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INTEGRAL INEQUALITIES RELATED TO WIRTINGER'S RESULT

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ABSTRACT. In this paper we establish some natural consequences of the Wirtinger integral inequality. Applications related to the trapezoid unweighted and weighted inequalities, of Fejér's inequality for convex functions and of Grüss' type inequalities are also provided.

1. Introduction

It is well known that, see for instance [4], or [7], if $u \in C^1([a,b],\mathbb{R})$ satisfies $u(a) = u(b) = 0$, then

$$\int_a^b u^2(t)\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.1}$$

with the equality holding if and only if $u(t) = K\sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant $K \in \mathbb{R}$.

If $u \in C^1([a,b],\mathbb{R})$ satisfies the condition $u(a) = 0$, then also

$$\int_a^b u^2(t)\,dt \le \frac{4(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.2}$$

and the equality holds if and only if $u(t) = L\sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant $L \in \mathbb{R}$.

If $h \in C^1([a,b],\mathbb{C})$ is a function with complex values and $h(a) = h(b) = 0$, then $\operatorname{Re} h(a) = \operatorname{Re} h(b) = 0$ and $\operatorname{Im} h(a) = \operatorname{Im} h(b) = 0$ and by writing (1.1) for $\operatorname{Re} h$ and $\operatorname{Im} h$ and adding the obtained inequalities, we get

$$\int_a^b |h(t)|^2\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b |h'(t)|^2\,dt \tag{1.3}$$

with the equality holding if and only if

$$h(t) = K\sin\left[\frac{\pi(t-a)}{b-a}\right]$$

for some complex constant $K \in \mathbb{C}$.

Similarly, if $h \in C^1([a,b],\mathbb{C})$ with $h(a) = 0$, then by (1.2) we have

$$\int_a^b |h(t)|^2\,dt \le \frac{4(b-a)^2}{\pi^2}\int_a^b |h'(t)|^2\,dt \tag{1.4}$$

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and the equality holds if and only if

$$h(t) = L \sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$$

for some complex constant $L \in \mathbb{R}$.

For some related Wirtinger type integral inequalities see [1], [2], [4] and [6]-[10].

Motivated by the above results, in this paper we establish some natural consequences of the Wirtinger integral inequality above. Applications related to the trapezoid unweighted and weighted inequalities, of Fejér's inequality for convex functions and of Grüss' type inequalities are also provided.

2. Some Applications

If $g \in C^1([a,b],\mathbb{C})$, then by taking

$$h(t) := \frac{g(t) + g(a+b-t)}{2} - \frac{g(a) + g(b)}{2},\ t \in [a,b]$$

we have $h(a) = h(b) = 0$ and by (1.3) we have

$$\int_a^b \left|\frac{g(t) + g(a+b-t)}{2} - \frac{g(a) + g(b)}{2}\right|^2 dt \leq \frac{(b-a)^2}{4\pi^2}\int_a^b |g'(t) - g'(a+b-t)|^2\, dt. \tag{2.1}$$

By Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$(b-a)\int_a^b \left|\frac{g(t) + g(a+b-t)}{2} - \frac{g(a) + g(b)}{2}\right|^2 dt$$
$$\geq \left|\int_a^b \left[\frac{g(t) + g(a+b-t)}{2} - \frac{g(a) + g(b)}{2}\right] dt\right|^2 = \left|\int_a^b g(t)\, dt - \frac{g(a) + g(b)}{2}(b-a)\right|^2,$$

which implies that

$$\left|\frac{g(a) + g(b)}{2}\int_a^b g(t)\, dt - \frac{1}{b-a}\right|^2 \leq \frac{1}{b-a}\int_a^b \left|\frac{g(t) + g(a+b-t)}{2} - \frac{g(a) + g(b)}{2}\right|^2 dt. \tag{2.2}$$

By utilising (2.1) and (2.2) we can state the following result:

Proposition 1. *Let $g \in C^1([a,b],\mathbb{C})$. Then*

$$\left|\frac{g(a) + g(b)}{2} - \frac{1}{b-a}\int_a^b g(t)\, dt\right| \tag{2.3}$$
$$\leq \left(\frac{1}{b-a}\int_a^b \left|\frac{g(t) + g(a+b-t)}{2} - \frac{g(a) + g(b)}{2}\right|^2 dt\right)^{1/2}$$
$$\leq \frac{\sqrt{b-a}}{2\pi}\left(\int_a^b |g'(t) - g'(a+b-t)|^2\, dt\right)^{1/2}.$$

If $g \in C^1([a,b],\mathbb{C})$, then by taking

$$h(t) := g(t) - \frac{g(a)(b-t) + g(b)(t-a)}{b-a}, \ t \in [a,b]$$

we have $h(a) = h(b) = 0$ and by (1.3) we have

$$\int_a^b \left| g(t) - \frac{g(a)(b-t) + g(b)(t-a)}{b-a} \right|^2 dt \le \frac{(b-a)^2}{\pi^2} \int_a^b \left| g'(t) - \frac{g(b) - g(a)}{b-a} \right|^2 dt. \tag{2.4}$$

By Cauchy-Bunyakovsky-Schwarz integral inequality we also have

$$\left| \frac{g(a) + g(b)}{2} \int_a^b g(t)\,dt - \frac{1}{b-a} \right|^2 \le \frac{1}{b-a} \int_a^b \left| g(t) - \frac{g(a)(b-t) + g(b)(t-a)}{b-a} \right|^2 dt. \tag{2.5}$$

By utilising (2.4) and (2.5) we can state the following result:

Proposition 2. *Let $g \in C^1([a,b],\mathbb{C})$. Then*

$$\begin{aligned} \left| \frac{g(a) + g(b)}{2} - \frac{1}{b-a} \int_a^b g(t)\,dt \right| &\le \left(\frac{1}{b-a} \int_a^b \left| g(t) - \frac{g(a)(b-t) + g(b)(t-a)}{b-a} \right|^2 dt \right)^{1/2} \\ &\le \frac{\sqrt{b-a}}{\pi} \left(\int_a^b \left| g'(t) - \frac{g(b) - g(a)}{b-a} \right|^2 dt \right)^{1/2}. \end{aligned} \tag{2.6}$$

Assume that $g : [a,b] \to \mathbb{C}$ is continuous, then by taking

$$h(t) := \int_a^t g(s)\,ds - \frac{t-a}{b-a} \int_a^b g(s)\,ds, \ t \in [a,b]$$

we have $h(a) = h(b) = 0$, $h \in C^1([a,b],\mathbb{C})$ and by (1.3) we get

$$\int_a^b \left| \int_a^t g(s)\,ds - \frac{t-a}{b-a} \int_a^b g(s)\,ds \right|^2 dt \le \frac{(b-a)^2}{\pi^2} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\,ds \right|^2 dt. \tag{2.7}$$

Observe that, integrating by parts, we have

$$\begin{aligned} \int_a^b \left(\int_a^t g(s)\,ds - \frac{t-a}{b-a} \int_a^b g(s)\,ds \right) dt &= \int_a^b \left(\int_a^t g(s)\,ds \right) dt - \frac{b-a}{2} \int_a^b g(s)\,ds \\ = b \int_a^b g(s)\,ds - \int_a^b t g(t)\,dt - \frac{b-a}{2} \int_a^b g(s)\,ds &= \frac{b+a}{2} \int_a^b g(s)\,ds - \int_a^b t g(t)\,dt. \end{aligned}$$

By Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$(b-a)\int_a^b \left|\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds\right|^2 dt \tag{2.8}$$

$$\geq \left|\int_a^b \left(\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds\right) dt\right|^2 = \left|\frac{b+a}{2}\int_a^b g(s)\,ds - \int_a^b tg(t)\,dt\right|^2.$$

Proposition 3. *Let $g \in C([a,b],\mathbb{C})$. Then*

$$\left|\frac{b+a}{2}\int_a^b g(s)\,ds - \int_a^b tg(t)\,dt\right| \tag{2.9}$$

$$\leq \left((b-a)\int_a^b \left|\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds\right|^2 dt\right)^{1/2}$$

$$\leq \frac{(b-a)^2}{\pi}\left[\frac{1}{b-a}\int_a^b |g(t)|^2\,dt - \left|\frac{1}{b-a}\int_a^b g(s)\,ds\right|^2\right]^{1/2}.$$

The proof follows by the inequalities (2.7) and (2.8) and by observing that

$$\frac{1}{b-a}\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 dt = \frac{1}{b-a}\int_a^b |g(t)|^2\,dt - \left|\frac{1}{b-a}\int_a^b g(s)\,ds\right|^2.$$

3. Composite Inequalities

We have:

Theorem 1. *Let $g : [a,b] \to [g(a), g(b)]$ be a continuous strictly increasing function that is of class C^1 on (a,b).*

(i) *If $f \in C^1([a,b],\mathbb{C})$ is a function with complex values and $f(a) = f(b) = 0$, then*

$$\int_a^b |f(t)|^2 g'(t)\,dt \leq \frac{[g(b)-g(a)]^2}{\pi^2}\int_a^b \frac{|f'(t)|^2}{g'(t)}dt. \tag{3.1}$$

The equality holds in (3.1) iff

$$f(t) = K\sin\left[\frac{\pi(g(t)-g(a))}{g(b)-g(a)}\right],\ K \in \mathbb{C}.$$

(ii) *If $f \in C^1([a,b],\mathbb{C})$ is a function with complex values and $f(a) = 0$, then*

$$\int_a^b |f(t)|^2 g'(t)\,dt \leq \frac{4[g(b)-g(a)]^2}{\pi^2}\int_a^b \frac{|f'(t)|^2}{g'(t)}dt. \tag{3.2}$$

The equality holds in (3.2) iff

$$f(t) = K\sin\left[\frac{\pi(g(t)-g(a))}{2(g(b)-g(a))}\right],\ K \in \mathbb{C}.$$

Proof. (i) We write the inequality (1.3) for the function $h = f \circ g^{-1}$ on the interval $[g(a), g(b)]$ to get

$$\int_{g(a)}^{g(b)} \left|\left(f \circ g^{-1}\right)(z)\right|^2 dz \le \frac{(g(b) - g(a))^2}{\pi^2} \int_{g(a)}^{g(b)} \left|\left(f \circ g^{-1}\right)'(z)\right|^2 dz. \tag{3.3}$$

If $f : [c, d] \to \mathbb{C}$ is absolutely continuous on $[c, d]$, then $f \circ g^{-1} : [g(c), g(d)] \to \mathbb{C}$ is absolutely continuous on $[g(c), g(d)]$ and using the chain rule and the derivative of inverse functions we have

$$\left(f \circ g^{-1}\right)'(z) = \left(f' \circ g^{-1}\right)(z) \left(g^{-1}\right)'(z) = \frac{\left(f' \circ g^{-1}\right)(z)}{\left(g' \circ g^{-1}\right)(z)} \tag{3.4}$$

for almost every (a.e.) $z \in [g(c), g(d)]$.

Using the inequality (3.3) we then get

$$\int_{g(a)}^{g(b)} \left|\left(f \circ g^{-1}\right)(z)\right|^2 dz \le \frac{(g(b) - g(a))^2}{\pi^2} \int_{g(a)}^{g(b)} \left|\frac{\left(f' \circ g^{-1}\right)(z)}{\left(g' \circ g^{-1}\right)(z)}\right|^2 dz, \tag{3.5}$$

provided $\left(f \circ g^{-1}\right)(g(a)) = f(a) = 0$ and $\left(f \circ g^{-1}\right)(g(b)) = f(b) = 0.$

Observe also that, by the change of variable $t = g^{-1}(z)$, $z \in [g(a), g(b)]$, we have $z = g(t)$ that gives $dz = g'(t)\, dt$ and

$$\int_{g(a)}^{g(b)} \left|\left(f \circ g^{-1}\right)(z)\right|^2 dz = \int_a^b |f(t)|^2 g'(t)\, dt. \tag{3.6}$$

We also have

$$\int_{g(a)}^{g(b)} \left|\frac{\left(f' \circ g^{-1}\right)(z)}{\left(g' \circ g^{-1}\right)(z)}\right|^2 dz = \int_a^b \left|\frac{f'(t)}{g'(t)}\right|^2 g'(t)\, dt = \int_a^b \frac{|f'(t)|^2}{g'(t)} dt.$$

By making use of (3.5) we get (3.1).

The equality holds in (3.5) provided

$$\left(f \circ g^{-1}\right)(z) = K \sin\left[\frac{\pi (z - g(a))}{g(b) - g(a)}\right],\ K \in \mathbb{C}$$

for $z \in [g(a), g(b)]$. If we take $t \in [a, b]$ and $z = g(t)$, we then get

$$f(t) = K \sin\left[\frac{\pi (g(t) - g(a))}{g(b) - g(a)}\right],\ K \in \mathbb{C}.$$

(ii) Follows in a similar way by (1.4). $\square$

Some examples are as follows:

a). If we take $g : [a, b] \subset (0, \infty) \to \mathbb{R}$, $g(t) = \ln t$ and assume that $f \in C^1([a, b], \mathbb{C})$ is a function with complex values and $f(a) = f(b) = 0$, then by (3.1) we get

$$\int_a^b \frac{|f(t)|^2}{t} dt \le \frac{\left[\ln\left(\frac{b}{a}\right)\right]^2}{\pi^2} \int_a^b |f'(t)|^2\, t dt. \tag{3.7}$$

The equality holds in (3.7) iff

$$f(t) = K \sin\left[\frac{\pi \ln\left(\frac{t}{a}\right)}{\ln\left(\frac{b}{a}\right)}\right],\ K \in \mathbb{C}.$$

If $f(a)=0$, then

$$\int_a^b \frac{|f(t)|^2}{t}dt \leq \frac{4\left[\ln\left(\frac{b}{a}\right)\right]^2}{\pi^2}\int_a^b |f'(t)|^2 t dt, \tag{3.8}$$

with equality iff

$$f(t)=K\sin\left[\frac{\pi\ln\left(\frac{t}{a}\right)}{2\ln\left(\frac{b}{a}\right)}\right],\ K\in\mathbb{C}.$$

b). If we take $g:[a,b]\subset\mathbb{R}\to(0,\infty)$, $g(t)=\exp t$ and assume that $f\in C^1([a,b],\mathbb{C})$ is a function with complex values and $f(a)=f(b)=0$, then by (3.1) we get

$$\int_a^b |f(t)|^2\exp t dt \leq \frac{(\exp b-\exp a)^2}{\pi^2}\int_a^b |f'(t)|^2\exp(-t)\,dt. \tag{3.9}$$

The equality holds in (3.9) iff

$$f(t)=K\sin\left[\frac{\pi(\exp t-\exp a)}{\exp b-\exp a}\right],\ K\in\mathbb{C}.$$

If $f(a)=0$, then

$$\int_a^b |f(t)|^2\exp t dt \leq \frac{4(\exp b-\exp a)^2}{\pi^2}\int_a^b |f'(t)|^2\exp(-t)\,dt \tag{3.10}$$

with equality iff

$$f(t)=K\sin\left[\frac{\pi(\exp t-\exp a)}{2(\exp b-\exp a)}\right],\ K\in\mathbb{C}.$$

c). If we take $g:[a,b]\subset(0,\infty)\to\mathbb{R}$, $g(t)=t^r$, $r>0$ and assume that $f\in C^1([a,b],\mathbb{C})$ is a function with complex values and $f(a)=f(b)=0$, then by (3.1) we get

$$\int_a^b |f(t)|^2 t^{r-1}dt \leq \frac{(b^r-a^r)^2}{r^2\pi^2}\int_a^b |f'(t)|^2 t^{1-r}dt. \tag{3.11}$$

The equality holds in (3.11) iff

$$f(t)=K\sin\left[\frac{\pi(t^r-a^r)}{b^r-a^r}\right],\ K\in\mathbb{C}.$$

If $f(a)=0$, then

$$\int_a^b |f(t)|^2 t^{r-1}dt \leq \frac{4(b^r-a^r)^2}{r^2\pi^2}\int_a^b |f'(t)|^2 t^{1-r}dt \tag{3.12}$$

with equality iff

$$f(t)=K\sin\left[\frac{\pi(t^r-a^r)}{2(b^r-a^r)}\right],\ K\in\mathbb{C}.$$

If $w:[a,b]\to\mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W:[a,b]\to[0,\infty)$, $W(x):=\int_a^x w(s)\,ds$ is strictly increasing and differentiable on (a,b). We have $W'(x)=w(x)$ for any $x\in(a,b)$.

Corollary 1. *Assume that $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ and $f \in C^1([a,b],\mathbb{C})$ is a function with complex values and $f(a) = f(b) = 0$, then*

$$\int_a^b |f(t)|^2 w(t)\,dt \le \frac{1}{\pi^2}\left(\int_a^b w(s)\,ds\right)^2 \int_a^b \frac{|f'(t)|^2}{w(t)}dt. \tag{3.13}$$

The equality holds in (3.13) iff

$$f(t) = K \sin\left[\frac{\pi \int_a^t w(s)\,ds}{\int_a^b w(s)\,ds}\right],\ K \in \mathbb{C}.$$

If $f(a) = 0$, then

$$\int_a^b |f(t)|^2 w(t)\,dt \le \frac{4}{\pi^2}\left(\int_a^b w(s)\,ds\right)^2 \int_a^b \frac{|f'(t)|^2}{w(t)}dt \tag{3.14}$$

with equality iff

$$f(t) = K \sin\left[\frac{\pi \int_a^t w(s)\,ds}{2\int_a^b w(s)\,ds}\right],\ K \in \mathbb{C}.$$

We observe that if in (3.14) we replace f by $g - g(a)$ we get the inequality

$$\int_a^b |g(t) - g(a)|^2 w(t)\,dt \le \frac{4}{\pi^2}\left(\int_a^b w(s)\,ds\right)^2 \int_a^b \frac{|g'(t)|^2}{w(t)}dt \tag{3.15}$$

for $w : [a,b] \to (0,\infty)$ that is continuous on $[a,b]$ and $g \in C^1([a,b],\mathbb{C})$.

Remark 1. *If f is a function with complex values and $f(a) = 0$, then the inequality (3.14) can be stated on the infinite interval $[a,\infty)$ as follows*

$$\int_a^\infty |f(t)|^2 w(t)\,dt \le \frac{4}{\pi^2}\left(\int_a^\infty w(s)\,ds\right)^2 \int_a^\infty \frac{|f'(t)|^2}{w(t)}dt \tag{3.16}$$

provided $f \in C^1([a,\infty),\mathbb{C})$, $w : [a,\infty) \to (0,\infty)$ is continuous on $[a,\infty)$ and the integrals above exist. The equality holds iff

$$f(t) = K \sin\left[\frac{\pi \int_a^t w(s)\,ds}{2\int_a^\infty w(s)\,ds}\right],\ K \in \mathbb{C}.$$

In probability theory and statistics, the beta prime distribution (also known as inverted beta distribution or beta distribution of the second kind) is an absolutely continuous probability distribution defined for $x \ge 0$ with two parameters α and β, having the probability density function:

$$w_{\alpha,\beta}(x) := \frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha,\beta)},$$

where B is Beta function

$$B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1},\ \alpha,\ \beta > 1.$$

The cumulative distribution function is

$$W_{\alpha,\beta}(x) = I_{\frac{x}{1+x}}(\alpha,\beta),$$

where I is the regularized incomplete beta function defined by

$$I_z(\alpha,\beta) := \frac{B(z;\alpha,\beta)}{B(\alpha,\beta)}.$$

Here $B(\cdot;\alpha,\beta)$ is the incomplete beta function defined by

$$B(z;\alpha,\beta) := \int_0^z t^{\alpha-1}(1-t)^{\beta-1},\ \alpha,\ \beta,\ z>0.$$

Now, if we replace w by $w_{\alpha,\beta}$ in (3.16) we get

$$\int_0^\infty |f(t)|^2 t^{\alpha-1}(1+t)^{-\alpha-\beta}\,dt \leq \frac{4}{\pi^2}B^2(\alpha,\beta)\int_0^\infty |f'(t)|^2 t^{-\alpha+1}(1+t)^{\alpha+\beta}\,dt \tag{3.17}$$

for $\alpha,\ \beta>1$ provided $f\in C^1([0,\infty),\mathbb{C})$, $f(0)=0$ and the integrals above exist.

4. Some Weighted Inequalities of Trapezoid Type

We have:

Theorem 2. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ and $g\in C^1([a,b],\mathbb{C})$ is a function with complex values, then*

$$\begin{aligned}&\left|\frac{1}{\int_a^b w(t)\,dt}\int_a^b \frac{w(t)+w(a+b-t)}{2}g(t)\,dt-\frac{g(a)+g(b)}{2}\right|\\ &\quad\leq\frac{1}{2\pi}\left(\int_a^b w(s)\,ds\right)^{1/2}\left(\int_a^b\frac{|g'(t)-g'(a+b-t)|^2}{w(t)}dt\right)^{1/2}\\ &\quad\leq\frac{1}{2\pi}\max_{t\in[a,b]}|g'(t)-g'(a+b-t)|\left(\int_a^b w(s)\,ds\right)^{1/2}\left(\int_a^b\frac{1}{w(t)}dt\right)^{1/2}.\end{aligned}\tag{4.1}$$

In particular, if w is symmetrical, i.e. $w(a+b-t)=w(t)$ for any $t\in[a,b]$, then we have

$$\begin{aligned}&\left|\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\,g(t)\,dt-\frac{g(a)+g(b)}{2}\right|\\ &\quad\leq\frac{1}{2\pi}\left(\int_a^b w(s)\,ds\right)^{1/2}\left(\int_a^b\frac{|g'(t)-g'(a+b-t)|^2}{w(t)}dt\right)^{1/2}\\ &\quad\leq\frac{1}{2\pi}\max_{t\in[a,b]}|g'(t)-g'(a+b-t)|\left(\int_a^b w(s)\,ds\right)^{1/2}\left(\int_a^b\frac{1}{w(t)}dt\right)^{1/2}.\end{aligned}\tag{4.2}$$

Proof. Consider the function

$$f(t) := \frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2},\ t\in[a,b],$$

we have $f(a) = f(b) = 0$ and by (3.13) we have

$$\int_a^b \left| \frac{g(t) + g(a+b-t)}{2} - \frac{g(a) + g(b)}{2} \right|^2 w(t)\,dt \tag{4.3}$$
$$\leq \frac{1}{4\pi^2} \left(\int_a^b w(s)\,ds \right)^2 \int_a^b \frac{|g'(t) - g'(a+b-t)|^2}{w(t)} dt.$$

By the weighted Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$\int_a^b w(s)\,ds \int_a^b \left| \frac{g(t) + g(a+b-t)}{2} - \frac{g(a) + g(b)}{2} \right|^2 w(t)\,dt \tag{4.4}$$
$$\geq \left| \int_a^b \frac{g(t) + g(a+b-t)}{2} w(t)\,dt - \frac{g(a) + g(b)}{2} \int_a^b w(t)\,dt \right|^2.$$

Observe that, by the change of variable $s = a + b - t$, $t \in [a, b]$ we have that

$$\int_a^b g(a+b-t)\,w(t)\,dt = \int_a^b g(s)\,w(a+b-s)\,ds$$

and then

$$\int_a^b \frac{g(t) + g(a+b-t)}{2} w(t)\,dt = \int_a^b \frac{w(t) + w(a+b-t)}{2} g(t)\,dt.$$

By making use of (4.3) and (4.4) we get the first inequality in (4.1). The second inequality in (4.1) is obvious. □

In 1906, Fejér [5], while studying trigonometric polynomials, obtained the following inequalities which generalize that of Hermite & Hadamard:

Theorem 3 (Fejér's Inequality)**.** *Consider the integral $\int_a^b h(x)\,w(x)\,dx$, where h is a convex function in the interval (a, b) and w is a positive function in the same interval such that*

$$w(x) = w(a+b-x), \text{ for any } x \in [a, b]$$

i.e., $y = w(x)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b), 0\right)$ and is normal to the x-axis. Under those conditions the following inequalities are valid:

$$h\left(\frac{a+b}{2}\right) \leq \frac{1}{\int_a^b w(x)\,dx} \int_a^b h(x)\,w(x)\,dx \leq \frac{h(a) + h(b)}{2}. \tag{4.5}$$

If h is concave on (a, b), then the inequalities reverse in (4.5).

Remark 2. *If $g : [a, b] \to \mathbb{R}$ is differentiable convex and $g'_-(b)$ and $g'_+(a)$ are finite and $w : [a, b] \to (0, \infty)$ is continuous on $[a, b]$ and symmetrical, then by (4.2) we get the following reverse of the second inequality in (4.5)*

$$0 \leq \frac{g(a) + g(b)}{2} - \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t)\,g(t)\,dt \tag{4.6}$$
$$\leq \frac{1}{2\pi} \left[g'_-(b) - g'_+(a) \right] \left(\int_a^b w(s)\,ds \right)^{1/2} \left(\int_a^b \frac{1}{w(t)} dt \right)^{1/2},$$

provided $\int_a^b \frac{1}{w(t)} dt < \infty$.

Remark 3. *Assume that $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ and $g \in C^1([a,b],\mathbb{C})$ is a function with complex values and such that g' is K-Lipschitzian on $[a,b]$, i.e. $|g'(t) - g'(s)| \le K|t-s|$ for any $[a,b]$, then by (4.1) we get*

$$\left|\frac{1}{\int_a^b w(t)\,dt}\int_a^b \frac{w(t)+w(a+b-t)}{2} g(t)\,dt - \frac{g(a)+g(b)}{2}\right| \le \frac{1}{\pi}K\left(\int_a^b w(s)\,ds\right)^{1/2}\left(\int_a^b \frac{\left(t-\frac{a+b}{2}\right)^2}{w(t)}dt\right)^{1/2} \le \frac{1}{2\pi}K(b-a)\left(\int_a^b w(s)\,ds\right)^{1/2}\left(\int_a^b \frac{1}{w(t)}dt\right)^{1/2}, \tag{4.7}$$

provided $\int_a^b \frac{1}{w(t)}dt < \infty$.

If $g : [a,b] \to \mathbb{R}$ is twice differentiable and convex with $\|g''\|_{[a,b],\infty} < \infty$ and $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ and symmetrical, then by (4.7) we get the following reverse of the second inequality in (4.5)

$$0 \le \frac{g(a)+g(b)}{2} - \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\,g(t)\,dt \le \frac{1}{\pi}\|g''\|_{[a,b],\infty}\left(\int_a^b w(s)\,ds\right)^{1/2}\left(\int_a^b \frac{\left(t-\frac{a+b}{2}\right)^2}{w(t)}dt\right)^{1/2} \le \frac{1}{2\pi}\|g''\|_{[a,b],\infty}(b-a)\left(\int_a^b w(s)\,ds\right)^{1/2}\left(\int_a^b \frac{1}{w(t)}dt\right)^{1/2}, \tag{4.8}$$

provided $\int_a^b \frac{1}{w(t)}dt < \infty$.

Another trapezoid type weighted inequality is as follows:

Theorem 4. *Assume that $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ and $g \in C^1([a,b],\mathbb{C})$ is a function with complex values, then*

$$\left|\frac{g(a)[b-E(w;[a,b])] + g(b)[E(w;[a,b])-a]}{b-a} - \frac{1}{\int_a^b w(s)\,ds}\int_a^b g(t)\,w(t)\,dt\right| \le \frac{1}{\pi}\left(\int_a^b w(s)\,ds\right)^{1/2}\left(\int_a^b \left|g'(t) - \frac{g(b)-g(a)}{b-a}\right|^2 \frac{1}{w(t)}dt\right)^{1/2} \le \frac{1}{\pi}\max_{t\in[a,b]}\left\{\left|g'(t) - \frac{g(b)-g(a)}{b-a}\right|\right\}\left(\int_a^b w(s)\,ds\right)^{1/2}\left(\int_a^b \frac{1}{w(t)}dt\right)^{1/2}, \tag{4.9}$$

provided $\int_a^b \frac{1}{w(t)}dt < \infty$, where

$$E(w;[a,b]) := \frac{1}{\int_a^b w(s)\,ds}\int_a^b t w(t)\,dt.$$

Proof. If $g \in C^1([a,b],\mathbb{C})$, then by taking

$$f(t) := g(t) - \frac{g(a)(b-t) + g(b)(t-a)}{b-a}, \ t \in [a,b]$$

we have $f(a) = f(b) = 0$ and by (3.13) we have

$$\int_a^b \left| g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a} \right|^2 w(t)\,dt \leq \frac{1}{\pi^2} \left(\int_a^b w(s)\,ds \right)^2 \int_a^b \left| g'(t) - \frac{g(b)-g(a)}{b-a} \right|^2 \frac{1}{w(t)} dt. \tag{4.10}$$

By the weighted Cauchy-Bunyakovsky-Schwarz (CBS) integral inequality we have

$$\begin{aligned} &\int_a^b w(s)\,ds \int_a^b \left| g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a} \right|^2 w(t)\,dt \\ &\geq \left| \int_a^b g(t)\,w(t)\,dt - \int_a^b \frac{g(a)(b-t)+g(b)(t-a)}{b-a} w(t)\,dt \right|^2 \\ &= \left| \int_a^b g(t)\,w(t)\,dt - \frac{1}{b-a} g(a) \int_a^b (b-t)\,w(t)\,dt - g(b) \int_a^b (t-a)\,w(t)\,dt \right|^2 \\ &= \left| \int_a^b g(t)\,w(t)\,dt - \frac{1}{b-a} g(a) \int_a^b (b-t)\,w(t)\,dt - g(b) \int_a^b (t-a)\,w(t)\,dt \right|^2 \\ &= \left(\int_a^b w(s)\,ds \right)^2 \\ &\times \left| \frac{1}{\int_a^b w(s)\,ds} \int_a^b g(t)\,w(t)\,dt - \frac{g(a)\left[b - E(w;[a,b])\right] + g(b)\left[E(w;[a,b]) - a\right]}{b-a} \right|^2. \end{aligned} \tag{4.11}$$

By using (4.10) and (4.11) we get

$$\begin{aligned} &\left| \frac{1}{\int_a^b w(s)\,ds} \int_a^b g(t)\,w(t)\,dt \right. \\ &\quad \left. - \frac{g(a)\left[b - E(w;[a,b])\right] + g(b)\left[E(w;[a,b]) - a\right]}{b-a} \right|^2 \\ &\quad \leq \frac{1}{\pi^2} \left(\int_a^b w(s)\,ds \right) \int_a^b \left| g'(t) - \frac{g(b)-g(a)}{b-a} \right|^2 \frac{1}{w(t)} dt, \end{aligned} \tag{4.12}$$

which is equivalent to the first inequality in (4.9).

The second inequality in (4.9) is obvious. □

The case of convex function is as follows:

Corollary 2. *If $g : [a,b] \to \mathbb{R}$ is continuously differentiable convex and $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$, then*

$$
\begin{aligned}
0 &\leq \frac{g(a)\left[b - E(w;[a,b])\right] + g(b)\left[E(w;[a,b]) - a\right]}{b-a} \\
&\quad - \frac{1}{\int_a^b w(s)\,ds} \int_a^b g(t)\,w(t)\,dt \\
&\leq \frac{1}{\pi} \left(\int_a^b w(s)\,ds \right)^{1/2} \left(\int_a^b \left| g'(t) - \frac{g(b)-g(a)}{b-a} \right|^2 \frac{1}{w(t)} dt \right)^{1/2} \\
&\leq \frac{1}{\pi} \max_{t\in[a,b]} \left\{ \left| g'(t) - \frac{g(b)-g(a)}{b-a} \right| \right\} \left(\int_a^b w(s)\,ds \right)^{1/2} \left(\int_a^b \frac{1}{w(t)} dt \right)^{1/2},
\end{aligned} \tag{4.13}
$$

provided $\int_a^b \frac{1}{w(t)} dt < \infty$.

The positivity follows by the fact that, for a convex function g on $[a,b]$ we have

$$
\frac{g(a)(b-t) + g(b)(t-a)}{b-a} \geq g(t)
$$

for any $t \in [a,b]$. The rest is obvious by Theorem 4.

5. Some Inequalities for the Weighted Čebyšev Functional

Consider now the *weighted Čebyšev functional*

$$
\begin{aligned}
C_w(f,g) &:= \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t)\,f(t)\,g(t)\,dt \\
&\quad - \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t)\,f(t)\,dt \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t)\,g(t)\,dt
\end{aligned} \tag{5.1}
$$

where f, g, $w : [a,b] \to \mathbb{R}$ and $w(t) \geq 0$ for a.e. $t \in [a,b]$ are measurable functions such that the involved integrals exist and $\int_a^b w(t)\,dt > 0$.

In [3], Cerone and Dragomir obtained, among others, the following inequalities:

$$
\begin{aligned}
&|C_w(f,g)| \\
&\leq \frac{1}{2}(M-m) \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) \left| g(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(s)\,g(s)\,ds \right| dt \\
&\leq \frac{1}{2}(M-m) \left[\frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) \left| g(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(s)\,g(s)\,ds \right|^p dt \right]^{\frac{1}{p}} \\
&\leq \frac{1}{2}(M-m) \operatorname*{essup}_{t\in[a,b]} \left| g(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(s)\,g(s)\,ds \right|
\end{aligned} \tag{5.2}
$$

for $p > 1$, provided $-\infty < m \leq f(t) \leq M < \infty$ for a.e. $t \in [a,b]$ and the corresponding integrals are finite. The constant $\frac{1}{2}$ is sharp in all the inequalities in (5.2) in the sense that it cannot be replaced by a smaller constant.

In addition, if $-\infty < n \leq g(t) \leq N < \infty$ for a.e. $t \in [a,b]$, then the following refinement of the celebrated Grüss inequality is obtained:

$$
\begin{aligned}
(5.3)\quad & |C_w(f,g)| \\
&\leq \frac{1}{2}(M-m)\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)g(s)\,ds\right|dt \\
&\leq \frac{1}{2}(M-m)\left[\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)g(s)\,ds\right|^2 dt\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}(M-m)(N-n).
\end{aligned}
$$

Here, the constants $\frac{1}{2}$ and $\frac{1}{4}$ are also sharp in the sense mentioned above.

Theorem 5. *Assume that* $w:[a,b]\to(0,\infty)$ *is continuous on* $[a,b]$, $f\in L_2([a,b],\mathbb{C})$ *and* $g\in C^1([a,b],\mathbb{C})$ *is a function with complex values, then*

$$
\begin{aligned}
(5.4)\quad |C_w(f,g)| &\leq \frac{b-a}{\pi}\left(\int_a^b |g'(x)|^2\,dx\right)^{1/2} \\
&\times \frac{1}{\int_a^b w(s)\,ds}\left(\int_a^b \left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)w(s)\,ds\right|^2 w^2(t)\,dt\right)^{1/2}.
\end{aligned}
$$

Proof. Integrating by parts, we have

$$
\begin{aligned}
&\frac{1}{\int_a^b w(s)\,ds}\int_a^b\left(\int_a^x f(t)w(t)\,dt-\frac{\int_a^x w(s)\,ds}{\int_a^b w(s)\,ds}\int_a^b f(s)w(s)\,ds\right)g'(x)\,dx \\
&=\frac{1}{\int_a^b w(s)\,ds}\left[\left(\int_a^x f(t)w(t)\,dt-\frac{\int_a^x w(s)\,ds}{\int_a^b w(s)\,ds}\int_a^b f(s)w(s)\,ds\right)g(x)\Big|_a^b\right. \\
&\left.-\int_a^b g(x)\left(f(x)w(x)-\frac{w(x)}{\int_a^b w(s)\,ds}\int_a^b f(s)w(s)\,ds\right)dx\right] \\
&=-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(x)g(x)w(x)\,dx \\
&+\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)w(s)\,ds\frac{1}{\int_a^b w(s)\,ds}\int_a^b g(x)w(x)\,dx,
\end{aligned}
$$

which gives that

$$
\begin{aligned}
(5.5)\quad C_w(f,g) &= \frac{1}{\int_a^b w(s)\,ds} \\
&\times\int_a^b\left(\frac{\int_a^x w(s)\,ds}{\int_a^b w(s)\,ds}\int_a^b f(s)w(s)\,ds-\int_a^x f(t)w(t)\,dt\right)g'(x)\,dx.
\end{aligned}
$$

Using (CBS) integral inequality we have

$$\begin{aligned}(5.6)\quad & \left|C_w(f,g)\right|^2 \\ &= \frac{1}{\left(\int_a^b w(s)\,ds\right)^2}\left|\int_a^b\left(\frac{\int_a^x w(s)\,ds}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds-\int_a^x f(t)\,w(t)\,dt\right)g\prime(x)\,dx\right|^2 \\ &\le \frac{1}{\left(\int_a^b w(s)\,ds\right)^2}\int_a^b\left|\frac{\int_a^x w(s)\,ds}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds-\int_a^x f(t)\,w(t)\,dt\right|^2\int_a^b\left|g\prime(x)\right|^2dx.\end{aligned}$$

If we take

$$h(x):=\frac{\int_a^x w(s)\,ds}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds-\int_a^x f(t)\,w(t)\,dt,\ \ x\in[a,b]$$

we observe that $h(a)=h(b)=0$ and $h\in C^1([a,b],\mathbb{C})$.

Then by (1.3) we get

$$\begin{aligned}(5.7)\quad & \int_a^b\left|\frac{\int_a^x w(s)\,ds}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds-\int_a^x f(t)\,w(t)\,dt\right|^2dx \\ &\le \frac{(b-a)^2}{\pi^2}\int_a^b\left|f(t)\,w(t)-\frac{w(t)}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^2dt \\ &= \frac{(b-a)^2}{\pi^2}\int_a^b\left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^2w^2(t)\,dt.\end{aligned}$$

On making use of (5.6) we get

$$\begin{aligned}\left|C_w(f,g)\right|^2 &\le \frac{(b-a)^2}{\pi^2}\int_a^b\left|g\prime(x)\right|^2dx \\ &\quad\times\frac{1}{\left(\int_a^b w(s)\,ds\right)^2}\int_a^b\left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^2w^2(t)\,dt,\end{aligned}$$

which is equivalent to (5.4). □

Remark 4. *If we take $w\equiv 1$ in (5.4), then we get the unweighted Grüss' type inequality*

$$\begin{aligned}(5.8)\quad & \left|C(f,g)\right| \\ &\le \frac{\sqrt{b-a}}{\pi}\left(\int_a^b\left|g\prime(x)\right|^2dx\right)^{1/2}\left(\frac{1}{b-a}\int_a^b\left|f(t)\right|^2dt-\left|\frac{1}{b-a}\int_a^b f(s)\,ds\right|^2\right)^{1/2}.\end{aligned}$$

The following result also holds:

Theorem 6. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$, $f\in L_2([a,b],\mathbb{C})$ and $g\in C^1([a,b],\mathbb{C})$ is a function with complex values and such that $\frac{|g'|^2}{w}\in$*

$L([a,b],\mathbb{R})$, *then*

$$|C_w(f,g)| \leq \frac{2}{\pi}\left(\int_a^b w(s)\,ds\right)^{1/2}\left(\int_a^b \frac{|g'(t)|^2}{w(t)}dt\right)^{1/2} \times\left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b |f(t)|^2 w(t)\,dt - \left|\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^2\right)^{1/2}. \tag{5.9}$$

Proof. We use the following Sonin type identity

$$C_w(f,g) = \frac{1}{\int_a^b w(s)\,ds}\int_a^b \left(f(t) - \frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right)(g(t)-g(a))\,w(t)\,dt, \tag{5.10}$$

which can be proved directly on calculating the integral from the right hand side.

By using the weighted (CBS) integral inequality, we have

$$\begin{aligned} |C_w(f,g)| &\leq \frac{1}{\int_a^b w(s)\,ds}\int_a^b \left|f(t) - \frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right| |g(t)-g(a)|\,w(t)\,dt \\ &\leq \frac{1}{\int_a^b w(s)\,ds}\left(\int_a^b \left|f(t) - \frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^2 w(t)\,dt\right)^{1/2} \\ &\qquad\times\left(\int_a^b |g(t)-g(a)|^2 w(t)\,dt\right)^{1/2}. \end{aligned} \tag{5.11}$$

Using (3.15) we have

$$\left(\int_a^b |g(t)-g(a)|^2 w(t)\,dt\right)^{1/2} \leq \frac{2}{\pi}\left(\int_a^b w(s)\,ds\right)\left(\int_a^b \frac{|g'(t)|^2}{w(t)}dt\right)^{1/2}. \tag{5.12}$$

By making use of (5.11) and (5.12) we get

$$\begin{aligned} |C_w(f,g)| &\leq \frac{1}{\int_a^b w(s)\,ds}\left(\int_a^b \left|f(t) - \frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^2 w(t)\,dt\right)^{1/2} \\ &\qquad\times\frac{2}{\pi}\left(\int_a^b w(s)\,ds\right)\left(\int_a^b \frac{|g'(t)|^2}{w(t)}dt\right)^{1/2} \end{aligned} \tag{5.13}$$

$$
\begin{aligned}
&= \frac{2}{\pi}\left(\int_a^b w(s)\,ds\right)^{1/2} \\
&\quad\times\left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b \left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^2 w(t)\,dt\right)^{1/2} \\
&\qquad\qquad\times\left(\int_a^b \frac{|g'(t)|^2}{w(t)}dt\right)^{1/2}
\end{aligned}
$$

and since

$$
\begin{aligned}
&\frac{1}{\int_a^b w(s)\,ds}\int_a^b \left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^2 w(t)\,dt \\
&= \frac{1}{\int_a^b w(s)\,ds}\int_a^b |f(t)|^2\,w(t)\,dt - \left|\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^2,
\end{aligned}
$$

hence by (5.13) we get the desired result (5.9). □

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