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WEIGHTED INTEGRAL INEQUALITIES RELATED TO WIRTINGER'S RESULT FOR p-NORMS WITH APPLICATIONS

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Abstract. In this paper we establish several natural consequences of some Wirtinger type integral inequalities for p-norms. The corresponding weighted versions and applications related to the weighted trapezoid inequalities, to weighted Grüss' type inequalities and reverses of Jensen's inequality are also provided.

1. Introduction

The following Wirtinger type inequalities are well known

$$\int_a^b u^2(t)dt \leq \frac{(b-a)^2}{\pi^2}\int_a^b \left[u'(t)\right]^2 dt \tag{1.1}$$

provided $u \in C^1([a,b],\mathbb{R})$ and $u(a) = u(b) = 0$ with equality holding if and only if $u(t) = K\sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant K, and, similarly, if $u \in C^1([a,b],\mathbb{R})$ satisfies $u(a) = 0$, then

$$\int_a^b u^2(t)dt \leq \frac{4(b-a)^2}{\pi^2}\int_a^b \left[u'(t)\right]^2 dt. \tag{1.2}$$

The equality holds in (1.2) if and only if $u(t) = K\sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant K.

For $p > 1$ put $\pi_{p-1} := \frac{2\pi}{p}\sin\left(\frac{\pi}{p}\right)$. In [11], J. Jaroš obtained, as a particular case of a more general inequality, the following generalization of (1.1)

$$\int_a^b |u(t)|^p\, dt \leq \frac{(b-a)^p}{(p-1)\,\pi_{p-1}^p}\int_a^b |u'(t)|^p\, dt \tag{1.3}$$

provided $u \in C^1([a,b],\mathbb{R})$ and $u(a) = u(b) = 0$, with equality if and only if $u(t) = K\sin_{p-1}\left[\frac{\pi_{p-1}(t-a)}{b-a}\right]$ for some $K \in \mathbb{R}$, where $\sin_{p-1}$ is the $2\pi_{p-1}$-periodic generalized sine function, see [18] or [5].

If $u(a) = 0$ and $u \in C^1([a,b],\mathbb{R})$, then

$$\int_a^b |u(t)|^p\, dt \leq \frac{[2(b-a)]^p}{(p-1)\,\pi_{p-1}^p}\int_a^b |u'(t)|^p\, dt \tag{1.4}$$

with equality iff $u(t) = K\sin_{p-1}\left[\frac{\pi_{p-1}(t-a)}{2(b-a)}\right]$ for some $K \in \mathbb{R}$.

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The inequalities (1.3) and (1.4) were obtained for $a=0$, $b=1$ and $q=p>1$ in [17] by the use of an elementary proof.

For some related Wirtinger type integral inequalities see [1], [2], [4], [9], [12], [11] and [15]-[17].

In this paper we establish several natural consequences of the Wirtinger type integral inequalities for p-norms from (1.3) and in (1.4). The corresponding weighted versions and applications related to the weighted trapezoid inequalities, to weighted Grüss' type inequalities and reverses of Jensen's inequality are also provided.

2. Some Weighted Inequalities

We have:

Theorem 1. *Let $g:[a,b]\to[g(a),g(b)]$ be a continuous strictly increasing function that is of class C^1 on (a,b) and $p>1$.*

(i) *If $f\in C^1([a,b],\mathbb{R})$ is such $f(a)=f(b)=0$, then*

$$\int_a^b |f(t)|^p g'(t)\,dt \le \frac{[g(b)-g(a)]^p}{(p-1)\,\pi_{p-1}^p}\int_a^b \frac{|f'(t)|^p}{|g'(t)|^{p-1}}dt. \tag{2.1}$$

The equality holds in (2.1) iff

$$f(t)=K\sin_{p-1}\left[\frac{\pi_{p-1}(g(t)-g(a))}{g(b)-g(a)}\right],\ K\in\mathbb{R}.$$

(ii) *If $f\in C^1([a,b],\mathbb{R})$ is such that $f(a)=0$, then*

$$\int_a^b |f(t)|^p g'(t)\,dt \le \frac{2^p[g(b)-g(a)]^p}{(p-1)\,\pi_{p-1}^p}\int_a^b \frac{|f'(t)|^p}{|g'(t)|^{p-1}}dt. \tag{2.2}$$

The equality holds in (2.2) iff

$$f(t)=K\sin_{p-1}\left[\frac{\pi_{p-1}(g(t)-g(a))}{2(g(b)-g(a))}\right],\ K\in\mathbb{R}.$$

Proof. (i) We write the inequality (1.3) for the function $u=f\circ g^{-1}$ on the interval $[g(a),g(b)]$ to get

$$\int_{g(a)}^{g(b)} \left|f\circ g^{-1}(z)\right|^p dz \le \frac{(g(b)-g(a))^p}{(p-1)\,\pi_{p-1}^p}\int_{g(a)}^{g(b)} \left|\left(f\circ g^{-1}\right)'(z)\right|^p dz. \tag{2.3}$$

If $f:[c,d]\to\mathbb{R}$ is absolutely continuous on $[c,d]$, then $f\circ g^{-1}:[g(c),g(d)]\to\mathbb{C}$ is absolutely continuous on $[g(c),g(d)]$ and using the chain rule and the derivative of inverse functions we have

$$\left(f\circ g^{-1}\right)'(z)=\left(f'\circ g^{-1}\right)(z)\left(g^{-1}\right)'(z)=\frac{\left(f'\circ g^{-1}\right)(z)}{\left(g'\circ g^{-1}\right)(z)} \tag{2.4}$$

for almost every (a.e.) $z\in[g(c),g(d)]$.

Using the inequality (2.3) we then get

$$\int_{g(a)}^{g(b)} \left|f\circ g^{-1}(z)\right|^p dz \le \frac{(g(b)-g(a))^p}{(p-1)\,\pi_{p-1}^p}\int_{g(a)}^{g(b)} \left|\frac{\left(f'\circ g^{-1}\right)(z)}{\left(g'\circ g^{-1}\right)(z)}\right|^p dz$$

provided $\left(f\circ g^{-1}\right)(g(a))=f(a)=0$ and $\left(f\circ g^{-1}\right)(g(b))=f(b)=0$.

Observe also that, by the change of variable $t = g^{-1}(z)$, $z \in [g(a), g(b)]$, we have $z = g(t)$ that gives $dz = g'(t)\,dt$ and

$$\int_{g(a)}^{g(b)} \left| f \circ g^{-1}(z) \right|^p dz = \int_a^b |f(t)|^p g'(t)\,dt.$$

We also have

$$\int_{g(a)}^{g(b)} \left| \frac{\left(f' \circ g^{-1}\right)(z)}{\left(g' \circ g^{-1}\right)(z)} \right|^p dz = \int_a^b \left| \frac{f'(t)}{g'(t)} \right|^p g'(t)\,dt = \int_a^b \frac{|f'(t)|^p}{|g'(t)|^{p-1}} dt. \tag{2.5}$$

By making use of (2.5) we get (2.1).

The case of equality follows by the case of equality in (1.3).

(ii) Follows in a similar way by (1.4). □

Remark 1. *For $p = 2$ we get from Theorem 1 the weighted integral inequalities from* [7].

Assume that $p > 1$. Some examples are as follows:

a). If we take $g : [a, b] \subset (0, \infty) \to \mathbb{R}$, $g(t) = \ln t$ and assume that $f \in C^1([a, b], \mathbb{R})$ is such that $f(a) = f(b) = 0$, then by (2.1) we get

$$\int_a^b \frac{|f(t)|^p}{t} dt \leq \frac{\left[\ln\left(\frac{b}{a}\right)\right]^p}{(p-1)\,\pi_{p-1}^p} \int_a^b |f'(t)|^p\, t^{p-1} dt. \tag{2.6}$$

The equality holds in (2.6) iff

$$f(t) = K \sin_{p-1}\left[\frac{\pi_{p-1} \ln\left(\frac{t}{a}\right)}{\ln\left(\frac{b}{a}\right)}\right],\ K \in \mathbb{R}.$$

b). If we take $g : [a, b] \subset \mathbb{R} \to (0, \infty)$, $g(t) = \exp t$ and assume that $f \in C^1([a, b], \mathbb{C})$ is a function with complex values and $f(a) = f(b) = 0$, then by (2.1) we get

$$\int_a^b |f(t)|^p \exp(t)\,dt \leq \frac{(\exp b - \exp a)^p}{(p-1)\,\pi_{p-1}^p} \int_a^b |f'(t)|^p \exp((1-p)\,t)\,dt. \tag{2.7}$$

The equality holds in (2.6) iff

$$f(t) = K \sin_{p-1}\left[\frac{\pi_{p-1}(\exp t - \exp a)}{\exp b - \exp a}\right],\ K \in \mathbb{R}.$$

c). If we take $g : [a, b] \subset (0, \infty) \to \mathbb{R}$, $g(t) = t^r$, $r > 0$ and assume that $f \in C^1([a, b], \mathbb{R})$ is such that $f(a) = f(b) = 0$, then by (2.1) we get

$$\int_a^b |f(t)|^p\, t^{r-1} dt \leq \frac{(b^r - a^r)^p}{r^p\,(p-1)\,\pi_{p-1}^p} \int_a^b \frac{|f'(t)|^p}{t^{(r-1)(p-1)}} dt. \tag{2.8}$$

The equality holds in (2.8) iff

$$f(t) = K \sin_{p-1}\left[\frac{\pi_{p-1}(t^r - a^r)}{b^r - a^r}\right],\ K \in \mathbb{R}.$$

If $w : [a, b] \to \mathbb{R}$ is continuous and positive on the interval $[a, b]$, then the function $W : [a, b] \to [0, \infty)$, $W(x) := \int_a^x w(s)\,ds$ is strictly increasing and differentiable on (a, b). We have $W'(x) = w(x)$ for any $x \in (a, b)$.

Corollary 1. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ and $f\in C^1([a,b],\mathbb{C})$ is a function with complex values and $f(a)=f(b)=0$, then*

$$\int_a^b |f(t)|^p w(t)\,dt \le \frac{1}{(p-1)\,\pi_{p-1}^p}\left(\int_a^b w(s)\,ds\right)^p \int_a^b \frac{|f'(t)|^p}{w^{p-1}(t)}dt. \tag{2.9}$$

The equality holds in (2.9) iff

$$f(t)=K\sin_{p-1}\left[\frac{\pi\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds}\right],\ K\in\mathbb{C}.$$

If $f(a)=0$, then

$$\int_a^b |f(t)|^p w(t)\,dt \le \frac{2^p}{(p-1)\,\pi_{p-1}^p}\left(\int_a^b w(s)\,ds\right)^p \int_a^b \frac{|f'(t)|^p}{w^{p-1}(t)}dt \tag{2.10}$$

with equality iff

$$f(t)=K\sin_{p-1}\left[\frac{\pi\int_a^t w(s)\,ds}{2\int_a^b w(s)\,ds}\right],\ K\in\mathbb{C}.$$

Remark 2. *If f is a function such that $f(a)=0$, then the inequality (2.10) can be stated on the infinite interval $[a,\infty)$ as follows*

$$\int_a^\infty |f(t)|^p w(t)\,dt \le \frac{2^p}{(p-1)\,\pi_{p-1}^p}\left(\int_a^\infty w(s)\,ds\right)^p \int_a^\infty \frac{|f'(t)|^p}{w^{p-1}(t)}dt \tag{2.11}$$

provided $f\in C^1([a,\infty),\mathbb{C})$, $w:[a,\infty)\to(0,\infty)$ is continuous on $[a,\infty)$ and the integrals above exist. The equality holds iff

$$f(t)=K\sin_{p-1}\left[\frac{\pi\int_a^t w(s)\,ds}{2\int_a^\infty w(s)\,ds}\right],\ K\in\mathbb{C}.$$

3. Some Weighted Inequalities of Trapezoid Type

We have:

Theorem 2. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ and $g\in C^1([a,b],\mathbb{R})$, then for $p,\,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,*

$$\begin{aligned}\left|\frac{1}{\int_a^b w(t)\,dt}\int_a^b \frac{w(t)+w(a+b-t)}{2}g(t)\,dt-\frac{g(a)+g(b)}{2}\right| &\le \frac{1}{2\,(p-1)^{1/p}\,\pi_{p-1}}\left(\int_a^b w(s)\,ds\right)^{1/q}\left(\int_a^b \frac{|g'(t)-g'(a+b-t)|^p}{w^{p-1}(t)}dt\right)^{1/p}\\ &\le \frac{1}{2\,(p-1)^{1/p}\,\pi_{p-1}}\max_{t\in[a,b]}|g'(t)-g'(a+b-t)|\\ &\quad\times\left(\int_a^b w(s)\,ds\right)^{1/q}\left(\int_a^b \frac{1}{w^{p-1}(t)}dt\right)^{1/p}.\end{aligned} \tag{3.1}$$

In particular, if w is symmetrical, i.e. $w(a+b-t)=w(t)$ for any $t\in[a,b]$, then we have

$$\begin{aligned}(3.2)\quad &\left|\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\,g(t)\,dt-\frac{g(a)+g(b)}{2}\right| \\ &\leq \frac{1}{2(p-1)^{1/p}\,\pi_{p-1}}\left(\int_a^b w(s)\,ds\right)^{1/q}\left(\int_a^b \frac{|g'(t)-g'(a+b-t)|^p}{w^{p-1}(t)}dt\right)^{1/p} \\ &\leq \frac{1}{2(p-1)^{1/p}\,\pi_{p-1}}\max_{t\in[a,b]}|g'(t)-g'(a+b-t)| \\ &\qquad\times\left(\int_a^b w(s)\,ds\right)^{1/q}\left(\int_a^b \frac{1}{w^{p-1}(t)}dt\right)^{1/p}.\end{aligned}$$

Proof. Consider the function

$$f(t):=\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2},\ t\in[a,b],$$

we have $f(a)=f(b)=0$ and by (2.9) we have

$$\begin{aligned}&\int_a^b\left|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right|^p w(t)\,dt \\ &\qquad\leq\frac{1}{2^p(p-1)\,\pi_{p-1}^p}\left(\int_a^b w(s)\,ds\right)^p\int_a^b\frac{|g'(t)-g'(a+b-t)|^p}{w^{p-1}(t)}dt,\end{aligned}$$

namely

$$\begin{aligned}(3.3)\quad &\left(\int_a^b\left|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right|^p w(t)\,dt\right)^{1/p} \\ &\leq\frac{1}{2(p-1)^{1/p}\,\pi_{p-1}}\left(\int_a^b w(s)\,ds\right)\left(\int_a^b\frac{|g'(t)-g'(a+b-t)|^p}{w^{p-1}(t)}dt\right)^{1/p}.\end{aligned}$$

By the weighted Hölder's integral inequality for $p,\,q>1$, $\frac{1}{p}+\frac{1}{q}=1$ we have

$$\begin{aligned}(3.4)\quad &\left(\int_a^b\left|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right|^p w(t)\,dt\right)^{1/p}\left(\int_a^b w(s)\,ds\right)^{1/q} \\ &\geq\left|\int_a^b\left[\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right]w(t)\,dt\right| \\ &=\left|\int_a^b\frac{g(t)+g(a+b-t)}{2}w(t)\,dt-\frac{g(a)+g(b)}{2}\int_a^b w(t)\,dt\right|.\end{aligned}$$

By making use of (3.3) and (3.4) we get

$$\left|\int_a^b \frac{g(t)+g(a+b-t)}{2}w(t)\,dt - \frac{g(a)+g(b)}{2}\int_a^b w(t)\,dt\right| \tag{3.5}$$
$$\leq \frac{1}{2(p-1)^{1/p}\pi_{p-1}}\left(\int_a^b w(s)\,ds\right)^{1+1/q}\left(\int_a^b \frac{|g'(t)-g'(a+b-t)|^p}{w^{p-1}(t)}dt\right)^{1/p}.$$

Observe that, by the change of variable $s = a+b-t$, $t\in[a,b]$ we have that

$$\int_a^b g(a+b-t)\,w(t)\,dt = \int_a^b g(s)\,w(a+b-s)\,ds$$

and then

$$\int_a^b \frac{g(t)+g(a+b-t)}{2}w(t)\,dt = \int_a^b \frac{w(t)+w(a+b-t)}{2}g(t)\,dt.$$

Utilising (3.5) we get the first part of (3.1). The second inequality in (3.1) is obvious. □

In 1906, Fejér [8], while studying trigonometric polynomials, obtained the following inequalities which generalize that of Hermite & Hadamard:

Theorem 3 (Fejér's Inequality)**.** *Consider the integral $\int_a^b h(x)\,w(x)\,dx$, where h is a convex function in the interval (a,b) and w is a positive function in the same interval such that*

$$w(x) = w(a+b-x),\ \text{for any } x\in[a,b]$$

i.e., $y = w(x)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b),0\right)$ and is normal to the x-axis. Under those conditions the following inequalities are valid:

$$h\left(\frac{a+b}{2}\right) \leq \frac{1}{\int_a^b w(x)\,dx}\int_a^b h(x)\,w(x)\,dx \leq \frac{h(a)+h(b)}{2}. \tag{3.6}$$

If h is concave on (a,b), then the inequalities reverse in (3.6).

Remark 3. *If $g:[a,b]\to\mathbb{R}$ is differentiable convex and $g'_-(b)$ and $g'_+(a)$ are finite and $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ and symmetrical, then by (3.2) we get the following reverse of the second inequality in (3.6)*

$$0 \leq \frac{g(a)+g(b)}{2} - \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\,g(t)\,dt \tag{3.7}$$
$$\leq \frac{1}{2(p-1)^{1/p}\pi_{p-1}}\left[g'_-(b)-g'_+(a)\right]\left(\int_a^b w(s)\,ds\right)^{1/q}\left(\int_a^b \frac{1}{w^{p-1}(t)}dt\right)^{1/p},$$

provided $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $\int_a^b \frac{1}{w^{p-1}(t)}dt<\infty$.

Remark 4. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ and $g\in C^1([a,b],\mathbb{C})$ is a function with complex values and such that g' is K-Lipschitzian*

on $[a,b]$, *i.e.* $|g'(t)-g'(s)|\le K|t-s|$ *for any* $[a,b]$, *then by (3.1) we get*

$$\begin{aligned}(3.8)\quad &\left|\frac{1}{\int_a^b w(t)\,dt}\int_a^b \frac{w(t)+w(a+b-t)}{2}g(t)\,dt-\frac{g(a)+g(b)}{2}\right|\\ &\le \frac{1}{(p-1)^{1/p}\,\pi_{p-1}}K\left(\int_a^b w(s)\,ds\right)^{1/q}\left(\int_a^b \frac{\left|t-\frac{a+b}{2}\right|^p}{w^{p-1}(t)}dt\right)^{1/p}\\ &\le \frac{b-a}{2(p-1)^{1/p}\,\pi_{p-1}}K\left(\int_a^b w(s)\,ds\right)^{1/q}\left(\int_a^b \frac{1}{w^{p-1}(t)}dt\right)^{1/p}\end{aligned}$$

provided $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $\int_a^b \frac{1}{w^{p-1}(t)}dt<\infty$.

If $g:[a,b]\to\mathbb{R}$ *is twice differentiable and convex with* $\|g''\|_{[a,b],\infty}<\infty$ *and* $w:[a,b]\to(0,\infty)$ *is continuous on* $[a,b]$ *and symmetrical, then by (3.8) we get the following reverse of the second inequality in (3.6)*

$$\begin{aligned}(3.9)\quad &0\le \frac{g(a)+g(b)}{2}-\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\,g(t)\,dt\\ &\le \frac{1}{(p-1)^{1/p}\,\pi_{p-1}}\|g''\|_{[a,b],\infty}\left(\int_a^b w(s)\,ds\right)^{1/q}\left(\int_a^b \frac{\left|t-\frac{a+b}{2}\right|^p}{w^{p-1}(t)}dt\right)^{1/p}\\ &\le \frac{b-a}{2(p-1)^{1/p}\,\pi_{p-1}}\|g''\|_{[a,b],\infty}\left(\int_a^b w(s)\,ds\right)^{1/q}\left(\int_a^b \frac{1}{w^{p-1}(t)}dt\right)^{1/p}\end{aligned}$$

provided $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $\int_a^b \frac{1}{w^{p-1}(t)}dt<\infty$.

Another trapezoid type weighted inequality is as follows:

Theorem 4. *Assume that* $w:[a,b]\to(0,\infty)$ *is continuous on* $[a,b]$ *and* $g\in C^1([a,b],\mathbb{R})$ *then*

$$\begin{aligned}(3.10)\quad &\left|\frac{g(a)\,[b-E(w;[a,b])]+g(b)\,[E(w;[a,b])-a]}{b-a}\right.\\ &\qquad\left.-\frac{1}{\int_a^b w(s)\,ds}\int_a^b g(t)\,w(t)\,dt\right|\\ &\le \frac{1}{(p-1)^{1/p}\,\pi_{p-1}}\left(\int_a^b w(s)\,ds\right)^{1/q}\left(\int_a^b \frac{\left|g'(t)-\frac{g(b)-g(a)}{b-a}\right|^p}{w^{p-1}(t)}dt\right)^{1/p}\\ &\le \frac{1}{\pi}\max_{t\in[a,b]}\left\{\left|g'(t)-\frac{g(b)-g(a)}{b-a}\right|\right\}\left(\int_a^b w(s)\,ds\right)^{1/q}\left(\int_a^b \frac{1}{w^{p-1}(t)}dt\right)^{1/p},\end{aligned}$$

provided $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $\int_a^b \frac{1}{w^{p-1}(t)}dt<\infty$, *where*

$$E(w;[a,b]):=\frac{1}{\int_a^b w(s)\,ds}\int_a^b tw(t)\,dt.$$

Proof. If $g \in C^1([a,b],\mathbb{R})$, then by taking

$$f(t) := g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a},\ t \in [a,b]$$

we have $f(a) = f(b) = 0$ and by (2.9) we have

$$\int_a^b \left| g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a} \right|^p w(t)\,dt$$
$$\leq \frac{1}{(p-1)\,\pi_{p-1}^p} \left(\int_a^b w(s)\,ds \right)^p \int_a^b \frac{\left| g'(t) - \frac{g(b)-g(a)}{b-a} \right|^p}{w^{p-1}(t)} dt,$$

namely

$$\left(\int_a^b \left| g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a} \right|^p w(t)\,dt \right)^{1/p} \tag{3.11}$$
$$\leq \frac{1}{(p-1)^{1/p}\,\pi_{p-1}} \left(\int_a^b w(s)\,ds \right) \left(\int_a^b \frac{\left| g'(t) - \frac{g(b)-g(a)}{b-a} \right|^p}{w^{p-1}(t)} dt \right)^{1/p}.$$

By the weighted Hölder's integral inequality for $p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1$ we have

$$\left(\int_a^b \left| g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a} \right|^p w(t)\,dt \right)^{1/p} \left(\int_a^b w(s)\,ds \right)^{1/q} \tag{3.12}$$
$$\geq \left| \int_a^b g(t)\,w(t)\,dt - \int_a^b \frac{g(a)(b-t)+g(b)(t-a)}{b-a} w(t)\,dt \right|$$
$$= \left| \int_a^b g(t)\,w(t)\,dt - \frac{1}{b-a} g(a) \int_a^b (b-t)\,w(t)\,dt - g(b) \int_a^b (t-a)\,w(t)\,dt \right|$$
$$= \left| \int_a^b g(t)\,w(t)\,dt - \frac{1}{b-a} g(a) \int_a^b (b-t)\,w(t)\,dt - g(b) \int_a^b (t-a)\,w(t)\,dt \right|$$
$$= \left(\int_a^b w(s)\,ds \right)$$
$$\times \left| \frac{1}{\int_a^b w(s)\,ds} \int_a^b g(t)\,w(t)\,dt - \frac{g(a)\,[b - E(w;[a,b])] + g(b)\,[E(w;[a,b]) - a]}{b-a} \right|.$$

By making use of (3.11) and (3.12) we have

$$\left(\int_a^b w(s)\,ds \right) \left| \frac{1}{\int_a^b w(s)\,ds} \int_a^b g(t)\,w(t)\,dt \right.$$
$$\left. - \frac{g(a)\,[b - E(w;[a,b])] + g(b)\,[E(w;[a,b]) - a]}{b-a} \right|$$
$$\leq \frac{1}{(p-1)^{1/p}\,\pi_{p-1}} \left(\int_a^b w(s)\,ds \right)^{1+1/q} \left(\int_a^b \frac{\left| g'(t) - \frac{g(b)-g(a)}{b-a} \right|^p}{w^{p-1}(t)} dt \right)^{1/p},$$

which is equivalent to the first inequality in (3.10).

The second inequality in (3.10) is obvious. □

The case of convex function is as follows:

Corollary 2. *If $g:[a,b]\to\mathbb{R}$ is continuously differentiable convex and $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$, then*

$$
\begin{aligned}
(3.13)\quad 0 \le{}& \frac{g(a)\left[b-E(w;[a,b])\right]+g(b)\left[E(w;[a,b])-a\right]}{b-a} \\
&\qquad -\frac{1}{\int_a^b w(s)\,ds}\int_a^b g(t)\,w(t)\,dt\Bigg| \\
\le{}& \frac{1}{(p-1)^{1/p}\,\pi_{p-1}}\left(\int_a^b w(s)\,ds\right)^{1/q}\left(\int_a^b \frac{\left|g'(t)-\frac{g(b)-g(a)}{b-a}\right|^p}{w^{p-1}(t)}dt\right)^{1/p} \\
\le{}& \frac{1}{\pi}\max_{t\in[a,b]}\left\{\left|g'(t)-\frac{g(b)-g(a)}{b-a}\right|\right\}\left(\int_a^b w(s)\,ds\right)^{1/q}\left(\int_a^b \frac{1}{w^{p-1}(t)}dt\right)^{1/p},
\end{aligned}
$$

provided $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $\int_a^b \frac{1}{w^{p-1}(t)}dt<\infty$.

The positivity follows by the fact that, for a convex function g on $[a,b]$ we have

$$\frac{g(a)(b-t)+g(b)(t-a)}{b-a}\ge g(t)$$

for any $t\in[a,b]$. The rest is obvious by Theorem 4.

4. Some Inequalities for the Weighted Čebyšev Functional

Consider now the *weighted Čebyšev functional*

$$
\begin{aligned}
(4.1)\quad C_w(f,g) :={}& \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\,f(t)\,g(t)\,dt \\
&-\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\,f(t)\,dt\,\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\,g(t)\,dt
\end{aligned}
$$

where $f,\ g,\ w:[a,b]\to\mathbb{R}$ and $w(t)\ge 0$ for a.e. $t\in[a,b]$ are measurable functions such that the involved integrals exist and $\int_a^b w(t)\,dt>0$.

In [**?**], Cerone and Dragomir obtained, among others, the following inequalities:

$$
\begin{aligned}
(4.2)\quad & |C_w(f,g)| \\
&\le \frac{1}{2}(M-m)\,\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)\,g(s)\,ds\right|dt \\
&\le \frac{1}{2}(M-m)\left[\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)\,g(s)\,ds\right|^p dt\right]^{\frac{1}{p}} \\
&\le \frac{1}{2}(M-m)\operatorname*{essup}_{t\in[a,b]}\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)\,g(s)\,ds\right|
\end{aligned}
$$

for $p>1$, provided $-\infty<m\leq f(t)\leq M<\infty$ for a.e. $t\in[a,b]$ and the corresponding integrals are finite. The constant $\frac{1}{2}$ is sharp in all the inequalities in (4.2) in the sense that it cannot be replaced by a smaller constant.

In addition, if $-\infty<n\leq g(t)\leq N<\infty$ for a.e. $t\in[a,b]$, then the following refinement of the celebrated Grüss inequality is obtained:

$$
\begin{aligned}
(4.3)\quad & |C_w(f,g)| \\
&\leq \frac{1}{2}(M-m)\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)g(s)\,ds\right|dt \\
&\leq \frac{1}{2}(M-m)\left[\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)g(s)\,ds\right|^2 dt\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}(M-m)(N-n).
\end{aligned}
$$

Here, the constants $\frac{1}{2}$ and $\frac{1}{4}$ are also sharp in the sense mentioned above.

We have:

Theorem 5. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $f\in L_p([a,b],\mathbb{R})$ and $g\in C^1([a,b],\mathbb{R})$, then*

$$
\begin{aligned}
(4.4)\quad |C_w(f,g)| &\leq \frac{b-a}{(p-1)^{1/p}\pi_{p-1}}\left(\int_a^b |g'(t)|^q\,dt\right)^{1/q} \\
&\quad\times\frac{1}{\int_a^b w(s)\,ds}\left(\int_a^b\left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)w(s)\,ds\right|^p w^p(t)\,dt\right)^{1/p}.
\end{aligned}
$$

In particular, if $p=q=2$, then we have [7]

$$
\begin{aligned}
(4.5)\quad |C_w(f,g)| &\leq \frac{b-a}{\pi}\left(\int_a^b |g'(t)|^2\,dt\right)^{1/2} \\
&\quad\times\frac{1}{\int_a^b w(s)\,ds}\left(\int_a^b\left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)w(s)\,ds\right|^2 w^2(t)\,dt\right)^{1/2}.
\end{aligned}
$$

Proof. Integrating by parts, we have

$$\begin{aligned}
&\frac{1}{\int_a^b w(s)\,ds}\int_a^b\left(\int_a^x f(t)\,w(t)\,dt-\frac{\int_a^x w(s)\,ds}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right)g'(x)\,dx\\
&=\frac{1}{\int_a^b w(s)\,ds}\left[\left(\int_a^x f(t)\,w(t)\,dt-\frac{\int_a^x w(s)\,ds}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right)g(x)\right|_a^b\\
&-\int_a^b g(x)\left(f(x)\,w(x)-\frac{w(x)}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right)dx\\
&=-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(x)\,g(x)\,w(x)\,dx\\
&+\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\frac{1}{\int_a^b w(s)\,ds}\int_a^b g(x)\,w(x)\,dx,
\end{aligned}$$

which gives that

$$\begin{aligned}
C_w(f,g)&=\frac{1}{\int_a^b w(s)\,ds}\\
&\times\int_a^b\left(\frac{\int_a^x w(s)\,ds}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds-\int_a^x f(t)\,w(t)\,dt\right)g'(x)\,dx.
\end{aligned}\tag{4.6}$$

Using the Hölder integral inequality for $p,\,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then

$$\begin{aligned}
|C_w(f,g)|&\le\frac{1}{\int_a^b w(s)\,ds}\\
&\times\int_a^b\left|\left(\frac{\int_a^x w(s)\,ds}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds-\int_a^x f(t)\,w(t)\,dt\right)g'(x)\right|dx\\
&\le\frac{1}{\int_a^b w(s)\,ds}\left(\int_a^b\left|\frac{\int_a^x w(s)\,ds}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds-\int_a^x f(t)\,w(t)\,dt\right|^p dx\right)^{1/p}\\
&\times\left(\int_a^b|g'(x)|^q\,dx\right)^{1/q}.
\end{aligned}\tag{4.7}$$

If we take

$$u(x):=\frac{\int_a^x w(s)\,ds}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds-\int_a^x f(t)\,w(t)\,dt,\ x\in[a,b]$$

we observe that $u(a)=u(b)=0$ and $u\in C^1([a,b],\mathbb{C})$.

Using the inequality (1.3) we then have

$$\int_a^b \left| \frac{\int_a^x w(s)\,ds}{\int_a^b w(s)\,ds} \int_a^b f(s)\,w(s)\,ds - \int_a^x f(t)\,w(t)\,dt \right|^p dx$$
$$\leq \frac{(b-a)^p}{(p-1)\,\pi_{p-1}^p} \int_a^b \left| \frac{w(x)}{\int_a^b w(s)\,ds} \int_a^b f(s)\,w(s)\,ds - f(x)\,w(x) \right|^p dx$$
$$= \frac{(b-a)^p}{(p-1)\,\pi_{p-1}^p} \int_a^b \left| \frac{1}{\int_a^b w(s)\,ds} \int_a^b f(s)\,w(s)\,ds - f(x) \right|^p w^p(x)\,dx$$

that is equivalent to

$$\left(\int_a^b \left| \frac{\int_a^x w(s)\,ds}{\int_a^b w(s)\,ds} \int_a^b f(s)\,w(s)\,ds - \int_a^x f(t)\,w(t)\,dt \right|^p dx \right)^{1/p} \tag{4.8}$$
$$\leq \frac{b-a}{(p-1)^{1/p}\,\pi_{p-1}} \left(\int_a^b \left| \frac{1}{\int_a^b w(s)\,ds} \int_a^b f(s)\,w(s)\,ds - f(x) \right|^p w^p(x)\,dx \right)^{1/p}.$$

By using (4.7) and (4.8) we get the desired result (4.4). □

By taking $w \equiv 1$, we get the following unweighted inequality:

Corollary 3. *Let $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L_p([a,b],\mathbb{R})$ and $g \in C^1([a,b],\mathbb{R})$, then*

$$|C(f,g)| \leq \frac{(b-a)^{1/p}}{(p-1)^{1/p}\,\pi_{p-1}} \left(\int_a^b |g'(t)|^q\,dt \right)^{1/q} \tag{4.9}$$
$$\times \left(\frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\,ds \right|^p dt \right)^{1/p}.$$

In particular, if $p = q = 2$, then we have [7]

$$|C(f,g)| \tag{4.10}$$
$$\leq \frac{\sqrt{b-a}}{\pi} \left(\int_a^b |g'(t)|^2\,dt \right)^{1/2} \left(\frac{1}{b-a} \int_a^b |f(t)|^2\,dt - \left| \frac{1}{b-a} \int_a^b f(s)\,ds \right|^2 \right)^{1/2}.$$

The following result also holds:

Theorem 6. *Assume that $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$, $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L_p([a,b],\mathbb{R})$ and $g \in C^1([a,b],\mathbb{C})$, then*

$$|C_w(f,g)| \leq \frac{2}{(q-1)^{1/q}\,\pi_{q-1}} \left(\int_a^b w(s)\,ds \right)^{1/p} \left(\int_a^b \frac{|g'(t)|^q}{w^{q-1}(t)} dt \right)^{1/q} \tag{4.11}$$
$$\times \left(\frac{1}{\int_a^b w(s)\,ds} \int_a^b \left| f(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b f(s)\,w(s)\,ds \right|^p w(t)\,dt \right)^{1/p}.$$

In particular, if $p = q = 2$, then we have [7]

$$|C_w(f,g)| \le \frac{2}{\pi}\left(\int_a^b w(s)\,ds\right)^{1/2}\left(\int_a^b \frac{|g'(t)|^2}{w(t)}dt\right)^{1/2} \times\left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b f^2(t)\,w(t)\,dt-\left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right)^2\right)^{1/2}. \tag{4.12}$$

Proof. We use the following Sonin type identity

$$C_w(f,g) = \frac{1}{\int_a^b w(s)\,ds}\int_a^b\left(f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right)(g(t)-g(a))\,w(t)\,dt, \tag{4.13}$$

which can be proved directly on calculating the integral from the right hand side.

Using the weighted Hölder's integral inequality $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\begin{aligned}|C_w(f,g)| &\le \frac{1}{\int_a^b w(s)\,ds}\int_a^b\left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right||g(t)-g(a)|\,w(t)\,dt\\ &\le \frac{1}{\int_a^b w(s)\,ds}\left(\int_a^b\left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^p w(t)\,dt\right)^{1/p}\\ &\qquad\times\left(\int_a^b |g(t)-g(a)|^q\,w(t)\,dt\right)^{1/q}.\end{aligned} \tag{4.14}$$

Using (2.10) for $f = g - g(a)$, we have

$$\int_a^b |g(t)-g(a)|^q\,w(t)\,dt \le \frac{2^q}{(q-1)\,\pi_{q-1}^q}\left(\int_a^b w(s)\,ds\right)^q\int_a^b\frac{|g'(t)|^q}{w^{q-1}(t)}dt \tag{4.15}$$

that is equivalent to

$$\left(\int_a^b |g(t)-g(a)|^q\,w(t)\,dt\right)^{1/q} \le \frac{2}{(q-1)^{1/q}\,\pi_{q-1}}\left(\int_a^b w(s)\,ds\right)\left(\int_a^b\frac{|g'(t)|^q}{w^{q-1}(t)}dt\right)^{1/q}. \tag{4.16}$$

By making use of (4.14) and (4.16) we get

$$
\begin{aligned}
(4.17)\quad & |C_w(f,g)| \\
& \leq \frac{1}{\int_a^b w(s)\,ds}\left(\int_a^b \left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^p w(t)\,dt\right)^{1/p} \\
& \qquad \times \frac{2}{(q-1)^{1/q}\,\pi_{q-1}}\left(\int_a^b w(s)\,ds\right)\left(\int_a^b \frac{|g'(t)|^q}{w^{q-1}(t)}dt\right)^{1/q} \\
& = \frac{2}{(q-1)^{1/q}\,\pi_{q-1}}\left(\int_a^b w(s)\,ds\right)^{1/p} \\
& \qquad \times \left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b \left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^p w(t)\,dt\right)^{1/p} \\
& \qquad\qquad \times \left(\int_a^b \frac{|g'(t)|^q}{w^{q-1}(t)}dt\right)^{1/q}
\end{aligned}
$$

hence by (4.17) we get the desired result (4.10). □

Remark 5. *If we take $w \equiv 1$, then by (4.11) we get*

$$
\begin{aligned}
(4.18)\quad |C(f,g)| & \leq \frac{2}{(q-1)^{1/q}\,\pi_{q-1}}(b-a)^{1/p}\left(\int_a^b |g'(t)|^q\,dt\right)^{1/q} \\
& \qquad \times \left(\frac{1}{b-a}\int_a^b \left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,w(s)\,ds\right|^p dt\right)^{1/p}.
\end{aligned}
$$

5. Reverses of Jensen's Inequality

Let $(\Omega,\mathcal{A},\mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a μ-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\geq 0$ for μ-a.e. (almost every) $x\in\Omega$, consider the *Lebesgue space*

$$L_w(\Omega,\mu) := \{f:\Omega\to\mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega w(x)\,|f(x)|\,d\mu(x)<\infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, S. S. Dragomir obtained in 2002 [6] the following result:

Theorem 7. *Let $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable convex function on (m,M) and $f:\Omega\to[m,M]$ so that $\Phi\circ f$, f, $\Phi'\circ f$, $(\Phi'\circ f)f\in L_w(\Omega,\mu)$, where $w\geq 0$*

μ-a.e. (almost everywhere) on Ω with $\int_\Omega w d\mu = 1$. Then we have the inequality:

$$0 \leq \int_\Omega w\,(\Phi \circ f)\,d\mu - \Phi\left(\int_\Omega w f d\mu\right) \leq \int_\Omega w\,(\Phi' \circ f)\,f d\mu - \int_\Omega w\,(\Phi' \circ f)\,d\mu \int_\Omega w f d\mu. \tag{5.1}$$

We have the following reverse of Jensen's inequality:

Theorem 8. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M), $w : [a, b] \to (0, \infty)$ be continuous on $[a, b]$ and $f : [a, b] \to [m, M]$ is absolutely continuous so that $\Phi \circ f$, f, $\Phi' \circ f$, $(\Phi' \circ f) f \in L_w [a, b]$. Assume that p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

(i) *If $(\Phi'' \circ f) f' \in L_q [a, b]$ and $f \in L_p [a, b]$, then*

$$\begin{aligned} 0 &\leq \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t)\,(\Phi \circ f)(t)\,dt - \Phi\left(\frac{\int_a^b w(t) f(t)\,dt}{\int_a^b w(s)\,ds}\right) \\ &\leq \frac{b-a}{(p-1)^{1/p}\,\pi_{p-1}} \left(\int_a^b |(\Phi'' \circ f)(t)|^q\,|f'(t)|^q\,dt\right)^{1/q} \\ &\times \frac{1}{\int_a^b w(s)\,ds} \left(\int_a^b \left| f(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b f(s)\,w(s)\,ds \right|^p w^p(t)\,dt\right)^{1/p}. \end{aligned} \tag{5.2}$$

(ii) *If $f' \in L_q [a, b]$ and $\Phi' \circ f \in L_p [a, b]$, then*

$$\begin{aligned} 0 &\leq \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t)\,(\Phi \circ f)(t)\,dt - \Phi\left(\frac{\int_a^b w(t) f(t)\,dt}{\int_a^b w(s)\,ds}\right) \\ &\leq \frac{b-a}{(p-1)^{1/p}\,\pi_{p-1}} \left(\int_a^b |f'(t)|^q\,dt\right)^{1/q} \\ &\times \frac{1}{\int_a^b w(s)\,ds} \left(\int_a^b \left| (\Phi' \circ f)(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b (\Phi' \circ f)(s)\,w(s)\,ds \right|^p w^p(t)\,dt\right)^{1/p}. \end{aligned} \tag{5.3}$$

Proof. (i). From (5.1) we have

$$\begin{aligned} 0 &\leq \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t)\,(\Phi \circ f)(t)\,dt - \Phi\left(\frac{\int_a^b w(t) f(t)\,dt}{\int_a^b w(s)\,ds}\right) \\ &\leq \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t)\,(\Phi' \circ f)(t)\,f(t)\,dt \\ &- \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t)\,(\Phi' \circ f)(t)\,dt \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t) f(t)\,dt = C_w(f, \Phi' \circ f). \end{aligned} \tag{5.4}$$

From the inequality (4.4) we have

$$|C_w(f,\Phi'\circ f)| \le \frac{b-a}{(p-1)^{1/p}\pi_{p-1}}\left(\int_a^b \left|(\Phi'\circ f)'(t)\right|^q dt\right)^{1/q}$$
$$\times \frac{1}{\int_a^b w(s)\,ds}\left(\int_a^b \left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^p w^p(t)\,dx\right)^{1/p}$$
$$= \frac{b-a}{(p-1)^{1/p}\pi_{p-1}}\left(\int_a^b |(\Phi''\circ f)(t)\,f'(t)|^q\,dt\right)^{1/q}$$
$$\times \frac{1}{\int_a^b w(s)\,ds}\left(\int_a^b \left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^p w^p(t)\,dx\right)^{1/p},$$

which together with (5.4) proves (5.2).

(ii). From (4.4) we also have

$$|C_w(f,\Phi'\circ f)| = |C_w(\Phi'\circ f, f)|$$
$$\le \frac{b-a}{(p-1)^{1/p}\pi_{p-1}}\left(\int_a^b |f'(t)|^q\,dt\right)^{1/q}$$
$$\times \frac{1}{\int_a^b w(s)\,ds}\left(\int_a^b \left|\Phi'\circ f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b \Phi'\circ f(s)\,w(s)\,ds\right|^p w^p(t)\,dx\right)^{1/p},$$

which together with (5.4) proves (5.3). □

Corollary 4. *Let $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable convex function on (m,M) and $f:[a,b]\to[m,M]$ is absolutely continuous so that $\Phi\circ f$, f, $\Phi'\circ f$, $(\Phi'\circ f)f\in L[a,b]$. Assume that $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.*

(i) *If $(\Phi''\circ f)f'\in L_q[a,b]$ and $f\in L_p[a,b]$, then*

$$0 \le \frac{1}{b-a}\int_a^b (\Phi\circ f)(t)\,dt - \Phi\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)$$
$$\le \frac{(b-a)^{1/p}}{(p-1)^{1/p}\pi_{p-1}}\left(\int_a^b |(\Phi''\circ f)(t)|^q\,|f'(t)|^q\,dt\right)^{1/q}$$
$$\times\left(\frac{1}{b-a}\int_a^b \left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{1/p}. \tag{5.5}$$

(ii) *If* $f' \in L_q[a,b]$ *and* $\Phi' \circ f \in L_p[a,b]$, *then*

$$0 \leq \frac{1}{b-a}\int_a^b (\Phi \circ f)(t)\,dt - \Phi\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \leq \frac{(b-a)^{1/p}}{(p-1)^{1/p}\pi_{p-1}}\left(\int_a^b |f'(t)|^q\,dt\right)^{1/q} \times \left(\frac{1}{b-a}\int_a^b \left|(\Phi' \circ f)(t) - \frac{1}{b-a}\int_a^b (\Phi' \circ f)(s)\,ds\right|^p dt\right)^{1/p}. \tag{5.6}$$

Corollary 5. *Let* $\Phi : [a,b] \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable convex function on* (a,b), $w : [a,b] \to (0,\infty)$ *be continuous on* $[a,b]$ *so that* Φ, Φ', $\Phi'\ell \in L_w[a,b]$, *where* $\ell(t) = t$. *Assume that* p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

(i) *If* $\Phi'' \in L_q[a,b]$, *then*

$$0 \leq \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\Phi(t)\,dt - \Phi\left(\frac{\int_a^b tw(t)\,dt}{\int_a^b w(s)\,ds}\right) \leq \frac{b-a}{(p-1)^{1/p}\pi_{p-1}}\left(\int_a^b |\Phi''(t)|^q\,dt\right)^{1/q} \times \frac{1}{\int_a^b w(s)\,ds}\left(\int_a^b \left|t - \frac{1}{\int_a^b w(s)\,ds}\int_a^b sw(s)\,ds\right|^p w^p(t)\,dt\right)^{1/p}. \tag{5.7}$$

(ii) *If* $\Phi' \in L_p[a,b]$, *then*

$$0 \leq \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\Phi(t)\,dt - \Phi\left(\frac{\int_a^b tw(t)\,dt}{\int_a^b w(s)\,ds}\right) \leq \frac{b-a}{(p-1)^{1/p}\pi_{p-1}} \times \frac{1}{\int_a^b w(s)\,ds}\left(\int_a^b \left|\Phi'(t) - \frac{1}{\int_a^b w(s)\,ds}\int_a^b \Phi'(s)w(s)\,ds\right|^p w^p(t)\,dt\right)^{1/p}. \tag{5.8}$$

We also have:

Theorem 9. *Let* $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable convex function on* (m,M), $w : [a,b] \to (0,\infty)$ *be continuous on* $[a,b]$ *and* $f : [a,b] \to [m,M]$ *is absolutely continuous so that* $\Phi \circ f$, f, $\Phi' \circ f$, $(\Phi' \circ f) f \in L_w[a,b]$. *Assume that* p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

(i) *If* $|\Phi'' \circ f| |f'|^q w^{1-q} \in L[a,b]$ *and* $f \in L_p[a,b]$, *then*

$$0 \leq \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t)(\Phi \circ f)(t)\,dt - \Phi\left(\frac{\int_a^b w(t) f(t)\,dt}{\int_a^b w(s)\,ds}\right)$$
$$\leq \frac{2}{(q-1)^{1/q} \pi_{q-1}} \left(\int_a^b w(s)\,ds\right)^{1/p} \left(\int_a^b \frac{|(\Phi'' \circ f)(t)|^q |f'(t)|^q}{w^{q-1}(t)} dt\right)^{1/q}$$
$$\times \left(\frac{1}{\int_a^b w(s)\,ds} \int_a^b \left| f(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b f(s) w(s)\,ds \right|^p w(t)\,dt\right)^{1/p}. \tag{5.9}$$

(ii) *If* $|f'|^q w^{1-q} \in L[a,b]$ *and* $\Phi' \circ f \in L_p[a,b]$, *then*

$$0 \leq \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t)(\Phi \circ f)(t)\,dt - \Phi\left(\frac{\int_a^b w(t) f(t)\,dt}{\int_a^b w(s)\,ds}\right)$$
$$\leq \frac{2}{(q-1)^{1/q} \pi_{q-1}} \left(\int_a^b w(s)\,ds\right)^{1/p} \left(\int_a^b \frac{|f'(t)|^q}{w^{q-1}(t)} dt\right)^{1/q}$$
$$\times \left(\frac{1}{\int_a^b w(s)\,ds} \int_a^b \left| (\Phi' \circ f)(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b (\Phi' \circ f)(s) w(s)\,ds \right|^p w(t)\,dt\right)^{1/p}. \tag{5.10}$$

The proof follows in a similar way by utilising the inequality (4.11). The details are not provided here.

Corollary 6. *Let* $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable convex function on* (m,M) *and* $f : [a,b] \to [m,M]$ *is absolutely continuous so that* $\Phi \circ f$, f, $\Phi' \circ f$, $(\Phi' \circ f) f \in L[a,b]$. *Assume that* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

(i) *If* $(\Phi'' \circ f) f' \in L_q[a,b]$ *and* $f \in L_p[a,b]$, *then*

$$0 \leq \frac{1}{b-a} \int_a^b (\Phi \circ f)(t)\,dt - \Phi\left(\frac{1}{b-a} \int_a^b f(t)\,dt\right)$$
$$\leq \frac{2}{(q-1)^{1/q} \pi_{q-1}} (b-a)^{1/p} \left(\int_a^b |(\Phi'' \circ f)(t)|^q |f'(t)|^q\,dt\right)^{1/q}$$
$$\times \left(\frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\,ds \right|^p dt\right)^{1/p}. \tag{5.11}$$

(ii) *If* $f' \in L_q[a,b]$ *and* $\Phi' \circ f \in L_p[a,b]$, *then*

$$0 \leq \frac{1}{b-a} \int_a^b (\Phi \circ f)(t)\,dt - \Phi\left(\frac{1}{b-a} \int_a^b f(t)\,dt\right)$$
$$\leq \frac{2}{(q-1)^{1/q} \pi_{q-1}} (b-a)^{1/p} \left(\int_a^b |f'(t)|^q\,dt\right)^{1/q}$$
$$\times \left(\frac{1}{b-a} \int_a^b \left| (\Phi' \circ f)(t) - \frac{1}{b-a} \int_a^b (\Phi' \circ f)(s)\,ds \right|^p dt\right)^{1/p}. \tag{5.12}$$

Finally, we have:

Corollary 7. *Let $\Phi : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (a,b), $w : [a,b] \to (0,\infty)$ be continuous on $[a,b]$ so that Φ, Φ', $\Phi'\ell \in L_w[a,b]$, where $\ell(t) = t$. Assume that p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

(i) *If $|\Phi''|^q w^{1-q} \in L[a,b]$, then*

$$
\begin{aligned}
(5.13)\quad 0 \leq{}& \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\,\Phi(t)\,dt - \Phi\left(\frac{\int_a^b tw(t)\,dt}{\int_a^b w(s)\,ds}\right) \\
\leq{}& \frac{2}{(q-1)^{1/q}\,\pi_{q-1}}\left(\int_a^b w(s)\,ds\right)^{1/p}\left(\int_a^b \frac{|\Phi''(t)|^q}{w^{q-1}(t)}dt\right)^{1/q} \\
&\times\left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b \left|t - \frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,sds\right|^p w(t)\,dt\right)^{1/p}.
\end{aligned}
$$

(ii) *If $w^{1-q} \in L[a,b]$ and $\Phi' \in L_p[a,b]$, then*

$$
\begin{aligned}
(5.14)\quad 0 \leq{}& \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\,\Phi(t)\,dt - \Phi\left(\frac{\int_a^b tw(t)\,dt}{\int_a^b w(s)\,ds}\right) \\
\leq{}& \frac{2}{(q-1)^{1/q}\,\pi_{q-1}}\left(\int_a^b w(s)\,ds\right)^{1/p}\left(\int_a^b \frac{1}{w^{q-1}(t)}dt\right)^{1/q} \\
&\times\left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b \left|\Phi'(t) - \frac{1}{\int_a^b w(s)\,ds}\int_a^b \Phi'(s)\,w(s)\,ds\right|^p w(t)\,dt\right)^{1/p}.
\end{aligned}
$$

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