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GENERALIZATIONS OF OPIAL'S INEQUALITIES FOR TWO FUNCTIONS AND APPLICATIONS

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Abstract. In this paper we establish some generalizations of Opial's inequalities for two functions. Applications related to the trapezoid weighted inequalities and to Fejér's inequality for convex functions are also provided. Some Grüss' type inequalities are also given.

1. Introduction

We recall the following Opial type inequalities:

Theorem 1. *Assume that* $u:[a,b]\subset\mathbb{R}\to\mathbb{R}$ *is an absolutely continuous function on the interval* $[a,b]$ *and such that* $u'\in L_2[a,b]$.

(i) *If* $u(a)=u(b)=0$, *then*

$$\int_a^b |u(t)\,u'(t)|\,dt\le\frac{1}{4}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.1}$$

with equality if and only if

$$u(t)=\begin{cases} c(t-a) & \text{if } a\le t\le \frac{a+b}{2},\\ c(b-t) & \text{if } \frac{a+b}{2}<t\le b,\end{cases}$$

where c *is an arbitrary constant;*

(ii) *If* $u(a)=0$, *then*

$$\int_a^b |u(t)\,u'(t)|\,dt\le\frac{1}{2}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.2}$$

with equality if and only if $u(t)=c(t-a)$ *for some constant* c.

The inequality (1.1) was obtained by Olech in [9] in which he gave a simplified proof of an inequality originally due in a slightly less general form to Zdzislaw Opial [10].

Embedded in Olech's proof is the half-interval form of Opial's inequality, also discovered by Beesack [1], which is satisfied by those u vanishing only at a.

For various proofs of the above inequalities, see [5]-[8] and [12].

In this paper we establish some generalizations of Opial's inequalities for two functions. Applications related to the trapezoid weighted inequalities and to Fejér's inequality for convex functions are also provided. Some Grüss' type inequalities are also given.

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2. The Main Results

We have:

Theorem 2. *Assume that* $f,\, g:[a,b]\to\mathbb{C}$ *are absolutely continuous on* $[a,b]$ *with* $f',\, g'\in L_2[a,b]$.

(i) *If* $g(a)=0$, *then*

$$\int_a^b |f'(t)\,g(t)|\,dt \le \left(\int_a^b (t-a)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b (b-t)\,|g'(t)|^2\,dt\right)^{1/2} \le \frac{1}{2}\left[\int_a^b (t-a)\,|f'(t)|^2\,dt+\int_a^b (b-t)\,|g'(t)|^2\,dt\right]. \tag{2.1}$$

(ii) *If* $g(b)=0$, *then*

$$\int_a^b |f'(t)\,g(t)|\,dt \le \left(\int_a^b (b-t)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b (t-a)\,|g'(t)|^2\,dt\right)^{1/2} \le \frac{1}{2}\left[\int_a^b (b-t)\,|f'(t)|^2\,dt+\int_a^b (t-a)\,|g'(t)|^2\,dt\right]. \tag{2.2}$$

(iii) *If* $g(a)=g(b)=0$, *then*

$$\int_a^b |f'(t)\,g(t)|\,dt \le \left(\int_a^b K(t)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b \left|\frac{a+b}{2}-t\right|\,|g'(t)|^2\,dt\right)^{1/2} \le \frac{1}{2}\left[\int_a^b K(t)\,|f'(t)|^2\,dt+\int_a^b \left|\frac{a+b}{2}-t\right|\,|g'(t)|^2\,dt\right], \tag{2.3}$$

where

$$K(t):=\begin{cases} t-a \text{ if } a\le t\le \frac{a+b}{2}, \\ b-t \text{ if } \frac{a+b}{2}<t\le b. \end{cases}$$

Proof. (i) Since $g(a)=0$, then $g(t)=\int_a^t g'(s)\,ds$ for $t\in[a,b]$. We have

$$\int_a^b |f'(t)\,g(t)|\,dt=\int_a^b |f'(t)|\,|g(t)|\,dt=\int_a^b (t-a)^{1/2}\,|f'(t)|\,(t-a)^{-1/2}\,|g(t)|\,dt = \int_a^b (t-a)^{1/2}\,|f'(t)|\,(t-a)^{-1/2}\left|\int_a^t g'(s)\,ds\right|dt=:A.$$

Using Cauchy-Bunyakovsky-Schwarz (CBS) inequality, we have

$$\begin{aligned}
A &\leq \left(\int_a^b \left[(t-a)^{1/2}\left|f'(t)\right|\right]^2 dt\right)^{1/2} \\
&\quad \times \left(\int_a^b \left[(t-a)^{-1/2}\left|\int_a^t g'(s)\,ds\right|\right]^2 dt\right)^{1/2} \\
&= \left(\int_a^b (t-a)\left|f'(t)\right|^2 dt\right)^{1/2}\left(\int_a^b (t-a)^{-1}\left|\int_a^t g'(s)\,ds\right|^2 dt\right)^{1/2} =: B.
\end{aligned} \tag{2.4}$$

By (CBS) inequality we also have

$$(t-a)^{-1}\left|\int_a^t g'(s)\,ds\right|^2 \leq \int_a^t \left|g'(s)\right|^2 ds,$$

which gives

$$B \leq \left(\int_a^b (t-a)\left|f'(t)\right|^2 dt\right)^{1/2}\left(\int_a^b \left(\int_a^t \left|g'(s)\right|^2 ds\right) dt\right)^{1/2}. \tag{2.5}$$

Using integration by parts, we have

$$\begin{aligned}
\int_a^b \left(\int_a^t \left|g'(s)\right|^2 ds\right) dt &= b\int_a^b \left|g'(s)\right|^2 ds - \int_a^b t\left|g'(t)\right|^2 dt \\
&= \int_a^b (b-t)\left|g'(t)\right|^2
\end{aligned}$$

and by (2.4) we get the first inequality in (2.1).

The last part follows by the elementary inequality

$$\sqrt{\alpha\beta} \leq \frac{1}{2}(\alpha+\beta),\ \ \alpha,\beta \geq 0. \tag{2.6}$$

(ii) Since $g(b) = 0$, then $g(t) = -\int_t^b g'(s)\,ds$ for $t \in [a,b]$. We have

$$\begin{aligned}
\int_a^b \left|f'(t)\,g(t)\right| dt &= \int_a^b \left|f'(t)\right|\left|g(t)\right| dt = \int_a^b (b-t)^{1/2}\left|f'(t)\right|(b-t)^{-1/2}\left|g(t)\right| dt \\
&= \int_a^b (b-t)^{1/2}\left|f'(t)\right|(b-t)^{-1/2}\left|\int_t^b g'(s)\,ds\right| dt =: C.
\end{aligned}$$

Using (CBS) inequality we also have

$$\begin{aligned}
C &\leq \left(\int_a^b \left[(b-t)^{1/2}\left|f'(t)\right|\right]^2 dt\right)^{1/2} \\
&\quad \times \left(\int_a^b \left[(b-t)^{-1/2}\left|\int_t^b g'(s)\,ds\right|\right]^2 dt\right)^{1/2} \\
&= \left(\int_a^b (b-t)\left|f'(t)\right|^2 dt\right)^{1/2}\left(\int_a^b (b-t)^{-1}\left|\int_t^b g'(s)\,ds\right|^2 dt\right)^{1/2} =: D.
\end{aligned} \tag{2.7}$$

By (CBS) inequality we also have

$$(b-t)^{-1}\left|\int_t^b g'(s)\,ds\right|^2 \leq \int_t^b |g'(s)|^2\,ds,$$

which gives

$$D \leq \left(\int_a^b (b-t)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b\left(\int_t^b |g'(s)|^2\,ds\right)dt\right)^{1/2}. \tag{2.8}$$

Using integration by parts, we have

$$\begin{aligned}\int_a^b\left(\int_t^b |g'(s)|^2\,ds\right)dt &= -a\int_a^b |g'(s)|^2\,ds_a^b + \int_a^b t\,|g'(t)|^2\,dt\\ &= \int_a^b (t-a)\,|g'(t)|^2\,dt,\end{aligned}$$

and by (2.7) and (2.8) we obtain (2.2).

(iii) If we write the inequality (2.1) on the interval $\left[a, \frac{a+b}{2}\right]$, we have

$$\begin{aligned}&\int_a^{\frac{a+b}{2}} |f'(t)\,g(t)|\,dt\\ &\quad\leq \left(\int_a^{\frac{a+b}{2}} (t-a)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)|g'(t)|^2\,dt\right)^{1/2}\end{aligned} \tag{2.9}$$

and if we write the inequality (2.2) on the interval $\left[\frac{a+b}{2}, b\right]$, we have

$$\begin{aligned}&\int_{\frac{a+b}{2}}^b |f'(t)\,g(t)|\,dt\\ &\quad\leq \left(\int_{\frac{a+b}{2}}^b (b-t)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)|g'(t)|^2\,dt\right)^{1/2}.\end{aligned} \tag{2.10}$$

If we add the inequalities (2.9) and (2.10) we get

$$\begin{aligned}&\int_a^b |f'(t)\,g(t)|\,dt\\ &\leq \left(\int_a^{\frac{a+b}{2}} (t-a)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)|g'(t)|^2\,dt\right)^{1/2}\\ &+\left(\int_{\frac{a+b}{2}}^b (b-t)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)|g'(t)|^2\,dt\right)^{1/2}\end{aligned}$$

$$\leq \left[\int_a^{\frac{a+b}{2}} (t-a)\left|f'(t)\right|^2 dt + \int_{\frac{a+b}{2}}^b (b-t)\left|f'(t)\right|^2 dt\right]^{1/2}$$

$$\times \left[\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-t\right)\left|g'(t)\right|^2 dt + \int_{\frac{a+b}{2}}^b \left(t-\frac{a+b}{2}\right)\left|g'(t)\right|^2 dt\right]^{1/2}$$

$$= \left[\int_a^b K(t)\left|f'(t)\right|^2 dt\right]^{1/2}\left[\int_a^b \left|\frac{a+b}{2}-t\right|\left|g'(t)\right|^2 dt\right]^{1/2},$$

where for the last inequality we used the elementary (CBS) inequality

$$\alpha\beta+\gamma\delta \leq \left(\alpha^2+\gamma^2\right)^{1/2}\left(\beta^2+\delta^2\right)^{1/2},\ \alpha,\ \beta,\ \gamma,\ \delta \geq 0.$$

The last part follows by (2.6). □

We have the following refinement of Opial inequalities (1.1) and (1.2):

Corollary 1. *Assume that $f:[a,b]\to\mathbb{C}$ are absolutely continuous on $[a,b]$ and $f'\in L_2[a,b]$.*

(i) *If either $f(a)=0$ or $f(b)=0$, then*

$$\int_a^b \left|f'(t)f(t)\right| dt \leq \left(\int_a^b (t-a)\left|f'(t)\right|^2 dt\right)^{1/2}\left(\int_a^b (b-t)\left|f'(t)\right|^2 dt\right)^{1/2} \leq \frac{1}{2}(b-a)\int_a^b \left|f'(t)\right|^2 dt. \tag{2.11}$$

(ii) *If $f(a)=f(b)=0$, then*

$$\int_a^b \left|f'(t)f(t)\right| dt \leq \left[\int_a^b K(t)\left|f'(t)\right|^2 dt\right]^{1/2}\left[\int_a^b \left|\frac{a+b}{2}-t\right|\left|f'(t)\right|^2 dt\right]^{1/2} \leq \frac{1}{4}(b-a)\int_a^b \left|f'(t)\right|^2 dt. \tag{2.12}$$

The proof follows by (i) and (ii) of Theorem 2 for $g=f$. The statement (iii) follows by (iii) of Theorem 2 for $g=f$ and observing that

$$K(t)+\left|\frac{a+b}{2}-t\right| = \frac{1}{2}(b-a) \text{ for any } t\in[a,b]. \tag{2.13}$$

Remark 1. *Since*

$$K(t)=\frac{1}{2}(b-a)-\left|\frac{a+b}{2}-t\right|, \textit{ for any } t\in[a,b],$$

then

$$\int_a^b K(t)\left|f'(t)\right|^2 dt + \int_a^b \left|\frac{a+b}{2}-t\right|\left|g'(t)\right|^2 dt$$

$$= \frac{1}{2}(b-a)\int_a^b \left|f'(t)\right|^2 dt + \int_a^b \left|\frac{a+b}{2}-t\right|\left(\left|g'(t)\right|^2-\left|f'(t)\right|^2\right) dt$$

and by (2.3) we get

$$\int_a^b |f'(t) g(t)| dt \leq \frac{1}{4}(b-a) \int_a^b |f'(t)|^2 dt + \frac{1}{2} \int_a^b \left| \frac{a+b}{2} - t \right| \left(|g'(t)|^2 - |f'(t)|^2 \right) dt, \tag{2.14}$$

which shows that, if

$$\int_a^b \left| \frac{a+b}{2} - t \right| |g'(t)|^2 dt \leq \int_a^b \left| \frac{a+b}{2} - t \right| |f'(t)|^2 dt, \tag{2.15}$$

then

$$\int_a^b |f'(t) g(t)| dt \leq \frac{1}{4}(b-a) \int_a^b |f'(t)|^2 dt. \tag{2.16}$$

A sufficient condition for (2.15) to happen is that $|g'(t)| \leq |f'(t)|$ *for a.e.* $t \in [a,b]$.

The following result also holds:

Corollary 2. *Assume that* $f : [a,b] \to \mathbb{C}$ *is absolutely continuous on* $[a,b]$ *with* $f' \in L_2[a,b]$ *and* $h \in L_2[a,b]$ *with* $\int_a^b h(t) dt = 0$. *Then*

$$\begin{aligned} \left| \int_a^b f(t) h(t) dt \right| &\leq \left(\int_a^b K(t) |f'(t)|^2 dt \right)^{1/2} \left(\int_a^b \left| \frac{a+b}{2} - t \right| |h(t)|^2 dt \right)^{1/2} \\ &\leq \frac{1}{2} \left[\int_a^b K(t) |f'(t)|^2 dt + \int_a^b \left| \frac{a+b}{2} - t \right| |h(t)|^2 dt \right]. \end{aligned} \tag{2.17}$$

Proof. If we take in (2.3) $g(t) = \int_a^t h(s) ds$, $t \in [a,b]$, then we get

$$\begin{aligned} \int_a^b \left| f'(t) \int_a^t h(s) ds \right| dt &\leq \left(\int_a^b K(t) |f'(t)|^2 dt \right)^{1/2} \left(\int_a^b \left| \frac{a+b}{2} - t \right| |h(t)|^2 dt \right)^{1/2} \\ &\leq \frac{1}{2} \left[\int_a^b K(t) |f'(t)|^2 dt + \int_a^b \left| \frac{a+b}{2} - t \right| |h(t)|^2 dt \right]. \end{aligned} \tag{2.18}$$

Also, by the modulus properties and integrating by parts, we have

$$\begin{aligned} \int_a^b \left| f'(t) \int_a^t h(s) ds \right| dt &\geq \left| \int_a^b f'(t) \left(\int_a^t h(s) ds \right) dt \right| \\ &= \left| f(t) \int_a^t h(s) ds \Big|_a^b - \int_a^b f(t) h(t) dt \right| = \left| \int_a^b f(t) h(t) dt \right|. \end{aligned} \tag{2.19}$$

By making use of (2.18) and (2.19) we get the desired result (2.17). $\square$

Remark 2. *If $f:[a,b]\to\mathbb{C}$ is absolutely continuous on $[a,b]$ with $\int_a^b f(t)\,dt=0$ and $f'\in L_2[a,b]$, then by taking $h=\bar{f}$ in (2.17) we get*

$$\int_a^b |f(t)|^2\,dt \le \left(\int_a^b K(t)|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b \left|\frac{a+b}{2}-t\right||f(t)|^2\,dt\right)^{1/2} \le \frac{1}{2}\left[\int_a^b K(t)|f'(t)|^2\,dt+\int_a^b \left|\frac{a+b}{2}-t\right||f(t)|^2\,dt\right]. \tag{2.20}$$

Since

$$\int_a^b \left|\frac{a+b}{2}-t\right||f(t)|^2\,dt \le \max_{t\in[a,b]}\left|\frac{a+b}{2}-t\right|\int_a^b |f(t)|^2\,dt = \frac{1}{2}(b-a)\int_a^b |f(t)|^2\,dt,$$

then by first inequality in (2.20) we get

$$\left(\int_a^b |f(t)|^2\,dt\right)^2 \le \int_a^b K(t)|f'(t)|^2\,dt\int_a^b \left|\frac{a+b}{2}-t\right||f(t)|^2\,dt \le \frac{1}{2}(b-a)\int_a^b K(t)|f'(t)|^2\,dt\int_a^b |f(t)|^2\,dt,$$

which gives that

$$\int_a^b |f(t)|^2\,dt \le \frac{1}{2}(b-a)\int_a^b K(t)|f'(t)|^2\,dt. \tag{2.21}$$

Also, since

$$\left(\int_a^b \left|\frac{a+b}{2}-t\right||f(t)|^2\,dt\right)^{1/2} \le \|f\|_{\infty,[a,b]}\left(\int_a^b \left|\frac{a+b}{2}-t\right|dt\right)^{1/2} = \frac{1}{2}(b-a)\|f\|_{\infty,[a,b]},$$

then by the first inequality in (2.20) we get

$$\int_a^b |f(t)|^2\,dt \le \frac{1}{2}(b-a)\|f\|_{\infty,[a,b]}\left(\int_a^b K(t)|f'(t)|^2\,dt\right)^{1/2}. \tag{2.22}$$

Corollary 3. *If $g(a)=g(b)=0$ and $h,\ g'\in L_2[a,b]$, then*

$$\int_a^b |h(t)g(t)|\,dt \le \left(\int_a^b K(t)|h(t)|^2\,dt\right)^{1/2}\left(\int_a^b \left|\frac{a+b}{2}-t\right||g'(t)|^2\,dt\right)^{1/2} \le \frac{1}{2}\left[\int_a^b K(t)|h(t)|^2\,dt+\int_a^b \left|\frac{a+b}{2}-t\right||g'(t)|^2\,dt\right]. \tag{2.23}$$

The proof follows by the statement (iii) of Theorem 2 for $f=\int_a h(s)\,ds$.

3. Some Trapezoid Type Inequalities

We have:

Proposition 1. *Let $h:[a,b]\to\mathbb{C}$ be absolutely continuous on $[a,b]$ with $h'\in L_2[a,b]$ and $w:[a,b]\to\mathbb{C}$ such that $w\in L_2[a,b]$, then*

$$\left|\int_a^b \frac{w(t)+w(a+b-t)}{2}h(t)\,dt-\frac{h(a)+h(b)}{2}\int_a^b w(t)\,dt\right| \tag{3.1}$$
$$\leq\frac{1}{2}\left(\int_a^b K(t)\left|w(t)\right|^2dt\right)^{1/2}\left(\int_a^b\left|\frac{a+b}{2}-t\right|\left|h'(t)-h'(a+b-t)\right|^2dt\right)^{1/2}.$$

Moreover, if w is symmetrical, namely $w(a+b-t)=w(t)$ for all $t\in[a,b]$, then

$$\left|\int_a^b w(t)\,h(t)\,dt-\frac{h(a)+h(b)}{2}\int_a^b w(t)\,dt\right| \tag{3.2}$$
$$\leq\frac{1}{2}\left(\int_a^b K(t)\left|w(t)\right|^2dt\right)^{1/2}\left(\int_a^b\left|\frac{a+b}{2}-t\right|\left|h'(t)-h'(a+b-t)\right|^2dt\right)^{1/2}.$$

Proof. Consider the function $g:[a,b]\to\mathbb{C}$ defined by

$$g(t):=\frac{h(t)+h(a+b-t)}{2}-\frac{h(a)+h(b)}{2},\ t\in[a,b].$$

We have $g(a)=g(b)=0$.

If we write the inequality (2.3) for $f=\int_a w(t)\,dt$, then we get

$$\int_a^b\left|w(t)\left[\frac{h(t)+h(a+b-t)}{2}-\frac{h(a)+h(b)}{2}\right]\right|dt \tag{3.3}$$
$$\leq\left(\int_a^b K(t)\left|w(t)\right|^2dt\right)^{1/2}\left(\int_a^b\left|\frac{a+b}{2}-t\right|\left|\frac{h'(t)-h'(a+b-t)}{2}\right|^2dt\right)^{1/2}$$
$$=\frac{1}{2}\left(\int_a^b K(t)\left|w(t)\right|^2dt\right)^{1/2}\left(\int_a^b\left|\frac{a+b}{2}-t\right|\left|h'(t)-h'(a+b-t)\right|^2dt\right)^{1/2}.$$

By the modulus property, we have

$$\int_a^b\left|w(t)\left[\frac{h(t)+h(a+b-t)}{2}-\frac{h(a)+h(b)}{2}\right]\right|dt \tag{3.4}$$
$$\geq\left|\int_a^b w(t)\left[\frac{h(t)+h(a+b-t)}{2}-\frac{h(a)+h(b)}{2}\right]dt\right|$$
$$=\left|\frac{1}{2}\left[\int_a^b w(t)\,h(t)\,dt+\int_a^b w(t)\,h(a+b-t)\,dt\right]-\frac{h(a)+h(b)}{2}\int_a^b w(t)\,dt\right|.$$

By the change of variable $u=a+b-t$, $t\in[a,b]$, we have

$$\int_a^b w(t)\,h(a+b-t)\,dt=\int_a^b w(a+b-t)\,h(t)\,dt$$

and then by (3.3) and (3.4) we get the desired result (3.1). □

Corollary 4. *With the assumptions of Proposition 1 and if h' is Lipschitzian with constant $L>0$, namely $|h'(t)-h'(s)| \le L|t-s|$ for any $t,s \in [a,b]$, then*

$$\left|\int_a^b \frac{w(t)+w(a+b-t)}{2} h(t)\,dt - \frac{h(a)+h(b)}{2}\int_a^b w(t)\,dt\right| \le \frac{\sqrt{2}}{8}(b-a)^2 L\left(\int_a^b K(t)|w(t)|^2\,dt\right)^{1/2}, \tag{3.5}$$

where $w \in L_2[a,b]$.

In the case of symmetry for w, we have

$$\left|\int_a^b w(t)h(t)\,dt - \frac{h(a)+h(b)}{2}\int_a^b w(t)\,dt\right| \le \frac{\sqrt{2}}{8}(b-a)^2 L\left(\int_a^b K(t)|w(t)|^2\,dt\right)^{1/2}. \tag{3.6}$$

In 1906, Fejér [3], while studying trigonometric polynomials, obtained the following inequalities which generalize that of Hermite & Hadamard:

Theorem 3 (Fejér's Inequality)**.** *Consider the integral $\int_a^b h(x)w(x)\,dx$, where h is a convex function in the interval (a,b) and w is a positive function in the same interval such that*

$$w(x) = w(a+b-x), \text{ for any } x \in [a,b]$$

i.e., $y = w(x)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b),0\right)$ and is normal to the x-axis. Under those conditions the following inequalities are valid:

$$h\left(\frac{a+b}{2}\right) \le \frac{1}{\int_a^b w(x)\,dx}\int_a^b h(x)w(x)\,dx \le \frac{h(a)+h(b)}{2}. \tag{3.7}$$

If h is concave on (a,b), then the inequalities reverse in (3.7).

If $w \equiv 1$, then (3.7) becomes the well known Hermite-Hadamard inequality

$$h\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b h(x)\,dx \le \frac{h(a)+h(b)}{2}. \tag{3.8}$$

We have the following reverse of Fejér's inequality:

Corollary 5. *Let $h:[a,b]\to\mathbb{R}$ be a convex function and $w:[a,b]\to(0,\infty)$ be continuous, symmetrical on $[a,b]$ and such that $h' \in L_2[a,b]$. Then*

$$0 \le \frac{h(a)+h(b)}{2} - \frac{1}{\int_a^b w(x)\,dx}\int_a^b h(x)w(x)\,dx \le \frac{1}{2}\left(\int_a^b K(t)|w(t)|^2\,dt\right)^{1/2}\left(\int_a^b \left|\frac{a+b}{2}-t\right| |h'(t)-h'(a+b-t)|^2\,dt\right)^{1/2}. \tag{3.9}$$

Moreover, if h' is L-Lipschitzian, then

$$0 \leq \frac{h(a)+h(b)}{2} - \frac{1}{\int_a^b w(x)\,dx} \int_a^b h(x)\,w(x)\,dx \leq \frac{\sqrt{2}}{8}(b-a)^2 L \left(\int_a^b K(t) |w(t)|^2 dt \right)^{1/2}. \tag{3.10}$$

We also have:

Proposition 2. *Let $h : [a,b] \to \mathbb{C}$ be absolutely continuous on $[a,b]$ with $h' \in L_2[a,b]$ and $w : [a,b] \to \mathbb{C}$ such that $w \in L_2[a,b]$, then*

$$\left| \frac{\left[h(a)\left(b\int_a^b w(t)\,dt - \int_a^b w(t)\,tdt \right) + h(b)\left(\int_a^b w(t)\,tdt - a\int_a^b w(t)\,dt \right) \right]}{b-a} - \int_a^b w(t)\,h(t)\,dt \right| \leq \left(\int_a^b K(t) |w(t)|^2 dt \right)^{1/2} \left(\int_a^b \left| \frac{a+b}{2} - t \right| \left| h'(t) - \frac{h(b)-h(a)}{b-a} \right|^2 dt \right)^{1/2}. \tag{3.11}$$

Proof. Consider the function $g : [a,b] \to \mathbb{C}$ defined by

$$g(t) := h(t) - \frac{h(a)(b-t) + h(b)(t-a)}{b-a}, \ t \in [a,b].$$

We have $g(a) = g(b) = 0$.

If we write the inequality (2.3) for $f = \int_a w(t)\,dt$, then we get

$$\int_a^b \left| w(t) \left[h(t) - \frac{h(a)(b-t) + h(b)(t-a)}{b-a} \right] \right| dt \leq \left(\int_a^b K(t) |w(t)|^2 dt \right)^{1/2} \left(\int_a^b \left| \frac{a+b}{2} - t \right| \left| h'(t) - \frac{h(b)-h(a)}{b-a} \right|^2 dt \right)^{1/2}. \tag{3.12}$$

By the modulus property, we have

$$\begin{aligned} &\int_a^b \left| w(t) \left[h(t) - \frac{h(a)(b-t) + h(b)(t-a)}{b-a} \right] \right| dt \\ &\geq \left| \int_a^b w(t) \left[h(t) - \frac{h(a)(b-t) + h(b)(t-a)}{b-a} \right] dt \right| \\ &= \left| \int_a^b w(t)\,h(t)\,dt - \frac{h(a)\left(b\int_a^b w(t)\,dt - \int_a^b w(t)\,tdt \right) + h(b)\left(\int_a^b w(t)\,tdt - a\int_a^b w(t)\,dt \right)}{b-a} \right|, \end{aligned}$$

which together with (3.12) produces the desired result (3.11). □

Corollary 6. *Let $h : [a,b] \to \mathbb{R}$ be a convex function and $w : [a,b] \to (0,\infty)$ be continuous and such that $h' \in L_2[a,b]$. Then*

$$0 \leq \frac{h(a)[b - E(w,[a,b])] + h(b)[E(w,[a,b]) - a]}{b-a} - \int_a^b w(t)h(t)\,dt \tag{3.13}$$

$$\leq \frac{1}{\int_a^b w(t)\,dt}\left(\int_a^b K(t)|w(t)|^2\,dt\right)^{1/2}\left(\int_a^b \left|\frac{a+b}{2} - t\right|\left|h'(t) - \frac{h(b)-h(a)}{b-a}\right|^2 dt\right)^{1/2},$$

where

$$E(w,[a,b]) := \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\,tdt.$$

4. Some Grüss' Type Inequalities

For two *Lebesgue integrable* functions $f,\ g : [a,b] \to \mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt. \tag{4.1}$$

In 1935, Grüss [4] showed that

$$|C(f,g)| \leq \frac{1}{4}(M-m)(N-n), \tag{4.2}$$

provided that there exists the real numbers m, M, n, N such that

$$m \leq f(t) \leq M \quad \text{and} \quad n \leq g(t) \leq N \quad \text{for a.e. } t \in [a,b]. \tag{4.3}$$

The constant $\frac{1}{4}$ is best possible in (4.2) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known result, even though it was obtained by Čebyšev in 1882, [2], states that

$$|C(f,g)| \leq \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2, \tag{4.4}$$

provided that f', g' exist and are continuous on $[a,b]$ and $\|f'\|_\infty = \sup_{t\in[a,b]}|f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (4.4) also holds if $f,\ g : [a,b] \to \mathbb{R}$ are assumed to be *absolutely continuous* and $f',\ g' \in L_\infty[a,b]$ while $\|f'\|_\infty = \operatorname{essup}_{t\in[a,b]}|f'(t)|$.

A mixture between Grüss' result (3.7) and Čebyšev's one (4.4) is the following inequality obtained by Ostrowski in 1970, [11]:

$$|C(f,g)| \leq \frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{4.5}$$

provided that f is *Lebesgue integrable* and satisfies (3.8) while g is absolutely continuous and $g' \in L_\infty[a,b]$. The constant $\frac{1}{8}$ is best possible in (4.5).

The case of *euclidean norms* of the derivative was considered by A. Lupaş in [7] in which he proved that

$$|C(f,g)| \leq \frac{1}{\pi^2}\|f'\|_2\|g'\|_2(b-a), \tag{4.6}$$

provided that f, g are absolutely continuous and $f',\ g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Consider

$$K(t) := \begin{cases} t-a \text{ if } a \le t \le \frac{a+b}{2}, \\ \\ b-t \text{ if } \frac{a+b}{2} < t \le b. \end{cases} = \frac{1}{2}(b-a) - \left|\frac{a+b}{2} - t\right|,$$

for $t \in [a,b]$.

We have:

Theorem 4. *If $f,\ g : [a,b] \to \mathbb{C}$ are such that f is absolutely continuous with $f' \in L_2[a,b]$ and $g \in L_2[a,b]$, then*

$$\begin{aligned} (4.7)\quad & |C(f,g)| \\ & \le \left(\int_a^b K(t)\,|f'(t)|^2\,dt\right)^{1/2} \left(\int_a^b \left|\frac{a+b}{2} - t\right| \left|g(t) - \frac{1}{b-a}\int_a^b g(s)ds\right|^2 dt\right)^{1/2} \\ & \le \left(\int_a^b K(t)\,|f'(t)|^2\,dt\right)^{1/2} B(g), \end{aligned}$$

where

$$B(g) := \begin{cases} \frac{\sqrt{2}}{2}(b-a)\left(\frac{1}{b-a}\int_a^b |g(t)|^2\,dt - \left|\frac{1}{b-a}\int_a^b g(t)\,dt\right|^2\right)^{1/2}, \\ \\ \frac{1}{2}(b-a)\left\|g - \frac{1}{b-a}\int_a^b g(s)ds\right\|_{[a,b],\infty} \quad \text{if } g \in L_\infty[a,b]. \end{cases}$$

Proof. We have the following Sonin identity

$$C(f,g) = \frac{1}{b-a}\int_a^b (f(t) - \gamma)\left(g(t) - \frac{1}{b-a}\int_a^b g(s)ds\right) dt \tag{4.8}$$

for any $\gamma \in \mathbb{C}$, that can be easily proved by developing the right hand side of (4.8).

Observe that, if we take $h(t) = g(t) - \frac{1}{b-a}\int_a^b g(s)ds$, then we have $\int_a^b h(t)\,dt = 0$ and by Corollary 2 we get

$$\begin{aligned} & |C(f,g)| \\ & \le \left(\int_a^b K(t)\,|f'(t)|^2\,dt\right)^{1/2} \left(\int_a^b \left|\frac{a+b}{2} - t\right| \left|g(t) - \frac{1}{b-a}\int_a^b g(s)ds\right|^2 dt\right)^{1/2}. \end{aligned}$$

Observe that

$$\begin{aligned} & \left(\int_a^b \left|\frac{a+b}{2} - t\right| \left|g(t) - \frac{1}{b-a}\int_a^b g(s)ds\right|^2 dt\right)^{1/2} \\ & \le \max_{t\in[a,b]}\left|\frac{a+b}{2} - t\right|^{1/2} (b-a)^{1/2} \left(\frac{1}{b-a}\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)ds\right|^2 dt\right)^{1/2} \\ & = \frac{\sqrt{2}}{2}(b-a)\left(\frac{1}{b-a}\int_a^b |g(t)|^2\,dt - \left|\frac{1}{b-a}\int_a^b g(t)\,dt\right|^2\right)^{1/2}, \end{aligned}$$

which proves the first branch in the second inequality in (4.7).

We also have

$$\left(\int_a^b \left|\frac{a+b}{2}-t\right| \left|g\left(t\right)-\frac{1}{b-a}\int_a^b g(s)ds\right|^2 dt\right)^{1/2}$$
$$\leq \frac{1}{2}\left(b-a\right)\left\|g-\frac{1}{b-a}\int_a^b g(s)ds\right\|_{[a,b],\infty},$$

which proves the second branch in the second inequality in (4.7). □

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