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WEIGHTED INEQUALITIES OF OSTROWSKI TYPE FOR ABSOLUTELY CONTINUOUS FUNCTIONS IN TERMS OF p-NORMS AND APPLICATIONS

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ABSTRACT. In this paper we establish some upper bounds in terms of Lebesgue p-norms for the quantity

$$\left|\left[g(b)-g(a)\right]f(x)-\int_a^b f(t)\,g'(t)\,dt\right|$$

under the assumptions that $g:[a,b]\to[g(a),g(b)]$ is a *continuous strictly increasing function* that is *differentiable* on (a,b) and $f:[a,b]\to\mathbb{C}$ is an absolutely continuous function on $[a,b]$. When g is an integral, namely $g(x)=\int_a^x w(s)\,ds$, where $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$, then some weighted inequalities that generalize the Ostrowski's inequality are provided. Applications for continuous probability density functions supported on finite and infinite intervals with two examples are also given.

1. Introduction

In 1998, Dragomir and Wang proved the following Ostrowski type inequality [3].

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f'\in L_p[a,b]$, then we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}\left\|f'\right\|_{[a,b],p}, \tag{1.1}$$

for all $x\in[a,b]$, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\|\cdot\|_{[a,b],p}$ is the p-Lebesgue norm on $L_p[a,b]$, i.e., we recall it

$$\|g\|_{[a,b],p}:=\left(\int_a^b |g(t)|^p\,dt\right)^{1/p}.$$

Note that the inequality (1.1) can also be obtained from a more general result obtained by A. M. Fink in [6] choosing $n=1$ and doing some appropriate computation. However the inequality (1.1) was not stated explicitly in [6].

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From (1.1) we get the following midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{2(q+1)^{1/q}}(b-a)^{1/q}\|f'\|_{[a,b],p}, \tag{1.2}$$

and $\frac{1}{2}$ is a best possible constant.

Indeed, if we take $f:[a,b]\to\mathbb{R}$ with $f(t)=\left|t-\frac{a+b}{2}\right|$, then f is absolutely continuous $\int_a^b f(t)\,dt=\frac{(b-a)^2}{4}$, $\|f'\|_{[a,b],p}=(b-a)^{1/p}$ and if we assume that (1.2) holds with a constant $C>0$ instead of $\frac{1}{2}$, then we get $\frac{1}{4}(b-a)\le\frac{C}{(q+1)^{1/q}}(b-a)$ for any $q>1$. Letting $q\to1+$, we obtain $C\ge\frac{1}{2}$, which proves the sharpness of the constant.

For related results, see [1], [5] and [8]. For a comprehensive survey on Ostrowski's inequality, see [4] and the references therein.

In this paper we establish some upper bounds in terms of Lebesgue p-norms$\|\cdot\|_p$ for the quantity

$$\left|[g(b)-g(a)]f(x)-\int_a^b f(t)g'(t)\,dt\right|$$

under the assumptions that $g:[a,b]\to[g(a),g(b)]$ is a *continuous strictly increasing function* that is *differentiable* on (a,b) and $f:[a,b]\to\mathbb{C}$ is an absolutely continuous function on $[a,b]$. When g is an integral, namely $g(x)=\int_a^x w(s)\,ds$, where $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$, then some weighted inequalities that generalize the Ostrowski's inequality are provided. Applications for continuous probability density functions supported on finite and infinite intervals with two examples are also given.

2. Some Preliminary Facts

The following new result, which is an improvement on the inequality (1.1), holds.

Theorem 2 (Dragomir, 2013, [2])**.** *Let $h:[c,d]\to\mathbb{C}$ be an absolutely continuous function on $[c,d]$. If $h'\in L_p[c,d]$, then*

$$\begin{aligned}\left|h(z)-\frac{1}{d-c}\int_c^d h(t)\,dt\right| &\le\frac{1}{(q+1)^{1/q}}\left[\left(\frac{z-c}{d-c}\right)^{\frac{q+1}{q}}\|h'\|_{[c,z],p}+\left(\frac{d-z}{d-c}\right)^{\frac{q+1}{q}}\|h'\|_{[z,d],p}\right](d-c)^{1/q}\\ &\le\frac{1}{(q+1)^{1/q}}\end{aligned} \tag{2.1}$$

$$\times\begin{cases}\frac{1}{2}\left[\|h'\|_{[c,z],p}+\|h'\|_{[z,d],p}+\left|\|h'\|_{[c,z],p}-\|h'\|_{[z,d],p}\right|\right] \\ \times\left[\left(\frac{z-c}{d-c}\right)^{\frac{q+1}{q}}+\left(\frac{d-z}{d-c}\right)^{\frac{q+1}{q}}\right](d-c)^{1/q} \\ \\ \left(\|h'\|^{\alpha}_{[c,z],p}+\|h'\|^{\alpha}_{[z,d],p}\right)^{\frac{1}{\alpha}}\left[\left(\frac{z-c}{d-c}\right)^{\frac{q+1}{q}\beta}+\left(\frac{d-z}{d-c}\right)^{\frac{q+1}{q}\beta}\right]^{\frac{1}{\beta}}(d-c)^{1/q} \\ \text{where } \alpha>1 \text{ and } \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \\ \left[\|h'\|_{[c,z],p}+\|h'\|_{[z,d],p}\right]\left[\frac{1}{2}+\left|\frac{z-\frac{c+d}{2}}{d-c}\right|\right]^{\frac{q+1}{q}}(d-c)^{1/q}\end{cases}$$

for all $z\in[c,d]$, *where* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $\|\cdot\|_{[m,n],p}$ *denotes the usual* p*-norm on* $L_p[m,n]$ *with* $m<n$, *i.e., we recall that*

$$\|g\|_{[m,n],p}:=\left(\int_m^n|g(t)|\,dt\right)^{1/p}<\infty.$$

Proof. For the sake of completeness, we give here a proof.

Using the integration by parts formula for absolutely continuous functions on $[c,d]$, we have

$$\int_c^z(t-c)\,h'(t)\,dt=(z-c)\,h(z)-\int_c^z h(t)\,dt \tag{2.2}$$

and

$$\int_z^d(t-d)\,h'(t)\,dt=(d-z)\,h(z)-\int_z^d h(t)\,dt \tag{2.3}$$

for all $z\in[c,d]$.

Adding the two inequalities, we obtain the *Montgomery identity* for absolutely continuous functions (see for example [7, p. 565])

$$(d-c)\,h(z)-\int_c^d h(t)\,dt=\int_c^z(t-c)\,h'(t)\,dt+\int_z^d(t-d)\,h'(t)\,dt \tag{2.4}$$

for all $z\in[c,d]$.

Taking the modulus, we deduce

$$\begin{aligned}&\left|(d-c)\,h(z)-\int_c^d h(t)\,dt\right| \\ &\leq\left|\int_c^z(t-c)\,h'(t)\,dt\right|+\left|\int_z^d(t-d)\,h'(t)\,dt\right| \\ &\leq\int_c^z(t-c)\,|h'(t)|\,dt+\int_z^d(d-t)\,|h'(t)|\,dt.\end{aligned} \tag{2.5}$$

Utilizing Hölder's integral inequality we have

$$\begin{aligned}
&\int_c^z (t-c)\left|h'(t)\right| dt + \int_z^d (d-t)\left|h'(t)\right| dt \\
&\le \left(\int_c^z (t-c)^q\, dt\right)^{1/q} \left(\int_c^z \left|h'(t)\right|^p dt\right)^{1/p} \\
&+ \left(\int_z^d (d-t)^q\, dt\right)^{1/q} \left(\int_z^d \left|h'(t)\right|^p dt\right)^{1/p} \\
&= \frac{1}{(d-c)(q+1)^{1/q}} \left[(z-c)^{\frac{q+1}{q}} \left\|h'\right\|_{[c,z],p} + (d-z)^{\frac{q+1}{q}} \left\|h'\right\|_{[z,d],p}\right]
\end{aligned}$$

for all $z \in [c,d]$, and the first inequality in (2.1) is proved.

Now, let us observe that

$$\begin{aligned}
&(z-c)^{\frac{q+1}{q}} \left\|h'\right\|_{[c,z],p} + (d-z)^{\frac{q+1}{q}} \left\|h'\right\|_{[z,d],p} \\
&\le \max\left\{\left\|h'\right\|_{[c,z],p}, \left\|h'\right\|_{[z,d],p}\right\} \left[(z-c)^{\frac{q+1}{q}} + (d-z)^{\frac{q+1}{q}}\right] \\
&= \frac{1}{2}\left[\left\|h'\right\|_{[c,z],p} + \left\|h'\right\|_{[z,d],p} + \left|\left\|h'\right\|_{[c,z],p} - \left\|h'\right\|_{[z,d],p}\right|\right] \\
&\times \left[(z-c)^{\frac{q+1}{q}} + (d-z)^{\frac{q+1}{q}}\right]
\end{aligned}$$

and the first part of the second inequality is proved.

For the second inequality, we employ the elementary inequality for real numbers which can be derived from Hölder's discrete inequality

$$0 \le ms + nt \le (m^\alpha + n^\alpha)^{\frac{1}{\alpha}} \times \left(s^\beta + t^\beta\right)^{\frac{1}{\beta}}, \tag{2.6}$$

provided that m, s, n, $t \ge 0$, $\alpha > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Using (2.6), we obtain

$$\begin{aligned}
&(z-c)^{\frac{q+1}{q}} \left\|h'\right\|_{[c,z],p} + (d-z)^{\frac{q+1}{q}} \left\|h'\right\|_{[z,d],p} \\
&\le \left(\left\|h'\right\|_{[c,z],p}^\alpha + \left\|h'\right\|_{[z,d],p}^\alpha\right)^{\frac{1}{\alpha}} \left[(z-c)^{\frac{q+1}{q}\beta} + (d-z)^{\frac{q+1}{q}\beta}\right]^{\frac{1}{\beta}}
\end{aligned}$$

and the second part of the second inequality in (2.1) is also obtained.

Finally, we observe that

$$\begin{aligned}
&(z-c)^{\frac{q+1}{q}} \left\|h'\right\|_{[c,z],p} + (d-z)^{\frac{q+1}{q}} \left\|h'\right\|_{[z,d],p} \\
&\le \max\left\{(z-c)^{\frac{q+1}{q}}, (d-z)^{\frac{q+1}{q}}\right\} \left[\left\|h'\right\|_{[c,z],p} + \left\|h'\right\|_{[z,d],p}\right] \\
&= \left[\frac{d-c}{2} + \left|z - \frac{c+d}{2}\right|\right]^{\frac{q+1}{q}} \left[\left\|h'\right\|_{[c,z],p} + \left\|h'\right\|_{[z,d],p}\right]
\end{aligned}$$

and the last part of the second inequality in (2.1) is proved. □

The following corollary is also natural.

Corollary 1. *Under the above assumptions, we have*

$$\begin{aligned}(2.7)\qquad &\left|h\left(\frac{c+d}{2}\right)-\frac{1}{d-c}\int_c^d h(t)\,dt\right| \\ &\leq \frac{1}{2^{(q+1)/q}(q+1)^{1/q}}\left[\|h'\|_{\left[c,\frac{c+d}{2}\right],p}+\|h'\|_{\left[\frac{c+d}{2},d\right],p}\right](d-c)^{1/q}.\end{aligned}$$

Another interesting result is the following one.

Corollary 2. *Under the above assumptions, and if there is an $z_0 \in [c,d]$ with*

$$(2.8)\qquad \int_c^{z_0}|h'(t)|^p\,dt=\int_{z_0}^d|h'(t)|^p\,dt,$$

then we have the inequality

$$\begin{aligned}(2.9)\qquad &\left|h(z_0)-\frac{1}{d-c}\int_c^d h(t)\,dt\right| \\ &\leq \frac{1}{(q+1)^{1/q}}\left[\left(\frac{z_0-c}{d-c}\right)^{\frac{q+1}{q}}+\left(\frac{d-z_0}{d-c}\right)^{\frac{q+1}{q}}\right]\|h'\|_{[c,z_0],p}(d-c)^{1/q}.\end{aligned}$$

Remark 1. *If we take in (2.1) $\alpha=p$ and $\beta=q$, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then we get the following refinement of (1.1)*

$$\begin{aligned}(2.10)\qquad &\left|h(z)-\frac{1}{d-c}\int_c^d h(t)\,dt\right| \\ &\leq \frac{1}{(q+1)^{1/q}}\left[\left(\frac{z-c}{d-c}\right)^{\frac{q+1}{q}}\|h'\|_{[c,z],p}+\left(\frac{d-z}{d-c}\right)^{\frac{q+1}{q}}\|h'\|_{[z,d],p}\right](d-c)^{1/q} \\ &\leq \frac{1}{(q+1)^{1/q}}\left[\left(\frac{z-c}{d-c}\right)^{q+1}+\left(\frac{d-z}{d-c}\right)^{q+1}\right]^{1/q}(d-c)^{1/q}\|h'\|_{[c,d],p},\end{aligned}$$

for all $z\in[c,d]$.

This is true, since for $\alpha=p$, we have

$$\left(\|h'\|_{[c,z],p}^p+\|h'\|_{[z,d],p}^p\right)^{\frac{1}{p}}=\left(\int_c^z|h'(t)|^p\,dt+\int_z^d|h'(t)|^p\right)^{1/p}=\|h'\|_{[c,d],p}.$$

3. Main Results

We have:

Theorem 3. *Let $g:[a,b]\to[g(a),g(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). If $f:[a,b]\to\mathbb{C}$ is absolutely continuous on $[a,b]$ and $\frac{f'}{g'}\in L_p[a,b]$, where p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have*

$$\left|f(x)-\frac{1}{g(b)-g(a)}\int_a^b f(t)\,g'(t)\,dt\right| \tag{3.1}$$
$$\leq \frac{1}{(q+1)^{1/q}}[g(b)-g(a)]^{1/q}$$
$$\times\left[\left(\frac{g(x)-g(a)}{g(b)-g(a)}\right)^{\frac{q+1}{q}}\left\|\frac{f'}{(g')^{1/q}}\right\|_{[a,x],p}+\left(\frac{g(b)-g(x)}{g(b)-g(a)}\right)^{\frac{q+1}{q}}\left\|\frac{f'}{(g')^{1/q}}\right\|_{[x,b],p}\right]$$
$$\leq \frac{1}{(q+1)^{1/q}}[g(b)-g(a)]^{1/q}\left[\left(\frac{g(x)-g(a)}{g(b)-g(a)}\right)^{q+1}+\left(\frac{g(b)-g(x)}{g(b)-g(a)}\right)^{q+1}\right]^{1/q}$$
$$\times\left\|\frac{f'}{(g')^{1/q}}\right\|_{[a,b],p},$$

for any $x\in[a,b]$.

Proof. Assume that $[c,d]\subset[a,b]$. If $f:[c,d]\to\mathbb{C}$ is absolutely continuous on $[c,d]$, then $f\circ g^{-1}:[g(c),g(d)]\to\mathbb{C}$ is absolutely continuous on $[g(c),g(d)]$ and using the chain rule and the derivative of inverse functions we have

$$\left(f\circ g^{-1}\right)'(z)=\left(f'\circ g^{-1}\right)(z)\left(g^{-1}\right)'(z)=\frac{\left(f'\circ g^{-1}\right)(z)}{\left(g'\circ g^{-1}\right)(z)} \tag{3.2}$$

for almost every (a.e.) $z\in[g(c),g(d)]$.

Now, if we use the inequality (2.10) for the function $h=f\circ g^{-1}$ on the interval $[g(a),g(b)]$, then we get for any $z\in[g(a),g(b)]$ that

$$\left|f\circ g^{-1}(z)-\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f\circ g^{-1}(t)\,dt\right| \tag{3.3}$$
$$\leq \frac{1}{(q+1)^{1/q}}(g(b)-g(a))^{1/q}$$
$$\times\left[\left(\frac{z-g(a)}{g(b)-g(a)}\right)^{\frac{q+1}{q}}\left\|\frac{f'\circ g^{-1}}{g'\circ g^{-1}}\right\|_{[g(a),z],p}+\left(\frac{g(b)-z}{g(b)-g(a)}\right)^{\frac{q+1}{q}}\left\|\frac{f'\circ g^{-1}}{g'\circ g^{-1}}\right\|_{[z,g(b)],p}\right]$$
$$\leq \frac{1}{(q+1)^{1/q}}\left[\left(\frac{z-g(a)}{g(b)-g(a)}\right)^{q+1}+\left(\frac{g(b)-z}{g(b)-g(a)}\right)^{q+1}\right]^{1/q}$$
$$\times(g(b)-g(a))^{1/q}\left\|\frac{f'\circ g^{-1}}{g'\circ g^{-1}}\right\|_{[g(a),g(b)],p}.$$

Taking $z=g(x)$, $x\in[a,b]$, in (3.3) we then get

$$
\begin{aligned}
&\left|f(x)-\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f\circ g^{-1}(t)\,dt\right| \qquad (3.4)\\
&\quad\le \frac{1}{(q+1)^{1/q}}\left(g(b)-g(a)\right)^{1/q}\\
&\times\left[\left(\frac{g(x)-g(a)}{g(b)-g(a)}\right)^{\frac{q+1}{q}}\left\|\frac{f'\circ g^{-1}}{g'\circ g^{-1}}\right\|_{[g(a),g(x)],p}+\left(\frac{g(b)-g(x)}{g(b)-g(a)}\right)^{\frac{q+1}{q}}\left\|\frac{f'\circ g^{-1}}{g'\circ g^{-1}}\right\|_{[g(x),g(b)],p}\right]\\
&\quad\le \frac{1}{(q+1)^{1/q}}\left[\left(\frac{g(x)-g(a)}{g(b)-g(a)}\right)^{q+1}+\left(\frac{g(b)-g(x)}{g(b)-g(a)}\right)^{q+1}\right]^{1/q}\\
&\qquad\times\left(g(b)-g(a)\right)^{1/q}\left\|\frac{f'\circ g^{-1}}{g'\circ g^{-1}}\right\|_{[g(a),g(b)],p}.
\end{aligned}
$$

Observe also that, by the change of variable $t=g^{-1}(u)$, $u\in[g(a),g(b)]$, we have $u=g(t)$ that gives $du=g'(t)\,dt$ and

$$\int_{g(a)}^{g(b)}\left(f\circ g^{-1}\right)(u)\,du=\int_a^b f(t)\,g'(t)\,dt. \qquad (3.5)$$

Also

$$
\begin{aligned}
\left\|\frac{f'\circ g^{-1}}{g'\circ g^{-1}}\right\|_{[g(a),g(x)],p} &= \left(\int_{g(a)}^{g(x)}\left|\frac{\left(f'\circ g^{-1}\right)(u)}{\left(g'\circ g^{-1}\right)(u)}\right|^p du\right)^{1/p}\\
&= \left(\int_a^x\left|\frac{f'(t)}{g'(t)}\right|^p g'(t)\,dt\right)^{1/p}=\left(\int_a^x\left|\frac{f'(t)}{(g'(t))^{1-1/p}}\right|^p dt\right)^{1/p}\\
&= \left(\int_a^x\left|\frac{f'(t)}{(g'(t))^{1/q}}\right|^p dt\right)^{1/p}=\left\|\frac{f'}{(g')^{1/q}}\right\|_{[a,x],p}
\end{aligned}
$$

and, similarly,

$$\left\|\frac{f'\circ g^{-1}}{g'\circ g^{-1}}\right\|_{[g(x),g(b)],p}=\left\|\frac{f'}{(g')^{1/q}}\right\|_{[x,b],p}$$

and

$$\left\|\frac{f'\circ g^{-1}}{g'\circ g^{-1}}\right\|_{[g(a),g(b)],p}=\left\|\frac{f'}{(g')^{1/q}}\right\|_{[a,b],p}.$$

By replacing these norms into (3.4) we get the desired result (3.1). □

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* $a, b\in I$ as

$$M_g(a,b):=g^{-1}\left(\frac{g(a)+g(b)}{2}\right). \qquad (3.6)$$

If $I=\mathbb{R}$ and $g(t)=t$ is the *identity function*, then $M_g(a,b)=A(a,b):=\frac{a+b}{2}$, the *arithmetic mean*. If $I=(0,\infty)$ and $g(t)=\ln t$, then $M_g(a,b)=G(a,b):=\sqrt{ab}$, the *geometric mean*. If $I=(0,\infty)$ and $g(t)=-\frac{1}{t}$, then $M_g(a,b)=H(a,b):=$

$\frac{2ab}{a+b}$, the *harmonic mean.* If $I=(0,\infty)$ and $g(t)=t^{p}$, $p\neq 0$, then $M_{g}(a,b)=M_{p}(a,b):=\left(\frac{a^{p}+b^{p}}{2}\right)^{1/p}$, the *power mean with exponent* p. Finally, if $I=\mathbb{R}$ and $g(t)=\exp t$, then

$$M_{g}(a,b)=LME(a,b):=\ln\left(\frac{\exp a+\exp b}{2}\right), \tag{3.7}$$

the *LogMeanExp function.*

Corollary 3. *With the assumptions of Theorem 3 we have*

$$\begin{aligned} &\left|f\left(M_{g}(a,b)\right)-\frac{1}{g(b)-g(a)}\int_{a}^{b}f(t)\,g'(t)\,dt\right| \\ &\leq\frac{1}{2^{\frac{q+1}{q}}(q+1)^{1/q}}\left[g(b)-g(a)\right]^{1/q}\left[\left\|\frac{f'}{(g')^{1/q}}\right\|_{[a,M_{g}(a,b)],p}+\left\|\frac{f'}{(g')^{1/q}}\right\|_{[M_{g}(a,b),b],p}\right] \\ &\leq\frac{1}{2(q+1)^{1/q}}\left[g(b)-g(a)\right]^{1/q}\left\|\frac{f'}{(g')^{1/q}}\right\|_{[a,b],p}. \end{aligned} \tag{3.8}$$

Remark 2. *With the assumptions of Theorem 3, we have*

$$\begin{aligned} &\left|f\left(\frac{a+b}{2}\right)-\frac{1}{g(b)-g(a)}\int_{a}^{b}f(t)\,g'(t)\,dt\right| \\ &\leq\frac{1}{(q+1)^{1/q}}\left[g(b)-g(a)\right]^{1/q}\left[\left(\frac{g\left(\frac{a+b}{2}\right)-g(a)}{g(b)-g(a)}\right)^{\frac{q+1}{q}}\left\|\frac{f'}{(g')^{1/q}}\right\|_{[a,\frac{a+b}{2}],p}\right. \\ &\qquad\left.+\left(\frac{g(b)-g\left(\frac{a+b}{2}\right)}{g(b)-g(a)}\right)^{\frac{q+1}{q}}\left\|\frac{f'}{(g')^{1/q}}\right\|_{[\frac{a+b}{2},b],p}\right] \\ &\leq\frac{1}{(q+1)^{1/q}}\left[g(b)-g(a)\right]^{1/q}\left[\left(\frac{g\left(\frac{a+b}{2}\right)-g(a)}{g(b)-g(a)}\right)^{q+1}+\left(\frac{g(b)-g\left(\frac{a+b}{2}\right)}{g(b)-g(a)}\right)^{q+1}\right]^{1/q} \\ &\qquad\times\left\|\frac{f'}{(g')^{1/q}}\right\|_{[a,b],p}. \end{aligned} \tag{3.9}$$

Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. We can give the following examples of interest.

a). If we take $g:[a,b]\subset(0,\infty)\to\mathbb{R}$, $g(t)=\ln t$, in (3.1) and assume that $f'\ell^{1/q}\in L_\infty[a,b]$ where $\ell(t):=t$, then we get

$$\left|f(x)-\frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b\frac{f(t)}{t}dt\right| \tag{3.10}$$
$$\leq\frac{1}{(q+1)^{1/q}}\left[\ln\left(\frac{b}{a}\right)\right]^{1/q}$$
$$\times\left[\left(\frac{\ln\left(\frac{x}{a}\right)}{\ln\left(\frac{b}{a}\right)}\right)^{\frac{q+1}{q}}\left\|f'\ell^{1/q}\right\|_{[a,x],p}+\left(\frac{\ln\left(\frac{b}{x}\right)}{\ln\left(\frac{b}{a}\right)}\right)^{\frac{q+1}{q}}\left\|f'\ell^{1/q}\right\|_{[x,b],p}\right]$$
$$\leq\frac{1}{(q+1)^{1/q}}\left[\ln\left(\frac{b}{a}\right)\right]^{1/q}\left[\left(\frac{\ln\left(\frac{x}{a}\right)}{\ln\left(\frac{b}{a}\right)}\right)^{q+1}+\left(\frac{\ln\left(\frac{b}{x}\right)}{\ln\left(\frac{b}{a}\right)}\right)^{q+1}\right]^{1/q}\left\|f'\ell^{1/q}\right\|_{[a,b],p},$$

for any $x\in[a,b]$.

In particular, we have

$$\left|f(G(a,b))-\frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b\frac{f(t)}{t}dt\right| \tag{3.11}$$
$$\leq\frac{1}{2^{\frac{q+1}{q}}(q+1)^{1/q}}\left[\ln\left(\frac{b}{a}\right)\right]^{1/q}\left[\left\|\frac{f'}{(g')^{1/q}}\right\|_{[a,G(a,b)],p}+\left\|\frac{f'}{(g')^{1/q}}\right\|_{[G(a,b),b],p}\right]$$
$$\leq\frac{1}{2(q+1)^{1/q}}\left[\ln\left(\frac{b}{a}\right)\right]^{1/q}\left\|\frac{f'}{(g')^{1/q}}\right\|_{[a,b],p},$$

where $G(a,b):=\sqrt{ab}$ is the *geometric mean* of $a,\ b>0$.

b). If we take $g:[a,b]\subset\mathbb{R}\to(0,\infty)$, $g(t)=\exp t$, in (3.1) and assume that $f'\exp\left(-\frac{1}{q}\ell\right)\in L_\infty[a,b]$, then we get

$$\left|f(x)-\frac{1}{\exp b-\exp a}\int_a^b f(t)\exp t\,dt\right| \tag{3.12}$$
$$\leq\frac{1}{(q+1)^{1/q}}(\exp b-\exp a)^{1/q}\left[\left(\frac{\exp x-\exp a}{\exp b-\exp a}\right)^{\frac{q+1}{q}}\left\|f'\exp\left(-\frac{1}{q}\ell\right)\right\|_{[a,x],p}\right.$$
$$\left.+\left(\frac{\exp b-\exp x}{\exp b-\exp a}\right)^{\frac{q+1}{q}}\left\|f'\exp\left(-\frac{1}{q}\ell\right)\right\|_{[x,b],p}\right]$$
$$\leq\frac{1}{(q+1)^{1/q}}(\exp b-\exp a)^{1/q}\left[\left(\frac{\exp x-\exp a}{\exp b-\exp a}\right)^{q+1}+\left(\frac{\exp b-\exp x}{\exp b-\exp a}\right)^{q+1}\right]^{1/q}$$
$$\times\left\|f'\exp\left(-\frac{1}{q}\ell\right)\right\|_{[a,b],p},$$

for any $x\in[a,b]$.

In particular, we have

$$
\begin{aligned}
(3.13)\quad & \left| f\left(LME\left(a,b\right)\right) - \frac{1}{\exp b - \exp a}\int_a^b f\left(t\right)\exp t dt \right| \\
& \leq \frac{1}{2^{\frac{q+1}{q}}\left(q+1\right)^{1/q}}\left(\exp b - \exp a\right)^{1/q} \\
& \in \left[\left\| f'\exp\left(-\frac{1}{q}\ell\right)\right\|_{[a,LME(a,b)],p} + \left\| f'\exp\left(-\frac{1}{q}\ell\right)\right\|_{[LME(a,b),b],p} \right] \\
& \leq \frac{1}{2\left(q+1\right)^{1/q}}\left(\exp b - \exp a\right)^{1/q}\left\| f'\exp\left(-\frac{1}{q}\ell\right)\right\|_{[a,b],p},
\end{aligned}
$$

where $LME\left(a,b\right) := \ln\left(\frac{\exp a + \exp b}{2}\right)$ is the *LogMeanExp function.*

c). If we take $g : [a,b] \subset (0,\infty) \to \mathbb{R}$, $g\left(t\right) = t^r$, $r > 0$ in (3.1) and assume that $\ell^{\frac{1-r}{q}} f' \in L_\infty\left[a,b\right]$ then we get

$$
\begin{aligned}
(3.14)\quad & \left| f\left(x\right) - \frac{r}{b^r - a^r}\int_a^b f\left(t\right) t^{r-1} dt \right| \\
& \leq \frac{1}{r\left(q+1\right)^{1/q}}\left(b^r - a^r\right)^{1/q} \\
& \times \left[\left(\frac{x^r - a^r}{b^r - a^r}\right)^{\frac{q+1}{q}} \left\| \ell^{\frac{1-r}{q}} f' \right\|_{[a,x],p} + \left(\frac{b^r - x^r}{b^r - a^r}\right)^{\frac{q+1}{q}} \left\| \ell^{\frac{1-r}{q}} f' \right\|_{[x,b],p} \right] \\
& \leq \frac{1}{r\left(q+1\right)^{1/q}}\left(b^r - a^r\right)^{1/q}\left[\left(\frac{x^r - a^r}{b^r - a^r}\right)^{q+1} + \left(\frac{b^r - x^r}{b^r - a^r}\right)^{q+1} \right]^{1/q} \left\| \ell^{\frac{1-r}{q}} f' \right\|_{[a,b],p},
\end{aligned}
$$

for any $x \in [a,b]$.

In particular, we have

$$
\begin{aligned}
(3.15)\quad & \left| f\left(M_r\left(a,b\right)\right) - \frac{r}{b^r - a^r}\int_a^b f\left(t\right) t^{r-1} dt \right| \\
& \leq \frac{1}{2^{\frac{q+1}{q}}\left(q+1\right)^{1/q} r}\left(b^r - a^r\right)^{1/q}\left[\left\| \ell^{\frac{1-r}{q}} f' \right\|_{[a,M_p(a,b)],p} + \left\| \ell^{\frac{1-r}{q}} f' \right\|_{[M_p(a,b),b],p} \right] \\
& \leq \frac{1}{2r\left(q+1\right)^{1/q}}\left(b^r - a^r\right)^{1/q}\left\| \ell^{\frac{1-r}{q}} f' \right\|_{[a,b],p},
\end{aligned}
$$

where $M_r\left(a,b\right) := \left(\frac{a^r + b^r}{2}\right)^{1/r}$, $r > 1$ is the *power mean with exponent* r.

4. Weighted Integral Inequalities and Probability Distributions

If $w : [a,b] \to \mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W : [a,b] \to [0,\infty)$, $W\left(x\right) := \int_a^x w\left(s\right) ds$ is strictly increasing and differentiable on (a,b). We have $W'\left(x\right) = w\left(x\right)$ for any $x \in (a,b)$.

Proposition 1. *Assume that* $w : [a,b] \to (0,\infty)$ *is continuous on* $[a,b]$ *and* $f : [a,b] \to \mathbb{C}$ *is absolutely continuous on* $[a,b]$ *with* $\frac{f'}{w} \in L_p\left[a,b\right]$, *where* p, $q > 1$ *with*

$\frac{1}{p}+\frac{1}{q}=1$, *then we have*

$$
\begin{aligned}
(4.1)\quad & \left|f(x)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(t)\,w(t)\,dt\right| \\
& \leq \frac{1}{(q+1)^{1/q}}\left(\int_a^b w(s)\,ds\right)^{1/q} \\
& \times\left[\left(\frac{\int_a^x w(s)\,ds}{\int_a^b w(s)\,ds}\right)^{\frac{q+1}{q}}\left\|\frac{f'}{w^{1/q}}\right\|_{[a,x],p}+\left(\frac{\int_x^b w(s)\,ds}{\int_a^b w(s)\,ds}\right)^{\frac{q+1}{q}}\left\|\frac{f'}{w^{1/q}}\right\|_{[x,b],p}\right] \\
& \leq \frac{1}{(q+1)^{1/q}}\left(\int_a^b w(s)\,ds\right)^{1/q}\left[\left(\frac{\int_a^x w(s)\,ds}{\int_a^b w(s)\,ds}\right)^{q+1}+\left(\frac{\int_x^b w(s)\,ds}{\int_a^b w(s)\,ds}\right)^{q+1}\right]^{1/q}\left\|\frac{f'}{w^{1/q}}\right\|_{[a,b],p},
\end{aligned}
$$

In particular, if

$$M_W(a,b):=W^{-1}\left(\frac{1}{2}\int_a^b w(s)\,ds\right),$$

then we have

$$
\begin{aligned}
(4.2)\quad & \left|f(M_W(a,b))-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(t)\,w(t)\,dt\right| \\
& \leq \frac{1}{2^{\frac{q+1}{q}}(q+1)^{1/q}}\left(\int_a^b w(s)\,ds\right)^{1/q}\left[\left\|\frac{f'}{w^{1/q}}\right\|_{[a,M_W(a,b)],p}+\left\|\frac{f'}{w^{1/q}}\right\|_{[M_W(a,b),b],p}\right] \\
& \leq \frac{1}{2(q+1)^{1/q}}\left(\int_a^b w(s)\,ds\right)^{1/q}\left\|\frac{f'}{w^{1/q}}\right\|_{[a,b],p}.
\end{aligned}
$$

The above result can be extended for infinite intervals I by assuming that the function $f:I\to\mathbb{C}$ is locally absolutely continuous on I.

For instance, if $I=[a,\infty)$, $f:[a,\infty)\to\mathbb{C}$ is locally absolutely continuous on $[a,\infty)$ and $w(s)>0$ for $s\in[a,\infty)$ with $\int_a^\infty w(s)\,ds=1$, namely w is a probability density function on $[a,\infty)$, and if $\frac{f'}{w}\in L_p[a,\infty)$, p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by (4.1) we get

$$
\begin{aligned}
(4.3)\quad & \left|f(x)-\int_a^\infty f(t)\,w(t)\,dt\right| \\
& \leq \frac{1}{(q+1)^{1/q}}\left[[W(x)]^{\frac{q+1}{q}}\left\|\frac{f'}{w^{1/q}}\right\|_{[a,x],p}+[1-W(x)]^{\frac{q+1}{q}}\left\|\frac{f'}{w^{1/q}}\right\|_{[x,\infty),p}\right] \\
& \leq \frac{1}{(q+1)^{1/q}}\left[[W(x)]^{q+1}+[1-W(x)]^{q+1}\right]^{1/q}\left\|\frac{f'}{w^{1/q}}\right\|_{[a,\infty),p},
\end{aligned}
$$

for any $x\in[a,\infty)$, where $W(x):=\int_a^x w(s)\,ds$ is the cumulative distribution function.

If $m \in (a,\infty)$ is the *median point* for w, namely $W(m) = \frac{1}{2}$, then by (4.3) we get

$$\left| f(m) - \int_a^\infty f(t) w(t) dt \right| \tag{4.4}$$
$$\leq \frac{1}{2^{\frac{q+1}{q}} (q+1)^{1/q}} \left[\left\| \frac{f'}{w^{1/q}} \right\|_{[a,m],p} + \left\| \frac{f'}{w^{1/q}} \right\|_{[m,\infty),p} \right] \leq \frac{1}{2(q+1)^{1/q}} \left\| \frac{f'}{w^{1/q}} \right\|_{[a,\infty),p}.$$

In probability theory and statistics, the *beta prime distribution* (also known as *inverted beta distribution* or *beta distribution of the second kind*) is an absolutely continuous probability distribution defined for $x > 0$ with two parameters α and β, having the probability density function:

$$w_{\alpha,\beta}(x) := \frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha,\beta)}$$

where B is *Beta function*

$$B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}, \ \alpha, \ \beta > 0.$$

The cumulative distribution function is

$$W_{\alpha,\beta}(x) = I_{\frac{x}{1+x}}(\alpha,\beta),$$

where I is the *regularized incomplete beta function* defined by

$$I_z(\alpha,\beta) := \frac{B(z;\alpha,\beta)}{B(\alpha,\beta)}.$$

Here $B(\cdot;\alpha,\beta)$ is the *incomplete beta function* defined by

$$B(z;\alpha,\beta) := \int_0^z t^{\alpha-1}(1-t)^{\beta-1}, \ \alpha, \ \beta, \ z > 0.$$

Assume that $f : [0,\infty) \to \mathbb{C}$ is locally absolutely continuous on $[0,\infty)$ with $\frac{f'}{\ell^{\frac{\alpha-1}{q}}(1+\ell)^{-\frac{\alpha+\beta}{q}}} \in L_p[0,\infty)$, were $\ell(t) = t$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Using the inequality (4.3) we have for $x > 0$ that

$$\left| f(x) - \frac{1}{B(\alpha,\beta)} \int_0^\infty f(t) t^{\alpha-1}(1+t)^{-\alpha-\beta} dt \right| \tag{4.5}$$
$$\leq \frac{1}{(q+1)^{1/q}} B^{1/q}(\alpha,\beta) \left\{ \left[I_{\frac{x}{1+x}}(\alpha,\beta) \right]^{\frac{q+1}{q}} \left\| \frac{f'}{\ell^{\frac{\alpha-1}{q}}(1+\ell)^{-\frac{\alpha+\beta}{q}}} \right\|_{[0,x],p} \right.$$
$$\left. + \left[1 - I_{\frac{x}{1+x}}(\alpha,\beta) \right]^{\frac{q+1}{q}} \left\| \frac{f'}{\ell^{\frac{\alpha-1}{q}}(1+\ell)^{-\frac{\alpha+\beta}{q}}} \right\|_{[x,\infty),p} \right\}$$
$$\leq \frac{1}{(q+1)^{1/q}} B^{1/q}(\alpha,\beta) \left[\left[I_{\frac{x}{1+x}}(\alpha,\beta) \right]^{q+1} + \left[1 - I_{\frac{x}{1+x}}(\alpha,\beta) \right]^{q+1} \right]^{1/q}$$
$$\times \left\| \frac{f'}{\ell^{\frac{\alpha-1}{q}}(1+\ell)^{-\frac{\alpha+\beta}{q}}} \right\|_{[0,\infty),p}.$$

Similar results may be stated for the probability distributions that are supported on the whole axis $\mathbb{R} = (-\infty, \infty)$. Namely, if $I = (-\infty, \infty)$, $f : \mathbb{R} \to \mathbb{C}$ is locally absolutely continuous on $\mathbb{R}$ and $w(s) > 0$ for $s \in \mathbb{R}$ with $\int_{-\infty}^{\infty} w(s)\, ds = 1$, namely w is a probability density function on $(-\infty, \infty)$, and if $\frac{f'}{w} \in L_\infty(-\infty, \infty)$ then by (4.1) we get

$$\begin{aligned}
&\left| f(x) - \int_{-\infty}^{\infty} f(t)\, w(t)\, dt \right| \\
&\leq \frac{1}{(q+1)^{1/q}} \left[[W(x)]^{\frac{q+1}{q}} \left\| \frac{f'}{w^{1/q}} \right\|_{(-\infty,x],p} + [1 - W(x)]^{\frac{q+1}{q}} \left\| \frac{f'}{w^{1/q}} \right\|_{[x,\infty),p} \right] \\
&\qquad \leq \frac{1}{(q+1)^{1/q}} \left[[W(x)]^{q+1} + [1 - W(x)]^{q+1} \right]^{1/q} \left\| \frac{f'}{w^{1/q}} \right\|_{(-\infty,\infty),p},
\end{aligned} \tag{4.6}$$

for all $x \in (-\infty, \infty)$.

In particular, if $m \in \mathbb{R}$ is the *median point* for w, namely $W(m) = \frac{1}{2}$, then by (4.6) we get

$$\begin{aligned}
&\left| f(m) - \int_{-\infty}^{\infty} f(t)\, w(t)\, dt \right| \\
&\leq \frac{1}{2^{\frac{q+1}{q}} (q+1)^{1/q}} \left[\left\| \frac{f'}{w^{1/q}} \right\|_{(-\infty,m],p} + \left\| \frac{f'}{w^{1/q}} \right\|_{[m,\infty),p} \right] \\
&\qquad \leq \frac{1}{2(q+1)^{1/q}} \left\| \frac{f'}{w^{1/q}} \right\|_{(-\infty,\infty),p}.
\end{aligned} \tag{4.7}$$

In what follows we give an example.

The probability density of the *normal distribution* on $(-\infty, \infty)$ is

$$w_{\mu,\sigma^2}(x) := \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} \right), \ x \in \mathbb{R},$$

where μ is the *mean* or *expectation* of the distribution (and also its *median* and *mode*), σ is the *standard deviation*, and σ^2 is the *variance*.

The cumulative distribution function is

$$W_{\mu,\sigma^2}(x) = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}} \right),$$

where the *error function* erf is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-t^2\right) dt.$$

If $f:\mathbb{R}\to\mathbb{R}$ is locally absolutely continuous with $\exp\left(\frac{(\ell-\mu)^2}{2\sigma^2 q}\right)f'\in L_\infty(-\infty,\infty)$, where $\ell(t)=t$, then from (4.6) we get

$$\left|f(x)-\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}f(t)\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)dt\right| \tag{4.8}$$
$$\leq\frac{\left(\sqrt{2\pi}\sigma\right)^{1/q}}{2^{\frac{q+1}{q}}(q+1)^{1/q}}\left\{\left[1+\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]^{\frac{q+1}{q}}\left\|\exp\left(\frac{(\ell-\mu)^2}{2\sigma^2 q}\right)f'\right\|_{(-\infty,x],p}\right.$$
$$\left.+\left[1-\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]^{\frac{q+1}{q}}\left\|\exp\left(\frac{(\ell-\mu)^2}{2\sigma^2 q}\right)f'\right\|_{[x,\infty),p}\right\}$$
$$\leq\frac{\left(\sqrt{2\pi}\sigma\right)^{1/q}}{2^{\frac{q+1}{q}}(q+1)^{1/q}}\left[\left[1+\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]^{q+1}+\left[1-\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]^{q+1}\right]^{1/q}$$
$$\times\left\|\exp\left(\frac{(\ell-\mu)^2}{2\sigma^2 q}\right)f'\right\|_{(-\infty,\infty),p},$$

for all $x\in(-\infty,\infty)$.

In particular, we have

$$\left|f(\mu)-\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}f(t)\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)dt\right| \tag{4.9}$$
$$\leq\frac{\left(\sqrt{2\pi}\sigma\right)^{1/q}}{2^{\frac{q+1}{q}}(q+1)^{1/q}}$$
$$\times\left[\left\|\exp\left(\frac{(\ell-\mu)^2}{2\sigma^2 q}\right)f'\right\|_{(-\infty,m],p}+\left\|\exp\left(\frac{(\ell-\mu)^2}{2\sigma^2 q}\right)f'\right\|_{[m,\infty),p}\right]$$
$$\leq\frac{\left(\sqrt{2\pi}\sigma\right)^{1/q}}{2(q+1)^{1/q}}\left\|\exp\left(\frac{(\ell-\mu)^2}{2\sigma^2 q}\right)f'\right\|_{(-\infty,\infty),p}.$$

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