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# WEIGHTED INTEGRAL INEQUALITIES RELATED TO OPIAL'S RESULT

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Abstract. In this paper we establish some weighted versions of certain Opial type integral inequalities. Applications related to the trapezoid unweighted and weighted inequalities and to Fejér's inequality for convex functions are also provided.

## 1. Introduction

We recall the following Opial type inequalities:

**Theorem 1.** *Assume that $u:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is an absolutely continuous function on the interval $[a,b]$ and such that $u'\in L_2[a,b]$.*

(i) *If $u(a)=u(b)=0$, then*

$$\int_a^b |u(t)\,u'(t)|\,dt \le \frac{1}{4}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.1}$$

*with equality if and only if*

$$u(t)=\begin{cases} c(t-a) & \text{if } a\le t\le \frac{a+b}{2},\\ c(b-t) & \text{if } \frac{a+b}{2}<t\le b\end{cases}$$

*where $c$ is an arbitrary constant;*

(ii) *If $u(a)=0$, then*

$$\int_a^b |u(t)\,u'(t)|\,dt \le \frac{1}{2}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.2}$$

*with equality if and only if $u(t)=c(t-a)$ for some constant $c$;*

(iii) *If $\int_a^b u(t)\,dt=0$, then the inequality (1.1) holds with equality if and only if*

$$u(t)=c\left(t-\frac{a+b}{2}\right)$$

*for any constant $c$.*

The inequality (1.1) was obtained by Olech in [6] in which he gave a simplified proof of an inequality originally due in a slightly less general form to Zdzislaw Opial [7].

Embedded in Olech's proof is the half-interval form of Opial's inequality, also discovered by Beesack [1], which is satisfied by those $u$ vanishing only at $a$.

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The inequality (1.1) in the case (iii), namely in the case that $u$ satisfies the condition $\int_a^b u(t)\,dt = 0$ was obtained by Brown and Plum in [4].

As mentioned in [4] the inequality (1.1) also holds if $u(a) + u(b) = 0$.

In 1969, D. W. Boyd [2] obtained the following generalization of Opial inequality. Let

$$K(p) := \begin{cases} \frac{1}{2} \text{ if } p = 1, \\ \frac{2-p}{2p}\left(\frac{1}{p}\right)^{2(p-1)} I^{-p}(p) \text{ if } p \in (1,2), \\ \frac{4}{\pi^2} \text{ if } p = 2, \end{cases} \tag{1.3}$$

where

$$I(p) := \int_0^1 \left[1 + \frac{2(p-1)}{2-p} t\right]^{-2} [1 + (p-1)t]^{\frac{1-p}{p}} dt.$$

We have:

**Theorem 2.** *Assume that $u : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ and $p \in [1,2]$.*

(i) *If either $u(a) = 0$ or $u(b) = 0$, then* [2]

$$\int_a^b |u(t)\,u'(t)|^p \le K(p)(b-a)\left(\int_a^b |u'(t)|^2\,dt\right)^p. \tag{1.4}$$

(ii) *If $u(a) = u(b) = 0$, then*

$$\int_a^b |u(t)\,u'(t)|^p \le \frac{1}{2} K(p)(b-a)\left(\int_a^b |u'(t)|^2\,dt\right)^p. \tag{1.5}$$

As observed in [3], the inequality (1.5) follows from (1.4) written on the intervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ and adding the corresponding inequalities.

In this paper we establish some weighted versions of the Opial type integral inequalities above. Applications related to the trapezoid unweighted and weighted inequalities and to Fejér's inequality for convex functions are also provided.

## 2. Some Composite Inequalities

We have:

**Theorem 3.** *Let $g : [a,b] \to [g(a), g(b)]$ be a continuous strictly increasing function that is of class $C^1$ on $(a,b)$. Assume that $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function on the interval $[a,b]$ and such that $\frac{f'}{[g']^{1/2}} \in L_2[a,b]$.*

(i) *If $f(a) = f(b) = 0$, then*

$$\int_a^b |f(t)\,f'(t)|\,dt \le \frac{1}{4}[g(b) - g(a)] \int_a^b \frac{[f'(t)]^2}{g'(t)} dt, \tag{2.1}$$

*with equality if and only if*

$$f(t) = \begin{cases} c(g(t) - g(a)) \ \text{ if } a \le t \le g^{-1}\left(\frac{g(a)+g(b)}{2}\right), \\ \\ c(g(b) - g(t)) \ \text{ if } g^{-1}\left(\frac{g(a)+g(b)}{2}\right) < t \le b \end{cases}$$

*where $c$ is an arbitrary constant;*

(ii) *If* $f(a) = 0$*, then*

$$\int_a^b |f(t) f'(t)|\, dt \le \frac{1}{2} [g(b) - g(a)] \int_a^b \frac{[f'(t)]^2}{g'(t)} dt, \tag{2.2}$$

*with equality if and only if* $f(t) = c(g(t) - g(a))$, $t \in [a,b]$ *for some constant* $c$;

(iii) *If* $\int_a^b f(t) g'(t)\, dt = 0$*, then the inequality (1.1) holds with equality if and only if*

$$f(t) = c\left(g(t) - \frac{g(a) + g(b)}{2}\right), \ t \in [a,b]$$

*for any constant* $c$.

*Proof.* (i) Consider the function $u := f \circ g^{-1} : [g(a), g(b)] \to \mathbb{R}$. The function $u$ is absolutely continuous on $[g(a), g(b)]$, $u(g(a)) = f \circ g^{-1}(g(a)) = f(a) = 0$ and $u(g(b)) = f \circ g^{-1}(g(b)) = f(b) = 0$.

Using the chain rule and the derivative of inverse functions we have

$$\left(f \circ g^{-1}\right)'(z) = \left(f' \circ g^{-1}\right)(z) \left(g^{-1}\right)'(z) = \frac{\left(f' \circ g^{-1}\right)(z)}{\left(g' \circ g^{-1}\right)(z)} \tag{2.3}$$

for almost every (a.e.) $z \in [g(a), g(b)]$.

If we apply the inequality (1.1) for the function $u = f \circ g^{-1}$ on the interval $[g(a), g(b)]$, then we get

$$\begin{aligned} \int_{g(a)}^{g(b)} \left| f \circ g^{-1}(z) \frac{\left(f' \circ g^{-1}\right)(z)}{\left(g' \circ g^{-1}\right)(z)} \right| dz \\ \le \frac{1}{4} [g(b) - g(a)] \int_{g(a)}^{g(b)} \left| \frac{\left(f' \circ g^{-1}\right)(z)}{\left(g' \circ g^{-1}\right)(z)} \right|^2 dz. \end{aligned} \tag{2.4}$$

If we make the change of variable $t = g^{-1}(z)$, $z \in [g(a), g(b)]$, then $z = g(t)$, $dz = g'(t)\, dt$,

$$\begin{aligned} \int_{g(a)}^{g(b)} \left| f \circ g^{-1}(z) \frac{\left(f' \circ g^{-1}\right)(z)}{\left(g' \circ g^{-1}\right)(z)} \right| dz &= \int_a^b \left| f(t) \frac{f'(t)}{g'(t)} \right| g'(t)\, dt \\ &= \int_a^b |f(t) f'(t)|\, dt \end{aligned}$$

and

$$\int_{g(a)}^{g(b)} \left| \frac{\left(f' \circ g^{-1}\right)(z)}{\left(g' \circ g^{-1}\right)(z)} \right|^2 dz = \int_a^b \left| \frac{f'(t)}{g'(t)} \right|^2 g'(t)\, dt = \int_a^b \frac{[f'(t)]^2}{g'(t)} dt.$$

By utilising (2.4), we then get the desired inequality (2.1).

Now, by Theorem 1, the equality case holds in (2.4) iff

$$f \circ g^{-1}(z) = \begin{cases} c(z - g(a)) \ \text{ if } g(a) \le z \le \frac{g(a)+g(b)}{2}, \\ c(g(b) - z) \ \text{ if } \frac{g(a)+g(b)}{2} < z \le g(b). \end{cases}$$

If we take in this equality $z = g(t)$, $t \in [a,b]$, then we have

$$f(t) = \begin{cases} c(g(t) - g(a)) & \text{if } g(a) \le g(t) \le \frac{g(a)+g(b)}{2}, \\ \\ c(g(b) - g(t)) & \text{if } \frac{g(a)+g(b)}{2} < g(t) \le g(b) \end{cases}$$
$$= \begin{cases} c(g(t) - g(a)) & \text{if } a \le t \le g^{-1}\left(\frac{g(a)+g(b)}{2}\right), \\ \\ c(g(b) - g(t)) & \text{if } g^{-1}\left(\frac{g(a)+g(b)}{2}\right) < t \le b \end{cases},$$

and the case of equality is proved.

(ii) and (iii) follow in a similar way and the details are omitted. □

In what follows we consider the identity function $\ell(t) = t$.

a). If we take $g : [a,b] \subset (0,\infty) \to \mathbb{R}$, $g(t) = \ln t$ and assume that $f$ is an absolutely continuous function with $f(a) = f(b) = 0$ and $\ell^{1/2} f' \in L_2[a,b]$, then by (2.1) we get

$$\int_a^b |f(t) f'(t)| \, dt \le \frac{1}{4} \ln\left(\frac{b}{a}\right) \int_a^b t [f'(t)]^2 \, dt, \tag{2.5}$$

with equality if and only if

$$f(t) = \begin{cases} c \ln\left(\frac{t}{a}\right) & \text{if } a \le t \le \sqrt{ab}, \\ \\ c \ln\left(\frac{b}{t}\right) & \text{if } \sqrt{ab} < t \le b. \end{cases}$$

where $c$ is an arbitrary constant.

If $f(a) = 0$, then by (2.2)

$$\int_a^b |f(t) f'(t)| \, dt \le \frac{1}{2} \ln\left(\frac{b}{a}\right) \int_a^b t [f'(t)]^2 \, dt, \tag{2.6}$$

with equality if and only if $f(t) = c \ln\left(\frac{t}{a}\right)$, $t \in [a,b]$ for some constant $c$.

If $\int_a^b \frac{f(t)}{t} dt = 0$, then by (2.1) we have

$$\int_a^b |f(t) f'(t)| \, dt \le \frac{1}{4} \ln\left(\frac{b}{a}\right) \int_a^b t [f'(t)]^2 \, dt, \tag{2.7}$$

with equality iff

$$f(t) = c \ln\left(\frac{t}{\sqrt{ab}}\right), \ t \in [a,b].$$

b). If we take $g : [a,b] \subset \mathbb{R} \to (0,\infty)$, $g(t) = \exp t$ and assume that $f$ is an absolutely continuous function with $f(a) = f(b) = 0$ and $\exp\left(-\frac{1}{2}\ell\right) f' \in L_2[a,b]$, then by (2.1) we get

$$\int_a^b |f(t) f'(t)| \, dt \le \frac{1}{4} (\exp b - \exp a) \int_a^b \exp(-t) [f'(t)]^2 \, dt, \tag{2.8}$$

with equality if and only if

$$f(t) = \begin{cases} c(\exp t - \exp a) & \text{if } a \le t \le \ln\left(\frac{\exp a + \exp b}{2}\right), \\ \\ c(\exp b - \exp t) & \text{if } \ln\left(\frac{\exp a + \exp b}{2}\right) < t \le b. \end{cases}$$

If $f(a)=0$, then by (2.2)

$$\int_a^b |f(t) f'(t)|\, dt \le \frac{1}{2} (\exp b - \exp a) \int_a^b \exp(-t) [f'(t)]^2\, dt, \tag{2.9}$$

with equality if and only if

$$f(t) = c(\exp t - \exp a), t \in [a,b].$$

If $\int_a^b f(t) \exp t dt = 0$, then the inequality (2.8) holds with equality if and only if

$$f(t) = c\left(\exp t - \frac{\exp a + \exp b}{2}\right),\ t \in [a,b].$$

c). If we take $g : [a,b] \subset (0,\infty) \to \mathbb{R}$, $g(t) = t^r$, $r > 0$ and assume that $f$ is an absolutely continuous function with $f(a) = f(b) = 0$ and $\ell^{(1-r)/2} f' \in L_2[a,b]$, then by (2.1) we get

$$\int_a^b |f(t) f'(t)|\, dt \le \frac{1}{4r} (b^r - a^r) \int_a^b \frac{[f'(t)]^2}{t^{r-1}} dt, \tag{2.10}$$

with equality iff

$$f(t) = \begin{cases} c(t^r - a^r) \text{ if } a \le t \le \left(\frac{a^r+b^r}{2}\right)^{1/r}, \\ \\ c(b^r - t^r) \text{ if } \left(\frac{a^r+b^r}{2}\right)^{1/r} < t \le b. \end{cases}$$

If $f(a) = 0$, then

$$\int_a^b |f(t) f'(t)|\, dt \le \frac{1}{2r} (b^r - a^r) \int_a^b \frac{[f'(t)]^2}{t^{r-1}} dt, \tag{2.11}$$

with equality if and only if $f(t) = c(t^r - a^r)$, $t \in [a,b]$ for some constant $c$.

If $\int_a^b f(t) t^{r-1} dt = 0$, then the inequality (2.10) holds with equality if and only if

$$f(t) = c\left(t^r - \frac{a^r + b^r}{2}\right),\ t \in [a,b]$$

for any constant $c$.

If $w : [a,b] \to \mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W : [a,b] \to [0,\infty)$, $W(x) := \int_a^x w(s)\, ds$ is strictly increasing and differentiable on $(a,b)$. We have $W'(x) = w(x)$ for any $x \in (a,b)$.

**Corollary 1.** *Assume that $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ and that $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function on the interval $[a,b]$ and such that $\frac{f'}{w^{1/2}} \in L_2[a,b]$.*

(i) *If $f(a) = f(b) = 0$, then*

$$\int_a^b |f(t) f'(t)|\, dt \le \frac{1}{4} \int_a^b w(s)\, ds \int_a^b \frac{[f'(t)]^2}{w(t)} dt, \tag{2.12}$$

*with equality if and only if*

$$f(t) = \begin{cases} c\int_a^t w(s)\, ds \text{ if } a \le t \le W^{-1}\left(\frac{1}{2}\int_a^b w(s)\, ds\right), \\ \\ c\int_t^b w(s)\, ds \text{ if } W^{-1}\left(\frac{1}{2}\int_a^b w(s)\, ds\right) < t \le b \end{cases}$$

*where $c$ is an arbitrary constant;*

(ii) *If* $f(a)=0$, *then*

$$\int_a^b |f(t)f'(t)|\,dt \le \frac{1}{2}\int_a^b w(s)\,ds \int_a^b \frac{[f'(t)]^2}{w(t)}dt, \tag{2.13}$$

*with equality if and only if* $f(t)=c\int_a^t w(s)\,ds$, $t\in[a,b]$ *for some constant* $c$;

(iii) *If* $\int_a^b f(t)w(t)\,dt=0$, *then the inequality (2.1) holds with equality if and only if*

$$f(t)=c\left(\int_a^t w(s)\,ds-\frac{1}{2}\int_a^b w(s)\,ds\right),\ t\in[a,b]$$

*for any constant* $c$.

**Remark 1.** *The inequalities (2.12) and (2.13) were obtained by a different argument and for* $w=\frac{1}{p}$ *in* [1].

Further we have:

**Theorem 4.** *Let* $g:[a,b]\to[g(a),g(b)]$ *be a continuous strictly increasing function that is of class* $C^1$ *on* $(a,b)$. *Assume that* $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ *is an absolutely continuous function on the interval* $[a,b]$ *and such that* $\frac{f'}{[g']^{1/2}}\in L_2[a,b]$ *and* $p\in[1,2]$.

(i) *If* $f(a)=f(b)=0$, *then*

$$\int_a^b \frac{|f(t)f'(t)|^p}{[g'(t)]^{p-1}}dt \le \frac{1}{2}[g(b)-g(a)]K(p)\left(\int_a^b \frac{[f'(t)]^2}{g'(t)}dt\right)^p, \tag{2.14}$$

*where* $K(p)$ *is defined by (1.3).*

(ii) *If either* $f(a)=0$ *or* $f(b)=0$, *then*

$$\int_a^b \frac{|f(t)f'(t)|^p}{[g'(t)]^{p-1}}dt \le [g(b)-g(a)]K(p)\left(\int_a^b \frac{[f'(t)]^2}{g'(t)}dt\right)^p. \tag{2.15}$$

*Proof.* (i) Consider the function $u:=f\circ g^{-1}:[g(a),g(b)]\to\mathbb{R}$. The function $u$ is absolutely continuous on $[g(a),g(b)]$, $u(g(a))=f\circ g^{-1}(g(a))=f(a)=0$ and $u(g(b))=f\circ g^{-1}(g(b))=f(b)=0$.

If we apply the inequality (1.5) for the function $u=f\circ g^{-1}$ on the interval $[g(a),g(b)]$, then we get

$$\begin{aligned}\int_{g(a)}^{g(b)} \left|f\circ g^{-1}(z)\frac{(f'\circ g^{-1})(z)}{(g'\circ g^{-1})(z)}\right|^p dz \\ \le \frac{1}{2}[g(b)-g(a)]K(p)\left(\int_{g(a)}^{g(b)}\left|\frac{(f'\circ g^{-1})(z)}{(g'\circ g^{-1})(z)}\right|^2 dz\right)^p.\end{aligned} \tag{2.16}$$

If we make the change of variable $t = g^{-1}(z)$, $z \in [g(a), g(b)]$, then $z = g(t)$, $dz = g'(t)\,dt$,

$$\int_{g(a)}^{g(b)} \left| f \circ g^{-1}(z) \frac{\left(f' \circ g^{-1}\right)(z)}{\left(g' \circ g^{-1}\right)(z)} \right|^p dz = \int_a^b \left| f(t) \frac{f'(t)}{g'(t)} \right|^p g'(t)\,dt$$
$$= \int_a^b \frac{|f(t) f'(t)|^p}{[g'(t)]^{p-1}} dt$$

and

$$\int_{g(a)}^{g(b)} \left| \frac{\left(f' \circ g^{-1}\right)(z)}{\left(g' \circ g^{-1}\right)(z)} \right|^2 dz = \int_a^b \left| \frac{f'(t)}{g'(t)} \right|^2 g'(t)\,dt = \int_a^b \frac{[f'(t)]^2}{g'(t)} dt.$$

By utilising (2.16) we get the desired result (2.14).

(ii) Follows in a similar way from (1.4). □

**Corollary 2.** *Assume that $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ and that $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function on the interval $[a,b]$ and such that $\frac{f'}{w^{1/2}} \in L_2[a,b]$.*

(i) *If $f(a) = f(b) = 0$, then*

$$\int_a^b \frac{|f(t) f'(t)|^p}{w^{p-1}(t)} dt \le \frac{1}{2} K(p) \int_a^b w(t)\,dt \left( \int_a^b \frac{[f'(t)]^2}{w(t)} dt \right)^p. \tag{2.17}$$

(ii) *If either $f(a) = 0$ or $f(b) = 0$, then*

$$\int_a^b \frac{|f(t) f'(t)|^p}{w^{p-1}(t)} dt \le K(p) \int_a^b w(t)\,dt \left( \int_a^b \frac{[f'(t)]^2}{w(t)} dt \right)^p. \tag{2.18}$$

## 3. Applications

We have:

**Proposition 1.** *Assume that $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ and that $h : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function on the interval $[a,b]$ with $h(b) \ne h(a)$ and such that $\frac{h'}{w^{1/2}} \in L_2[a,b]$. Then*

$$\left| \frac{h(a) + h(b)}{2} - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(s) h(s)\,ds \right| \le \frac{1}{4} \frac{\int_a^b w(s)\,ds}{|h(b) - h(a)|} \int_a^b \frac{[h'(t)]^2}{w(t)} dt. \tag{3.1}$$

*Proof.* Consider the function

$$f(t) := h(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(s) h(s)\,ds, \ \ t \in [a,b].$$

Then

$$\int_a^b \left( h(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(s) h(s)\,ds \right) w(t)\,dt = 0,$$

and by the statement (iii) of Corollary 1 we have

$$\int_a^b \left| \left( h(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(s)\,h(s)\,ds \right) h'(t) \right| dt \leq \frac{1}{4} \int_a^b w(s)\,ds \int_a^b \frac{[h'(t)]^2}{w(t)} dt. \tag{3.2}$$

By the modulus and integral properties, we also have

$$\begin{aligned} &\int_a^b \left| \left( h(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(s)\,h(s)\,ds \right) h'(t) \right| dt \\ &\geq \left| \int_a^b \left( h(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(s)\,h(s)\,ds \right) h'(t)\,dt \right| \\ &= \left| \int_a^b h(t)\,h'(t)\,dt - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(s)\,h(s)\,ds \int_a^b h'(t)\,dt \right| \\ &= \left| \frac{1}{2} \left( h^2(b) - h^2(a) \right) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(s)\,h(s)\,ds\,(h(b) - h(a)) \right| \\ &= |h(b) - h(a)| \left| \frac{h(a) + h(b)}{2} - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(s)\,h(s)\,ds \right|. \end{aligned} \tag{3.3}$$

By utilising (3.2) and (3.3) we get the desired result (3.1). □

**Corollary 3.** *Assume that $h : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function on the interval $[a,b]$ with $h(b) \neq h(a)$ and such that $h' \in L_2[a,b]$. Then*

$$\left| \frac{h(a) + h(b)}{2} - \frac{1}{b-a} \int_a^b h(s)\,ds \right| \leq \frac{1}{4} \frac{b-a}{|h(b) - h(a)|} \int_a^b [h'(t)]^2\,dt. \tag{3.4}$$

In 1906, Fejér [5], while studying trigonometric polynomials, obtained the following inequalities which generalize that of Hermite & Hadamard:

**Theorem 5** (Fejér's Inequality)**.** *Consider the integral $\int_a^b h(x)\,w(x)\,dx$, where $h$ is a convex function in the interval $(a,b)$ and $w$ is a positive function in the same interval such that*

$$w(x) = w(a + b - x), \text{ for any } x \in [a,b]$$

*i.e., $y = w(x)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b), 0\right)$ and is normal to the $x$-axis. Under those conditions the following inequalities are valid:*

$$h\left(\frac{a+b}{2}\right) \leq \frac{1}{\int_a^b w(x)\,dx} \int_a^b h(x)\,w(x)\,dx \leq \frac{h(a) + h(b)}{2}. \tag{3.5}$$

*If $h$ is concave on $(a,b)$, then the inequalities reverse in (3.5).*

*If $w \equiv 1$, then (3.5) becomes the well known Hermite-Hadamard inequality*

$$h\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b h(x)\,dx \leq \frac{h(a)+h(b)}{2}. \tag{3.6}$$

We have the following reverse of Fejér's inequality:

**Corollary 4.** *Let $h:[a,b]\to\mathbb{R}$ be a convex function with $h(b)\neq h(a)$ and $w:[a,b]\to(0,\infty)$ be continuous, symmetrical on $[a,b]$ and such that $\frac{h'}{w^{1/2}}\in L_2[a,b]$. Then*

$$0 \leq \frac{h(a)+h(b)}{2} - \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\,h(t)\,dt \leq \frac{1}{4}\frac{\int_a^b w(s)\,ds}{|h(b)-h(a)|}\int_a^b \frac{[h'(t)]^2}{w(t)}dt. \tag{3.7}$$

*In particular, we have the following reverse of the Hermite-Hadamard inequality*

$$0 \leq \frac{h(a)+h(b)}{2} - \frac{1}{b-a}\int_a^b h(t)\,dt \leq \frac{1}{4}\frac{b-a}{|h(b)-h(a)|}\int_a^b [h'(t)]^2\,dt. \tag{3.8}$$

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