{\frac{1}{q}}}(h(b)-h(a))^{1+\frac{1}{q}}\left\|\frac{f''h'-f'h''}{[h']^{3-1/p}}\right\|_{[a,b],p} & \text{if } \frac{f''h'-f'h''}{[h']^{3-1/p}}\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{h(b)-h(a)}{8}\left\|\frac{f''h'-f'h''}{[h']^2}\right\|_{[a,b],1}. & \text{if } \frac{f''h'-f'h''}{[h']^2}\in L_1[a,b].\end{cases}$$

a) If we consider the function $h:[a,b]\subset(0,\infty)\to\mathbb{R}$, $h(t)=\ln t$, $t>0$, then by (3.1) for $\ell(t)=t$, we get

$$\left|f\left(G(a,b)\right)-\frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b\frac{f(t)}{t}dt\right| \tag{3.12}$$

$$\leq\begin{cases}\dfrac{\left(\ln\left(\frac{b}{a}\right)\right)^2}{24}\left\|f''\ell^2+f'\ell\right\|_{[a,b],\infty} & \text{if } f''\ell^2+f'\ell\in L_\infty[a,b];\\[2ex] \dfrac{1}{8(2q+1)^{\frac{1}{q}}}\left(\ln\left(\frac{b}{a}\right)\right)^{1+\frac{1}{q}} & \text{if } f''\ell^{2-1/p}+f'\ell^{1-1/p}\in L_p[a,b],\\ \times\left\|f''\ell^{2-1/p}+f'\ell^{1-1/p}\right\|_{[a,b],p} & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{\ln\left(\frac{b}{a}\right)}{8}\left\|f''\ell+f'\right\|_{[a,b],1} & \text{if } f''\ell+f'\in L_1[a,b],\end{cases}$$

where $G(a,b):=\sqrt{ab}$ is the geometric mean.

b). If we consider the function $h:[a,b]\subset(0,\infty)\to\mathbb{R}$, $h(t)=\exp t$, $t\in\mathbb{R}$, then by (3.1) for $\ell(t)=t$, we get

$$\left|f\left(LME(a,b)\right)-\frac{1}{\exp b-\exp a}\int_a^b f(t)\exp t\,dt\right| \tag{3.13}$$

$$\leq\begin{cases}\dfrac{(\exp b-\exp a)^2}{24}\left\|\frac{f''-f'}{\exp(2\ell)}\right\|_{[a,b],\infty} & \text{if } \frac{f''-f'}{\exp(2\ell)}\in L_\infty[a,b];\\[2ex] \dfrac{1}{8(2q+1)^{\frac{1}{q}}}(\exp b-\exp a)^{1+\frac{1}{q}} & \text{if } \frac{f''-f'}{\exp[(2-1/p)\ell]}\in L_p[a,b],\\ \times\left\|\frac{f''-f'}{\exp[(2-1/p)\ell]}\right\|_{[a,b],p} & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{\exp b-\exp a}{8}\left\|\frac{f''-f'}{\exp(\ell)}\right\|_{[a,b],1}. & \text{if } \frac{f''-f'}{\exp(\ell)}\in L_1[a,b],\end{cases}$$

$LME(a,b)$ is the *LogMeanExp function.*

c). If we consider the function $h:[a,b]\subset(0,\infty)\to\mathbb{R}$, $h(t)=t^r$, $t>0$, $r>0$ then by (3.1) for $\ell(t)=t$, we get

$$\left|f(M_p(a,b))-\frac{r}{b^r-a^r}\int_a^b f(t)t^{r-1}dt\right| \tag{3.14}$$

$$\leq\begin{cases}\dfrac{(b^r-a^r)^2}{24}\left\|\frac{f''-(r-1)\ell^{-1}f'}{r^2\ell^{2(r-1)}}\right\|_{[a,b],\infty} & \text{if } \frac{f''-(r-1)\ell^{-1}f'}{r^2\ell^{2(r-1)}}\in L_\infty[a,b]; \\ \dfrac{1}{8(2q+1)^{\frac{1}{q}}}(b^r-a^r)^{1+\frac{1}{q}} & \text{if } \frac{f''-f'(r-1)\ell^{-1}}{r^{2-1/p}\ell^{(r-1)(2-1/p)}}\in L_p[a,b], \\ \times\left\|\frac{f''-f'(r-1)\ell^{-1}}{r^{2-1/p}\ell^{(r-1)(2-1/p)}}\right\|_{[a,b],p} & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{b^r-a^r}{8}\left\|\frac{f''-f'(r-1)\ell^{-1}}{r\ell^{r-1}}\right\|_{[a,b],1} & \text{if } \frac{f''-f'(r-1)\ell^{-1}}{r\ell^{r-1}}\in L_1[a,b],\end{cases}$$

where $M_p(a,b)$ is the *power mean with exponent* p.

We also have:

Corollary 3. *Assume that* $w:[a,b]\to(0,\infty)$ *is absolutely continuous on* $[a,b]$ *and* $f:[a,b]\subset I\to\mathbb{C}$ *is differentiable on* $\mathring{I}$ *and the derivative is absolutely continuous on* $[a,b]$, *then*

$$\left|f\left(W^{-1}\left(\frac{1}{2}\int_a^b w(s)\,ds\right)\right)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(t)\,w(t)\,dt\right| \tag{3.15}$$

$$\leq\begin{cases}\dfrac{1}{24}\left(\int_a^b w(s)\,ds\right)^2\left\|\frac{f''w-f'w'}{w^3}\right\|_{[a,b],\infty} & \textit{if } \frac{f''w-f'w'}{w^3}\in L_\infty[a,b]; \\ \dfrac{1}{8(2q+1)^{\frac{1}{q}}}\left(\int_a^b w(s)\,ds\right)^{1+\frac{1}{q}}\left\|\frac{f''w-f'w'}{w^{3-1/p}}\right\|_{[a,b],p} & \textit{if } \frac{f''w-f'w'}{w^{3-1/p}}\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{1}{8}\int_a^b w(s)\,ds\left\|\frac{f''w-f'w'}{w^2}\right\|_{[a,b],1}. & \textit{if } \frac{f''w-f'w'}{w^2}\in L_1[a,b].\end{cases}$$

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