Received 28/05/18

NEW INEQUALITIES FOR THE ČEBYŠEV FUNCTIONAL

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper, by employing some classical results due to Ostrowski, Čebyšev and Lupaş, we establish some new inequalities for the *Čebyšev functional*

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt,$$

of two *Lebesgue integrable* functions $f,\ g : [a,b] \to \mathbb{R}$.

1. Introduction

For two *Lebesgue integrable* functions $f,\ g : [a,b] \to \mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt. \tag{1.1}$$

In 1935, Grüss [18] showed that

$$|C(f,g)| \le \frac{1}{4}(M-m)(N-n), \tag{1.2}$$

provided that there exists the real numbers m, M, n, N such that

$$m \le f(t) \le M \quad \text{and} \quad n \le g(t) \le N \quad \text{for a.e. } t \in [a,b]. \tag{1.3}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known result, even though it was obtained by Čebyšev in 1882, [4], states that

$$|C(f,g)| \le \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2, \tag{1.4}$$

provided that f', g' exist and are continuous on $[a,b]$ and $\|f'\|_\infty = \sup_{t\in[a,b]}|f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (1.4) also holds if $f,\ g : [a,b] \to \mathbb{R}$ are assumed to be *absolutely continuous* and $f',\ g' \in L_\infty[a,b]$ while $\|f'\|_\infty = \operatorname{essup}_{t\in[a,b]}|f'(t)|$.

A mixture between Grüss' result (1.2) and Čebyšev's one (1.4) is the following inequality obtained by Ostrowski in 1970, [25]:

$$|C(f,g)| \le \frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{1.5}$$

provided that f is *Lebesgue integrable* and satisfies (1.3) while g is absolutely continuous and $g' \in L_\infty[a,b]$. The constant $\frac{1}{8}$ is best possible in (1.5).

1991 *Mathematics Subject Classification.* 26D15; 26D10.

Key words and phrases. Ostrowski's inequality, Čebyšev inequality, Lupaş inequality,

The case of *euclidean norms* of the derivative was considered by A. Lupaş in [22] in which he proved that

$$|C(f,g)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b-a), \tag{1.6}$$

provided that f, g are absolutely continuous and $f', g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Consider now the *weighted Čebyšev functional*

$$C_w(f,g) := \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) f(t) g(t)\,dt - \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) f(t)\,dt \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) g(t)\,dt \tag{1.7}$$

where $f, g, w : [a,b] \to \mathbb{R}$ and $w(t) \geq 0$ for a.e. $t \in [a,b]$ are measurable functions such that the involved integrals exist and $\int_a^b w(t)\,dt > 0$.

In [6], Cerone and Dragomir obtained, among others, the following inequalities:

$$\begin{aligned} |C_w(f,g)| &\leq \frac{1}{2}(M-m) \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) \left| g(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(s) g(s)\,ds \right| dt \\ &\leq \frac{1}{2}(M-m) \left[\frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) \left| g(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(s) g(s)\,ds \right|^p dt \right]^{\frac{1}{p}} \\ &\leq \frac{1}{2}(M-m) \operatorname*{essup}_{t\in[a,b]} \left| g(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(s) g(s)\,ds \right| \end{aligned} \tag{1.8}$$

for $p > 1$, provided $-\infty < m \leq f(t) \leq M < \infty$ for a.e. $t \in [a,b]$ and the corresponding integrals are finite. The constant $\frac{1}{2}$ is sharp in all the inequalities in (1.8) in the sense that it cannot be replaced by a smaller constant.

In addition, if $-\infty < n \leq g(t) \leq N < \infty$ for a.e. $t \in [a,b]$, then the following refinement of the celebrated Grüss inequality is obtained:

$$\begin{aligned} |C_w(f,g)| &\leq \frac{1}{2}(M-m) \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) \left| g(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(s) g(s)\,ds \right| dt \\ &\leq \frac{1}{2}(M-m) \left[\frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) \left| g(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(s) g(s)\,ds \right|^2 dt \right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}(M-m)(N-n). \end{aligned} \tag{1.9}$$

Here, the constants $\frac{1}{2}$ and $\frac{1}{4}$ are also sharp in the sense mentioned above.

For other inequality of Grüss' type see [1]-[5], [7]-[17], [19]-[24] and [26]-[29].

In this paper we establish some new inequalities for the *Čebyšev functional* $C(f,g)$ under several conditions for the integrable functions f, $g : [a,b] \to \mathbb{R}$.

2. Preliminary Results

We have:

Lemma 1. *Let $f : [a,b] \to \mathbb{R}$ be a Lebesgue integrable function such that there exists $m < M$ with*

$$m \leq f(t) \leq M \text{ for a.e. } t \in [a,b] \tag{2.1}$$

and so that $F(b) = 0$, where $F(x) := \int_a^x f(t)\,dt$. Then we have

$$\|F\|_{[a,b],2}^2 \leq \frac{1}{8}(b-a)^2 (M-m)\|F\|_{[a,b],\infty}. \tag{2.2}$$

Proof. Using integration by parts we have

$$\begin{aligned} \int_a^b F^2(x)\,dx &= \int_a^b F(x)F(x)\,dx = \int_a^b F(x)\,d\left(\int_a^x F(s)\,ds\right) \\ &= F(x)\int_a^x F(s)\,ds\Big|_a^b - \int_a^b F'(x)\left(\int_a^x F(s)\,ds\right)dx \\ &= F(b)\int_a^b F(s)\,ds - \int_a^b f(x)\left(\int_a^x F(s)\,ds\right)dx \\ &= -\int_a^b f(x)\left(\int_a^x F(s)\,ds\right)dx = \left|\int_a^b f(x)\left(\int_a^x F(s)\,ds\right)dx\right|. \end{aligned} \tag{2.3}$$

Using Ostrowski's inequality (1.5) we have

$$\begin{aligned} \left|\frac{1}{b-a}\int_a^b f(x)\left(\int_a^x F(s)\,ds\right)dx\right| &= \left|\frac{1}{b-a}\int_a^b f(x)\left(\int_a^x F(s)\,ds\right)dx\right. \\ &\quad \left. -\frac{1}{b-a}\int_a^b f(x)\,dx \frac{1}{b-a}\int_a^b \left(\int_a^x F(s)\,ds\right)dx\right| \\ &\leq \frac{1}{8}(b-a)(M-m)\|F\|_{[a,b],\infty}, \end{aligned}$$

which implies (2.2). □

Corollary 1. *Let $h : [a,b] \to \mathbb{R}$ be a Lebesgue integrable function on $[a,b]$ such that*

$$\gamma \leq h(x) \leq \Gamma \text{ for a.e. on } [a,b], \tag{2.4}$$

then we have the inequality

$$\begin{aligned} &\int_a^b \left|\int_a^x h(t)\,dt - \frac{x-a}{b-a}\int_a^b h(s)\,ds\right|^2 dx \\ &\leq \frac{1}{8}(b-a)^2(\Gamma-\gamma)\max_{x\in[a,b]}\left|\int_a^x h(t)\,dt - \frac{x-a}{b-a}\int_a^b h(s)\,ds\right|. \end{aligned} \tag{2.5}$$

Proof. Follows from Lemma 1 by taking $f(t) = h(t) - \frac{1}{b-a}\int_a^b h(s)\,ds$ and observing that $\int_a^b f(t)\,dt = 0$,

$$\begin{aligned} m = \gamma - \frac{1}{b-a}\int_a^b h(s)\,ds &\leq h(t) - \frac{1}{b-a}\int_a^b h(s)\,ds \\ &\leq \Gamma - \frac{1}{b-a}\int_a^b h(s)\,ds = M \end{aligned}$$

and

$$F(x) = \int_a^x h(t)\,dt - \frac{x-a}{b-a}\int_a^b h(s)\,ds,\ \ x \in [a,b].$$

□

We also have by Ostrowski's inequality (1.5) that

Lemma 2. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function with $f' \in L_\infty[a,b]$ and such that there exists $n < 0 < N$ with*

$$n \leq F(x) \leq N \text{ for a.e. } x \in [a,b] \tag{2.6}$$

and so that $F(b) = 0$, where $F(x) := \int_a^x f(t)\,dt$. Then we have

$$\|F\|_{[a,b],2}^2 \leq \frac{1}{8}(b-a)^2 (N-n)\|f'\|_{[a,b],\infty}. \tag{2.7}$$

Corollary 2. *Let $h : [a,b] \to \mathbb{R}$ be an absolutely continuous function and such that there exists $\phi < 0 < \Phi$ with*

$$\phi \leq \int_a^x h(t)\,dt - \frac{x-a}{b-a}\int_a^b h(s)\,ds \leq \Phi \text{ for a.e. on } [a,b], \tag{2.8}$$

then we have the inequality

$$\int_a^b \left|\int_a^x h(t)\,dt - \frac{x-a}{b-a}\int_a^b h(s)\,ds\right|^2 dx \leq \frac{1}{8}(b-a)^2(\Phi - \phi)\|h'\|_{[a,b],\infty}. \tag{2.9}$$

We also have:

Lemma 3. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ such that $f' \in L_\infty[a,b]$. Then*

$$\|F\|_{[a,b],2}^2 \leq \frac{1}{12}(b-a)^2 \|f'\|_{[a,b],\infty}\|F\|_{[a,b],\infty}. \tag{2.10}$$

Proof. Using Čebyšev's inequality for the functions f and $\int_a F(s)\,ds$ we have

$$\begin{aligned} \left|\frac{1}{b-a}\int_a^b f(x)\left(\int_a^x F(s)\,ds\right)dx\right| &= \left|\frac{1}{b-a}\int_a^b f(x)\left(\int_a^x F(s)\,ds\right)dx\right. \\ &\quad \left. - \frac{1}{b-a}\int_a^b f(x)\,dx \frac{1}{b-a}\int_a^b \left(\int_a^x F(s)\,ds\right)dx\right| \\ &\leq \frac{1}{12}(b-a)^2\|f'\|_{[a,b],\infty}\|F\|_{[a,b],\infty}. \end{aligned}$$

Using the equality (2.3) we get the inequality (2.10). □

Corollary 3. *Let $h:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ with $h'\in L_\infty[a,b]$, then we have the inequality*

$$\int_a^b\left|\int_a^x h(t)\,dt-\frac{x-a}{b-a}\int_a^b h(s)\,ds\right|^2 dx \tag{2.11}$$
$$\leq\frac{1}{12}(b-a)^2\left\|h'\right\|_{[a,b],\infty}\max_{x\in[a,b]}\left|\int_a^x h(t)\,dt-\frac{x-a}{b-a}\int_a^b h(s)\,ds\right|.$$

We have

Lemma 4. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ such that $f'\in L_2[a,b]$. Then*

$$\|F\|_{[a,b],2}\leq\frac{1}{\pi^2}(b-a)^2\left\|f'\right\|_{[a,b],2} \tag{2.12}$$

Proof. Using Lupaş's inequality for the functions f and $\int_a F(s)\,ds$ we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\left(\int_a^x F(s)\,ds\right)dx\right|=\left|\frac{1}{b-a}\int_a^b f(x)\left(\int_a^x F(s)\,ds\right)dx\right.$$
$$\left.-\frac{1}{b-a}\int_a^b f(x)\,dx\frac{1}{b-a}\int_a^b\left(\int_a^x F(s)\,ds\right)dx\right|$$
$$\leq\frac{1}{\pi^2}(b-a)^2\left\|f'\right\|_{[a,b],2}\|F\|_{[a,b],2}.$$

Using the equality (2.3) we get

$$\|F\|_{[a,b],2}^2\leq\frac{1}{\pi^2}(b-a)^2\left\|f'\right\|_{[a,b],2}\|F\|_{[a,b],2},$$

which is equivalent to (2.12). □

Corollary 4. *Let $h:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ with $h'\in L_2[a,b]$, then we have the inequality*

$$\left(\int_a^b\left|\int_a^x h(t)\,dt-\frac{x-a}{b-a}\int_a^b h(s)\,ds\right|^2 dx\right)^{1/2}\leq\frac{1}{\pi^2}(b-a)^2\left\|h'\right\|_{[a,b],2}. \tag{2.13}$$

Consider a function $g:[a,b]\to\mathbb{R}$ and assume that it is bounded on $[a,b]$. The chord that connects its end points $A=(a,g(a))$ and $B=(b,g(b))$ has the equation

$$d_g:[a,b]\to\mathbb{R},\ d_g(t)=\frac{(b-t)\,g(a)+(t-a)\,g(b)}{b-a}.$$

We consider the error in approximation the function g by d_g denoted by E_g and defined by

$$E_g(t):=g(t)-d_g(t)=g(t)-\frac{(b-t)\,g(a)+(t-a)\,g(b)}{b-a},\ t\in[a,b].$$

Sharp bounds for Φ_g under various assumptions for g and including absolute continuity, convexity, bounded variation, and monotonicity, were given in [14]. Some applications for weighted means and f-divergence measures in information theory were also provided.

We observe that if $g:[a,b]\to\mathbb{R}$ is aboslutely continuous on $[a,b]$, then

$$\int_a^x g'(t)\,dt-\frac{x-a}{b-a}\int_a^b g'(s)\,ds=g(x)-g(a)-\frac{x-a}{b-a}[g(b)-g(a)]$$
$$=g(x)-\frac{(x-a)g(b)+(b-x)g(a)}{b-a}$$
$$=E_g(x)$$

for $x\in[a,b]$

Using the above results we can state the following propositions:

Proposition 1. *Assume that $g:[a,b]\to\mathbb{R}$ is aboslutely continuous on $[a,b]$ and there exists the constants γ, Γ so that*

$$\gamma\le g'(x)\le\Gamma \text{ for a.e. on } [a,b], \tag{2.14}$$

then we have the inequality

$$\left|\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}\right|^2 \le\frac{1}{b-a}\int_a^b |E_g(x)|^2\,dx\le\frac{1}{8}(b-a)(\Gamma-\gamma)\max_{x\in[a,b]}|E_g(x)|. \tag{2.15}$$

Proof. If we use the inequality (2.5) for $h=g'$ then we get the second inequality in (2.15). For the first inequality, we use the Cauchy-Bunyakovsky-Schwarz integral inequality to get

$$\frac{1}{b-a}\int_a^b |E_g(x)|^2\,dx\ge\left|\frac{1}{b-a}\int_a^b E_g(x)\,dx\right|^2$$
$$=\left|\frac{1}{b-a}\int_a^b\left[g(x)-\frac{(x-a)g(b)+(b-x)g(a)}{b-a}\right]dx\right|^2$$
$$=\left|\frac{1}{b-a}\int_a^b g(x)\,dx-\frac{g(a)+g(b)}{2}\right|^2.$$

□

Proposition 2. *Assume that $g:[a,b]\to\mathbb{R}$ is differentiable and the derivative g' is aboslutely continuous on $[a,b]$, $g''\in L_\infty[a,b]$, and there exists the constants ϕ, Φ so that*

$$\phi\le E_g(x)\le\Phi \text{ for a.e. } x\in[a,b], \tag{2.16}$$

then

$$\left|\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}\right|^2 \le\frac{1}{b-a}\int_a^b |E_g(x)|^2\,dx\le\frac{1}{8}(b-a)(\Phi-\phi)\|g''\|_{[a,b],\infty}. \tag{2.17}$$

The proof follows by Corollary 2.

Proposition 3. *Assume that $g : [a,b] \to \mathbb{R}$ is differentiable and the derivative g' is aboslutely continuous on $[a,b]$ and $g'' \in L_\infty [a,b]$, then we have the inequality*

$$\left| \frac{1}{b-a} \int_a^b g(t)\,dt - \frac{g(a)+g(b)}{2} \right|^2 \le \frac{1}{b-a} \int_a^b |E_g(x)|^2\,dx \le \frac{1}{12}(b-a)\|g''\|_{[a,b],\infty} \max_{x\in[a,b]} |E_g(x)|. \tag{2.18}$$

The proof follows by Corollary 3.

Proposition 4. *Assume that $g : [a,b] \to \mathbb{R}$ is differentiable and the derivative g' is absolutely continuous on $[a,b]$ and $g'' \in L_2[a,b]$, then*

$$\left| \frac{1}{b-a} \int_a^b g(t)\,dt - \frac{g(a)+g(b)}{2} \right|^2 \le \frac{1}{b-a} \int_a^b |E_g(x)|^2\,dx \le \frac{1}{\pi^4}(b-a)^3 \|g''\|^2_{[a,b],2}. \tag{2.19}$$

The proof follows by Corollary 4.

3. Bounds for Čebyšev's Functional

We have:

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a Lebesgue integrable function on $[a,b]$ such that there exists the real numbers $m < M$ with the property*

$$-\infty < m \le f(x) \le M < \infty \text{ for a.e. on } [a,b], \tag{3.1}$$

and $g : [a,b] \to \mathbb{R}$ is absolutely continuous with $g' \in L_2[a,b]$, then we have the inequality

$$|C(f,g)|^2 \le \frac{1}{(b-a)^2} \|g'\|^2_{[a,b],2} \int_a^b \left| \int_a^x f(t)\,dt - \frac{x-a}{b-a}\int_a^b f(s)\,ds \right|^2 \le \frac{1}{8}(M-m)\|g'\|^2_{[a,b],2} \max_{x\in[a,b]} \left| \int_a^x f(t)\,dt - \frac{x-a}{b-a}\int_a^b f(s)\,ds \right|. \tag{3.2}$$

Proof. Integrating by parts, we have

$$\begin{aligned} &\frac{1}{b-a}\int_a^b \left(\int_a^x f(t)\,dt - \frac{x-a}{b-a}\int_a^b f(s)\,ds \right) g\prime(x)\,dx \\ &= \frac{1}{b-a}\left[\left(\int_a^x f(t)\,dt - \frac{x-a}{b-a}\int_a^b f(s)\,ds \right) g(x) \Big|_a^b \right. \\ &\left. - \int_a^b g(x) \left(f(x) - \frac{1}{b-a}\int_a^b f(s)\,ds \right) dx \right. \\ &= -\frac{1}{b-a}\int_a^b f(x)g(x)\,dx + \frac{1}{b-a}\int_a^b f(s)\,ds \frac{1}{b-a}\int_a^b g(x)\,dx, \end{aligned}$$

which gives that

$$C(f,g)=\frac{1}{b-a}\int_a^b\left(\frac{x-a}{b-a}\int_a^b f(s)\,ds-\int_a^x f(t)\,dt\right)g\prime(x)\,dx. \tag{3.3}$$

Using (CBS) integral inequality and the inequality (2.5), we have

$$\begin{aligned}|C(f,g)|^2&=\frac{1}{(b-a)^2}\left|\int_a^b\left(\frac{x-a}{b-a}\int_a^b f(s)\,ds-\int_a^x f(t)\,dt\right)g\prime(x)\,dx\right|^2 \\ &\le\frac{1}{(b-a)^2}\int_a^b\left|\frac{x-a}{b-a}\int_a^b f(s)\,ds-\int_a^x f(t)\,dt\right|^2\|g'\|_{[a,b],2}^2\end{aligned} \tag{3.4}$$

Using the inequality (2.5) for the function f we have

$$\begin{aligned}&\frac{1}{(b-a)^2}\int_a^b\left|\int_a^x f(t)\,dt-\frac{x-a}{b-a}\int_a^b f(s)\,ds\right|^2 dx \\ &\le\frac{1}{8}(M-m)\max_{x\in[a,b]}\left|\int_a^x f(t)\,dt-\frac{x-a}{b-a}\int_a^b f(s)\,ds\right|.\end{aligned} \tag{3.5}$$

Using (3.4) and (3.5) we get (3.2). □

We also have:

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ is absolutely continuous with $g'\in L_2[a,b]$.*

(i) *If there exists $\phi<0<\Phi$ with*

$$\phi\le\int_a^x f(t)\,dt-\frac{x-a}{b-a}\int_a^b f(s)\,ds\le\Phi \text{ for a.e. on } [a,b], \tag{3.6}$$

then

$$\begin{aligned}|C(f,g)|^2&\le\frac{1}{(b-a)^2}\|g'\|_{[a,b],2}^2\int_a^b\left|\int_a^x f(t)\,dt-\frac{x-a}{b-a}\int_a^b f(s)\,ds\right|^2 \\ &\le\frac{1}{8}(\Phi-\phi)\|f'\|_{[a,b],\infty}\|g'\|_{[a,b],2}^2,\end{aligned} \tag{3.7}$$

provided $f'\in L_\infty[a,b]$.

(ii) *If $f'\in L_\infty[a,b]$, then we have the inequality*

$$\begin{aligned}|C(f,g)|^2&\le\frac{1}{(b-a)^2}\|g'\|_{[a,b],2}^2\int_a^b\left|\int_a^x f(t)\,dt-\frac{x-a}{b-a}\int_a^b f(s)\,ds\right|^2 dx \\ &\le\frac{1}{12}(b-a)^2\|f'\|_{[a,b],\infty}\|g'\|_{[a,b],2}^2\max_{x\in[a,b]}\left|\int_a^x f(t)\,dt-\frac{x-a}{b-a}\int_a^b f(s)\,ds\right|.\end{aligned} \tag{3.8}$$

(iii) *If* $f' \in L_2[a,b]$, *then we have the inequality*

$$|C(f,g)| \leq \frac{1}{b-a}\|g'\|_{[a,b],2}\left(\int_a^b \left|\int_a^x f(t)\,dt - \frac{x-a}{b-a}\int_a^b f(s)\,ds\right|^2 dx\right)^{1/2} \leq \frac{1}{\pi^2}(b-a)\|f'\|_{[a,b],2}\|g'\|_{[a,b],2} \tag{3.9}$$

The proof follows along the lines of the proof of Theorem 1 by making use of the inequalities (2.9), (2.11) and (2.13) written for the function f. The details are however omitted.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa