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p-NORMS GENERALIZATIONS OF OPIAL'S INEQUALITIES FOR TWO FUNCTIONS AND APPLICATIONS

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Abstract. In this paper we establish some p-norms generalizations of Opial's inequalities for two functions. Applications related to the trapezoid weighted inequalities and to Fejér's inequality for convex functions are also provided. Some Grüss' type inequalities for p-norms are given as well.

1. Introduction

We recall the following Opial type inequalities:

Theorem 1. *Assume that* $u:[a,b]\subset\mathbb{R}\to\mathbb{R}$ *is an absolutely continuous function on the interval* $[a,b]$ *and such that* $u'\in L_2[a,b]$.

(i) *If* $u(a)=u(b)=0$, *then*

$$\int_a^b |u(t)\,u'(t)|\,dt\le \frac{1}{4}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.1}$$

with equality if and only if

$$u(t)=\begin{cases} c(t-a) & \text{if } a\le t\le \frac{a+b}{2},\\ c(b-t) & \text{if } \frac{a+b}{2}<t\le b,\end{cases}$$

where c *is an arbitrary constant;*

(ii) *If* $u(a)=0$, *then*

$$\int_a^b |u(t)\,u'(t)|\,dt\le \frac{1}{2}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.2}$$

with equality if and only if $u(t)=c(t-a)$ *for some constant* c.

The inequality (1.1) was obtained by Olech in [10] in which he gave a simplified proof of an inequality originally due in a slightly less general form to Zdzislaw Opial [11].

Embedded in Olech's proof is the half-interval form of Opial's inequality, also discovered by Beesack [1], which is satisfied by those u vanishing only at a.

For various proofs of the above inequalities, see [6]-[9] and [13].

In the recent paper [3] we obtained the following generalization of Opial's inequalities for two functions:

Theorem 2. *Assume that* $f,\ g:[a,b]\to\mathbb{C}$ *are absolutely continuous on* $[a,b]$ *with* $f',\ g'\in L_2[a,b]$.

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(i) *If* $g(a)=0$, *then*

$$\int_a^b |f'(t)g(t)|\,dt \le \left(\int_a^b (t-a)|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b (b-t)|g'(t)|^2\,dt\right)^{1/2} \le \frac{1}{2}\left[\int_a^b (t-a)|f'(t)|^2\,dt + \int_a^b (b-t)|g'(t)|^2\,dt\right]. \tag{1.3}$$

(ii) *If* $g(b)=0$, *then*

$$\int_a^b |f'(t)g(t)|\,dt \le \left(\int_a^b (b-t)|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b (t-a)|g'(t)|^2\,dt\right)^{1/2} \le \frac{1}{2}\left[\int_a^b (b-t)|f'(t)|^2\,dt + \int_a^b (t-a)|g'(t)|^2\,dt\right]. \tag{1.4}$$

(iii) *If* $g(a)=g(b)=0$, *then*

$$\int_a^b |f'(t)g(t)|\,dt \le \left(\int_a^b K(t)|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b \left|\frac{a+b}{2}-t\right||g'(t)|^2\,dt\right)^{1/2} \le \frac{1}{2}\left[\int_a^b K(t)|f'(t)|^2\,dt + \int_a^b \left|\frac{a+b}{2}-t\right||g'(t)|^2\,dt\right], \tag{1.5}$$

where

$$K(t) := \begin{cases} t-a \text{ if } a\le t\le \frac{a+b}{2}, \\ b-t \text{ if } \frac{a+b}{2} < t \le b. \end{cases}$$

In this paper we establish some p-norms generalizations of Opial's inequalities for two functions. Applications related to the trapezoid weighted inequalities and to Fejér's inequality for convex functions are also provided. Some Grüss' type inequalities for p-norms are given as well.

2. The Main Results

We have the following natural generalization of Theorem 2:

Theorem 3. *Assume that* $f, g : [a,b] \to \mathbb{C}$ *are absolutely continuous on* $[a,b]$ *with* $f' \in L_p[a,b]$ *and* $g' \in L_q[a,b]$ *for* $p, q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

(i) *If* $g(a)=0$, *then*

$$\int_a^b |f'(t)g(t)|\,dt \le \left(\int_a^b (t-a)|f'(t)|^p\,dt\right)^{1/p}\left(\int_a^b (b-t)|g'(t)|^q\,dt\right)^{1/q} \le \frac{1}{p}\int_a^b (t-a)|f'(t)|^p\,dt + \frac{1}{q}\int_a^b (b-t)|g'(t)|^q\,dt. \tag{2.1}$$

(ii) *If* $g(b)=0$, *then*

$$\int_a^b |f'(t)\,g(t)|\,dt \le \left(\int_a^b (b-t)\,|f'(t)|^p\,dt\right)^{1/p}\left(\int_a^b (t-a)\,|g'(t)|^q\,dt\right)^{1/q} \le \frac{1}{p}\int_a^b (b-t)\,|f'(t)|^p\,dt + \frac{1}{q}\int_a^b (t-a)\,|g'(t)|^q\,dt. \tag{2.2}$$

(iii) *If* $g(a)=g(b)=0$, *then*

$$\int_a^b |f'(t)\,g(t)|\,dt \le \left(\int_a^b K(t)\,|f'(t)|^p\,dt\right)^{1/p}\left(\int_a^b \left|\frac{a+b}{2}-t\right||g'(t)|^q\,dt\right)^{1/q} \le \frac{1}{p}\int_a^b K(t)\,|f'(t)|^p\,dt + \frac{1}{q}\int_a^b \left|\frac{a+b}{2}-t\right||g'(t)|^q\,dt, \tag{2.3}$$

where K *is defined in Theorem 2.*

Proof. (i) Since $g(a)=0$, then $g(t)=\int_a^t g'(s)\,ds$ for $t\in[a,b]$. We have

$$\int_a^b |f'(t)\,g(t)|\,dt = \int_a^b |f'(t)|\,|g(t)|\,dt = \int_a^b (t-a)^{1/p}\,|f'(t)|\,(t-a)^{-1/p}\,|g(t)|\,dt = \int_a^b (t-a)^{1/p}\,|f'(t)|\,(t-a)^{-1/p}\left|\int_a^t g'(s)\,ds\right|dt =: A.$$

Using Hölder's inequality for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have

$$A \le \left(\int_a^b \left[(t-a)^{1/p}\,|f'(t)|\right]^p dt\right)^{1/p} \times \left(\int_a^b \left[(t-a)^{-1/p}\left|\int_a^t g'(s)\,ds\right|\right]^q dt\right)^{1/q} = \left(\int_a^b (t-a)\,|f'(t)|^p\,dt\right)^{1/p}\left(\int_a^b \left[(t-a)^{-1/p}\left|\int_a^t g'(s)\,ds\right|\right]^q dt\right)^{1/q} =: B. \tag{2.4}$$

By Hölder's inequality for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ we also have

$$(t-a)^{-1/p}\left|\int_a^t g'(s)\,ds\right| \le \left(\int_a^t |g'(s)|^q\,ds\right)^{1/q}$$

that implies

$$\left[(t-a)^{-1/p}\left|\int_a^t g'(s)\,ds\right|\right]^q \le \int_a^t |g'(s)|^q\,ds,$$

which gives

$$B \le \left(\int_a^b (t-a)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b \left(\int_a^t |g'(s)|^q\,ds\right)dt\right)^{1/2}. \tag{2.5}$$

Using integration by parts, we have

$$\int_a^b \left(\int_a^t |g'(s)|^q \, ds \right) dt = \int_a^b (b-t) |g'(t)|^q$$

and by (2.4) we get the first inequality in (2.1).

The last part follows by the elementary *Young's inequality*

$$\alpha^{1/p}\beta^{1/q} \le \frac{1}{p}\alpha + \frac{1}{q}\beta, \ \alpha, \ \beta \ge 0. \tag{2.6}$$

(ii) Since $g(b) = 0$, then $g(t) = -\int_t^b g'(s) \, ds$ for $t \in [a,b]$. We have

$$\begin{aligned} \int_a^b |f'(t) g(t)| \, dt &= \int_a^b |f'(t)| |g(t)| \, dt = \int_a^b (b-t)^{1/p} |f'(t)| (b-t)^{-1/p} |g(t)| \, dt \\ &= \int_a^b (b-t)^{1/p} |f'(t)| (b-t)^{-1/p} \left| \int_t^b g'(s) \, ds \right| dt =: C. \end{aligned}$$

Using Hölder's inequality for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we also have

$$\begin{aligned} C &\le \left(\int_a^b \left[(b-t)^{1/p} |f'(t)| \right]^p dt \right)^{1/p} \\ &\quad \times \left(\int_a^b \left[(b-t)^{-1/p} \left| \int_t^b g'(s) \, ds \right| \right]^q dt \right)^{1/q} \\ &= \left(\int_a^b (b-t) |f'(t)|^p \, dt \right)^{1/p} \left(\left[(b-t)^{-1/p} \left| \int_t^b g'(s) \, ds \right| \right]^q dt \right)^{1/2} =: D. \end{aligned} \tag{2.7}$$

By Hölder's inequality for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we also have

$$(b-t)^{-1/p} \left| \int_t^b g'(s) \, ds \right| \le \left(\int_t^b |g'(s)|^q \, ds \right)^{1/q},$$

which gives

$$D \le \left(\int_a^b (b-t) |f'(t)|^2 \, dt \right)^{1/2} \left(\int_a^b \left(\int_t^b |g'(s)|^q \, ds \right) dt \right)^{1/2}. \tag{2.8}$$

Using integration by parts, we have

$$\int_a^b \left(\int_t^b |g'(s)|^q \, ds \right) dt = \int_a^b (t-a) |g'(t)|^q \, dt,$$

and by (2.7) and (2.8) we obtain (2.2).

(iii) If we write the inequality (2.1) on the interval $\left[a, \frac{a+b}{2}\right]$, we have

$$\begin{aligned} &\int_a^{\frac{a+b}{2}} |f'(t) g(t)| \, dt \\ &\le \left(\int_a^{\frac{a+b}{2}} (t-a) |f'(t)|^p \, dt \right)^{1/p} \left(\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t \right) |g'(t)|^q \, dt \right)^{1/q} \end{aligned} \tag{2.9}$$

and if we write the inequality (2.2) on the interval $\left[\frac{a+b}{2}, b\right]$, we have

$$\int_{\frac{a+b}{2}}^{b} |f'(t)\, g(t)|\, dt \tag{2.10}$$
$$\leq \left(\int_{\frac{a+b}{2}}^{b} (b-t)\, |f'(t)|^p\, dt\right)^{1/p} \left(\int_{\frac{a+b}{2}}^{b} \left(t - \frac{a+b}{2}\right) |g'(t)|^q\, dt\right)^{1/q}.$$

If we add the inequalities (2.9) and (2.10) we get

$$\begin{aligned}
&\int_a^b |f'(t)\, g(t)|\, dt \\
&\leq \left(\int_a^{\frac{a+b}{2}} (t-a)\, |f'(t)|^p\, dt\right)^{1/p} \left(\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right) |g'(t)|^q\, dt\right)^{1/q} \\
&+ \left(\int_{\frac{a+b}{2}}^{b} (b-t)\, |f'(t)|^p\, dt\right)^{1/p} \left(\int_{\frac{a+b}{2}}^{b} \left(t - \frac{a+b}{2}\right) |g'(t)|^q\, dt\right)^{1/q} \\
&\leq \left[\int_a^{\frac{a+b}{2}} (t-a)\, |f'(t)|^p\, dt + \int_{\frac{a+b}{2}}^{b} (b-t)\, |f'(t)|^p\, dt\right]^{1/p} \\
&\times \left[\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right) |g'(t)|^q\, dt + \int_{\frac{a+b}{2}}^{b} \left(t - \frac{a+b}{2}\right) |g'(t)|^q\, dt\right]^{1/q} \\
&= \left[\int_a^b K(t)\, |f'(t)|^p\, dt\right]^{1/p} \left[\int_a^b \left|\frac{a+b}{2} - t\right| |g'(t)|^q\, dt\right]^{1/q},
\end{aligned}$$

where for the last inequality we used the elementary Hölder inequality for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$\alpha\beta + \gamma\delta \leq (\alpha^p + \gamma^p)^{1/p} (\beta^q + \delta^q)^{1/q},\ \alpha,\ \beta,\ \gamma,\ \delta \geq 0.$$

The last part follows by (2.6). □

Remark 1. *If we take $p = q = 2$ in Theorem 3, then we get Theorem 2.*

Corollary 1. *Assume that $f : [a,b] \to \mathbb{C}$ are absolutely continuous on $[a,b]$ and $f' \in L_p[a,b] \cap L_q[a,b]$ for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

(i) *If $f(a) = 0$, then*

$$\int_a^b |f'(t)\, f(t)|\, dt \leq \left(\int_a^b (t-a)\, |f'(t)|^p\, dt\right)^{1/p} \left(\int_a^b (b-t)\, |f'(t)|^q\, dt\right)^{1/q} \tag{2.11}$$
$$\leq \frac{1}{p} \int_a^b (t-a)\, |f'(t)|^p\, dt + \frac{1}{q} \int_a^b (b-t)\, |f'(t)|^q\, dt.$$

(ii) *If* $f(b) = 0$, *then*

$$\int_a^b |f'(t) g(t)| \, dt \le \left(\int_a^b (b-t) |f'(t)|^p \, dt \right)^{1/p} \left(\int_a^b (t-a) |f'(t)|^q \, dt \right)^{1/q} \le \frac{1}{p} \int_a^b (b-t) |f'(t)|^p \, dt + \frac{1}{q} \int_a^b (t-a) |f'(t)|^q \, dt. \tag{2.12}$$

(iii) *If* $f(a) = f(b) = 0$, *then*

$$\int_a^b |f'(t) f(t)| \, dt \le \left(\int_a^b K(t) |f'(t)|^p \, dt \right)^{1/p} \left(\int_a^b \left| \frac{a+b}{2} - t \right| |f'(t)|^q \, dt \right)^{1/q} \le \frac{1}{p} \int_a^b K(t) |f'(t)|^p \, dt + \frac{1}{q} \int_a^b \left| \frac{a+b}{2} - t \right| |f'(t)|^q \, dt. \tag{2.13}$$

Remark 2. *If we take in Corollary 1* $p = q = 2$, *then we get the refinement of Opial's inequality (2.1)*

$$\int_a^b |f'(t) f(t)| \, dt \le \left(\int_a^b (t-a) |f'(t)|^2 \, dt \right)^{1/2} \left(\int_a^b (b-t) |f'(t)|^2 \, dt \right)^{1/2} \le \frac{1}{2} (b-a) \int_a^b |f'(t)|^2 \, dt, \tag{2.14}$$

if either $f(a) = 0$ *or* $f(b) = 0$.

If $f(a) = f(b) = 0$, *then we have the refinement of (1.1)*

$$\int_a^b |f'(t) f(t)| \, dt \le \left[\int_a^b K(t) |f'(t)|^2 \, dt \right]^{1/2} \left[\int_a^b \left| \frac{a+b}{2} - t \right| |f'(t)|^2 \, dt \right]^{1/2} \le \frac{1}{4} (b-a) \int_a^b |f'(t)|^2 \, dt. \tag{2.15}$$

Corollary 2. *Assume that* $f : [a,b] \to \mathbb{C}$ *is absolutely continuous on* $[a,b]$ *with* $f' \in L_p[a,b]$ *and* $h \in L_q[a,b]$ *with* $\int_a^b h(t) \, dt = 0$ *for* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\left| \int_a^b f(t) h(t) \, dt \right| \le \left(\int_a^b K(t) |f'(t)|^p \, dt \right)^{1/p} \left(\int_a^b \left| \frac{a+b}{2} - t \right| |h(t)|^q \, dt \right)^{1/q} \le \frac{1}{p} \int_a^b K(t) |f'(t)|^p \, dt + \frac{1}{q} \int_a^b \left| \frac{a+b}{2} - t \right| |h(t)|^q \, dt. \tag{2.16}$$

Proof. If we take in (2.3) $g(t)=\int_a^t h(s)\,ds,\ t\in[a,b]$, then we get

$$\begin{aligned}(2.17)\quad &\int_a^b \left|f'(t)\int_a^t h(s)\,ds\right| dt\\ &\le \left(\int_a^b K(t)\left|f'(t)\right|^p dt\right)^{1/p}\left(\int_a^b \left|\frac{a+b}{2}-t\right|\left|h(t)\right|^q dt\right)^{1/q}\\ &\le \frac{1}{p}\int_a^b K(t)\left|f'(t)\right|^p dt+\frac{1}{q}\int_a^b \left|\frac{a+b}{2}-t\right|\left|h(t)\right|^q dt.\end{aligned}$$

Also, by the modulus properties and integrating by parts, we have

$$\begin{aligned}(2.18)\quad &\int_a^b \left|f'(t)\int_a^t h(s)\,ds\right| dt\ge\left|\int_a^b f'(t)\left(\int_a^t h(s)\,ds\right)dt\right|\\ &=\left|f(t)\int_a^t h(s)\,ds\bigg|_a^b-\int_a^b f(t)\,h(t)\,dt\right|=\left|\int_a^b f(t)\,h(t)\,dt\right|.\end{aligned}$$

By making use of (2.17) and (2.18) we get the desired result (2.16). □

Corollary 3. *If $g(a)=g(b)=0$ and $h\in L_p[a,b]$, $g'\in L_q[a,b]$ for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\begin{aligned}(2.19)\quad &\int_a^b \left|h(t)\,g(t)\right| dt\\ &\le \left(\int_a^b K(t)\left|h(t)\right|^p dt\right)^{1/p}\left(\int_a^b \left|\frac{a+b}{2}-t\right|\left|g'(t)\right|^q dt\right)^{1/q}\\ &\le \frac{1}{p}\int_a^b K(t)\left|h(t)\right|^p dt+\frac{1}{q}\int_a^b \left|\frac{a+b}{2}-t\right|\left|g'(t)\right|^q dt.\end{aligned}$$

The proof follows by the statement (iii) of Theorem 3 for $f=\int_a h(s)\,ds$.

3. Some Trapezoid Type Inequalities

We have:

Proposition 1. *Let $h:[a,b]\to\mathbb{C}$ be absolutely continuous on $[a,b]$ nd $p,\ q>1$with $\frac{1}{p}+\frac{1}{q}=1$. If $h'\in L_q[a,b]$ and $w:[a,b]\to\mathbb{C}$ with $w\in L_p[a,b]$, then*

$$\begin{aligned}(3.1)\quad &\left|\int_a^b \frac{w(t)+w(a+b-t)}{2}h(t)\,dt-\frac{h(a)+h(b)}{2}\int_a^b w(t)\,dt\right|\\ &\le \frac{1}{2}\left(\int_a^b K(t)\left|w(t)\right|^p dt\right)^{1/p}\left(\int_a^b \left|\frac{a+b}{2}-t\right|\left|h'(t)-h'(a+b-t)\right|^q dt\right)^{1/q}.\end{aligned}$$

Moreover, if w is symmetrical, namely $w(a+b-t) = w(t)$ for all $t \in [a,b]$, then

$$\left| \int_a^b w(t) h(t) dt - \frac{h(a)+h(b)}{2} \int_a^b w(t) dt \right| \leq \frac{1}{2} \left(\int_a^b K(t) |w(t)|^p dt \right)^{1/p} \left(\int_a^b \left| \frac{a+b}{2} - t \right| |h'(t) - h'(a+b-t)|^q dt \right)^{1/q}. \tag{3.2}$$

Proof. Consider the function $g : [a,b] \to \mathbb{C}$ defined by

$$g(t) := \frac{h(t) + h(a+b-t)}{2} - \frac{h(a)+h(b)}{2}, \ t \in [a,b].$$

We have $g(a) = g(b) = 0$.

If we write the inequality (2.3) for $f = \int_a w(t) dt$, then we get

$$\begin{aligned} &\int_a^b \left| w(t) \left[\frac{h(t)+h(a+b-t)}{2} - \frac{h(a)+h(b)}{2} \right] \right| dt \\ &\leq \left(\int_a^b K(t) |w(t)|^p dt \right)^{1/p} \left(\int_a^b \left| \frac{a+b}{2} - t \right| \left| \frac{h'(t) - h'(a+b-t)}{2} \right|^q dt \right)^{1/q} \\ &= \frac{1}{2} \left(\int_a^b K(t) |w(t)|^p dt \right)^{1/p} \left(\int_a^b \left| \frac{a+b}{2} - t \right| |h'(t) - h'(a+b-t)|^q dt \right)^{1/q}. \end{aligned} \tag{3.3}$$

By the modulus property, we have

$$\begin{aligned} &\int_a^b \left| w(t) \left[\frac{h(t)+h(a+b-t)}{2} - \frac{h(a)+h(b)}{2} \right] \right| dt \\ &\geq \left| \int_a^b w(t) \left[\frac{h(t)+h(a+b-t)}{2} - \frac{h(a)+h(b)}{2} \right] dt \right| \\ &= \left| \frac{1}{2} \left[\int_a^b w(t) h(t) dt + \int_a^b w(t) h(a+b-t) dt \right] - \frac{h(a)+h(b)}{2} \int_a^b w(t) dt \right|. \end{aligned} \tag{3.4}$$

By the change of variable $u = a+b-t$, $t \in [a,b]$, we have

$$\int_a^b w(t) h(a+b-t) dt = \int_a^b w(a+b-t) h(t) dt$$

and then by (3.3) and (3.4) we get the desired result (3.1). □

Corollary 4. *With the assumptions of Proposition 1 and if h' is Lipschitzian with constant $L > 0$, namely $|h'(t) - h'(s)| \leq L|t-s|$ for any $t, s \in [a,b]$, then*

$$\left| \int_a^b \frac{w(t) + w(a+b-t)}{2} h(t) dt - \frac{h(a)+h(b)}{2} \int_a^b w(t) dt \right| \leq \frac{(b-a)^{1+2/q}}{2^{1+1/q} (q+2)^{1/q}} L \left(\int_a^b K(t) |w(t)|^p dt \right)^{1/p}. \tag{3.5}$$

In the case of symmetry for w, we have

$$\left|\int_a^b w(t)\,h(t)\,dt - \frac{h(a)+h(b)}{2}\int_a^b w(t)\,dt\right| \leq \frac{(b-a)^{1+2/q}}{2^{1+1/q}(q+2)^{1/q}} L\left(\int_a^b K(t)\,|w(t)|^p\,dt\right)^{1/p}. \tag{3.6}$$

In 1906, Fejér [4], while studying trigonometric polynomials, obtained the following inequalities which generalize that of Hermite & Hadamard:

Theorem 4 (Fejér's Inequality)**.** *Consider the integral $\int_a^b h(x)\,w(x)\,dx$, where h is a convex function in the interval (a,b) and w is a positive function in the same interval such that*

$$w(x) = w(a+b-x)\,,\ \textit{for any } x\in[a,b]$$

i.e., $y = w(x)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b),0\right)$ and is normal to the x-axis. Under those conditions the following inequalities are valid:

$$h\left(\frac{a+b}{2}\right) \leq \frac{1}{\int_a^b w(x)\,dx}\int_a^b h(x)\,w(x)\,dx \leq \frac{h(a)+h(b)}{2}. \tag{3.7}$$

If h is concave on (a,b), then the inequalities reverse in (3.7).

If $w\equiv 1$, then (3.7) becomes the well known Hermite-Hadamard inequality

$$h\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b h(x)\,dx \leq \frac{h(a)+h(b)}{2}. \tag{3.8}$$

We have the following reverse of Fejér's inequality:

Corollary 5. *Let $h:[a,b]\to\mathbb{R}$ be a convex function and $w:[a,b]\to(0,\infty)$ be continuous, symmetrical on $[a,b]$ and such that $h'\in L_q[a,b]$, where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$0 \leq \frac{h(a)+h(b)}{2} - \frac{1}{\int_a^b w(x)\,dx}\int_a^b h(x)\,w(x)\,dx \leq \frac{1}{2}\left(\int_a^b K(t)\,|w(t)|^p\,dt\right)^{1/p}\left(\int_a^b \left|\frac{a+b}{2}-t\right| |h'(t)-h'(a+b-t)|^q\,dt\right)^{1/q}. \tag{3.9}$$

Moreover, if h' is L-Lipschitzian, then

$$0 \leq \frac{h(a)+h(b)}{2} - \frac{1}{\int_a^b w(x)\,dx}\int_a^b h(x)\,w(x)\,dx \leq \frac{(b-a)^{1+2/q}}{2^{1+1/q}(q+2)^{1/q}} L\left(\int_a^b K(t)\,|w(t)|^p\,dt\right)^{1/p}. \tag{3.10}$$

We also have:

Proposition 2. *Assume that $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $h:[a,b]\to\mathbb{C}$ be absolutely continuous on $[a,b]$ with $h'\in L_q[a,b]$ and $w:[a,b]\to\mathbb{C}$ such that $w\in L_p[a,b]$, then*

$$\left|\frac{\left[h(a)\left(b\int_a^b w(t)\,dt-\int_a^b w(t)\,tdt\right)+h(b)\left(\int_a^b w(t)\,tdt-a\int_a^b w(t)\,dt\right)\right]}{b-a}-\int_a^b w(t)\,h(t)\,dt\right| \le\left(\int_a^b K(t)\,|w(t)|^p\,dt\right)^{1/p}\left(\int_a^b\left|\frac{a+b}{2}-t\right|\left|h'(t)-\frac{h(b)-h(a)}{b-a}\right|^q dt\right)^{1/q}. \tag{3.11}$$

Proof. Consider the function $g:[a,b]\to\mathbb{C}$ defined by

$$g(t):=h(t)-\frac{h(a)(b-t)+h(b)(t-a)}{b-a},\ \ t\in[a,b].$$

We have $g(a)=g(b)=0$.

If we write the inequality (2.3) for $f=\int_a w(t)\,dt$, then we get

$$\int_a^b\left|w(t)\left[h(t)-\frac{h(a)(b-t)+h(b)(t-a)}{b-a}\right]\right|dt \le\left(\int_a^b K(t)\,|w(t)|^p\,dt\right)^{1/p}\left(\int_a^b\left|\frac{a+b}{2}-t\right|\left|h'(t)-\frac{h(b)-h(a)}{b-a}\right|^q dt\right)^{1/q}. \tag{3.12}$$

By the modulus property, we have

$$\begin{aligned}\int_a^b\left|w(t)\left[h(t)-\frac{h(a)(b-t)+h(b)(t-a)}{b-a}\right]\right|dt &\ge\left|\int_a^b w(t)\left[h(t)-\frac{h(a)(b-t)+h(b)(t-a)}{b-a}\right]dt\right|\\ &=\left|\int_a^b w(t)\,h(t)\,dt-\frac{h(a)\left(b\int_a^b w(t)\,dt-\int_a^b w(t)\,tdt\right)+h(b)\left(\int_a^b w(t)\,tdt-a\int_a^b w(t)\,dt\right)}{b-a}\right|,\end{aligned}$$

which together with (3.12) produces the desired result (3.11). □

Corollary 6. *Assume that $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $h:[a,b]\to\mathbb{R}$ be a convex function and $w:[a,b]\to(0,\infty)$ be continuous and such that $h'\in L_q[a,b]$. Then*

$$0\le\frac{h(a)[b-E(w,[a,b])]+h(b)[E(w,[a,b])-a]}{b-a}-\int_a^b w(t)\,h(t)\,dt \le\frac{1}{\int_a^b w(t)\,dt}\left(\int_a^b K(t)\,|w(t)|^p\,dt\right)^{1/p}\left(\int_a^b\left|\frac{a+b}{2}-t\right|\left|h'(t)-\frac{h(b)-h(a)}{b-a}\right|^q dt\right)^{1/q}, \tag{3.13}$$

where

$$E(w,[a,b]) := \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\,tdt.$$

4. Some Grüss' Type Inequalities

For two *Lebesgue integrable* functions $f,\ g : [a,b] \to \mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt. \tag{4.1}$$

In 1935, Grüss [5] showed that‘

$$|C(f,g)| \le \frac{1}{4}(M-m)(N-n), \tag{4.2}$$

provided that there exists the real numbers m, M, n, N such that

$$m \le f(t) \le M \quad \text{and} \quad n \le g(t) \le N \quad \text{for a.e. } t \in [a,b]. \tag{4.3}$$

The constant $\frac{1}{4}$ is best possible in (4.2) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known result, even though it was obtained by Čebyšev in 1882, [2], states that

$$|C(f,g)| \le \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2, \tag{4.4}$$

provided that f', g' exist and are continuous on $[a,b]$ and $\|f'\|_\infty = \sup_{t\in[a,b]}|f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (4.4) also holds if $f,\ g : [a,b] \to \mathbb{R}$ are assumed to be *absolutely continuous* and $f',\ g' \in L_\infty[a,b]$ while $\|f'\|_\infty = \text{essup}_{t\in[a,b]}|f'(t)|$.

A mixture between Grüss' result (3.7) and Čebyšev's one (4.4) is the following inequality obtained by Ostrowski in 1970, [12]:

$$|C(f,g)| \le \frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{4.5}$$

provided that f is *Lebesgue integrable* and satisfies (3.8) while g is absolutely continuous and $g' \in L_\infty[a,b]$. The constant $\frac{1}{8}$ is best possible in (4.5).

The case of *euclidean norms* of the derivative was considered by A. Lupaş in [8] in which he proved that

$$|C(f,g)| \le \frac{1}{\pi^2}\|f'\|_2 \|g'\|_2 (b-a), \tag{4.6}$$

provided that $f,\ g$ are absolutely continuous and $f',\ g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Consider

$$K(t) := \begin{cases} t-a \text{ if } a \le t \le \frac{a+b}{2}, \\ b-t \text{ if } \frac{a+b}{2} < t \le b \end{cases} = \frac{1}{2}(b-a) - \left|\frac{a+b}{2} - t\right|,$$

for $t \in [a,b]$.

We have:

Theorem 5. *Assume that* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$. *If* $f,\ g:[a,b]\to\mathbb{C}$ *are such that* f *is absolutely continuous with* $f'\in L_p[a,b]$ *and* $g\in L_q[a,b]$, *then*

$$|C(f,g)| \le \left(\int_a^b K(t)|f'(t)|^p dt\right)^{1/p}\left(\int_a^b \left|\frac{a+b}{2}-t\right|\left|g(t)-\frac{1}{b-a}\int_a^b g(s)ds\right|^q dt\right)^{1/q} \le \left(\int_a^b K(t)|f'(t)|^p dt\right)^{1/p} B(g), \tag{4.7}$$

where

$$B(g):=\begin{cases} \frac{1}{2^{1/q}}(b-a)^{2/q}\left(\frac{1}{b-a}\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)ds\right|^q dt\right)^{1/q}, \\ \frac{1}{2^{2/q}}(b-a)^{2/q}\left\|g-\frac{1}{b-a}\int_a^b g(s)ds\right\|_{[a,b],\infty} \quad \text{if } g\in L_\infty[a,b]. \end{cases}$$

Proof. We have the following Sonin identity

$$C(f,g)=\frac{1}{b-a}\int_a^b (f(t)-\gamma)\left(g(t)-\frac{1}{b-a}\int_a^b g(s)ds\right)dt \tag{4.8}$$

for any $\gamma\in\mathbb{C}$, that can be easily proved by developing the right hand side of (4.8).

Observe that, if we take $h(t)=g(t)-\frac{1}{b-a}\int_a^b g(s)ds$, then we have $\int_a^b h(t)dt=0$ and by Corollary 2 we get

$$|C(f,g)| \le \left(\int_a^b K(t)|f'(t)|^p dt\right)^{1/p}\left(\int_a^b \left|\frac{a+b}{2}-t\right|\left|g(t)-\frac{1}{b-a}\int_a^b g(s)ds\right|^q dt\right)^{1/q}.$$

Observe that

$$\left(\int_a^b \left|\frac{a+b}{2}-t\right|\left|g(t)-\frac{1}{b-a}\int_a^b g(s)ds\right|^q dt\right)^{1/q} \le \max_{t\in[a,b]}\left|\frac{a+b}{2}-t\right|^{1/q}(b-a)^{1/q}\left(\frac{1}{b-a}\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)ds\right|^q dt\right)^{1/q}$$
$$=\frac{1}{2^{1/q}}(b-a)^{2/q}\left(\frac{1}{b-a}\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)ds\right|^q dt\right)^{1/q},$$

which proves the first branch in the second inequality in (4.7).

We also have

$$\left(\int_a^b \left|\frac{a+b}{2}-t\right|\left|g(t)-\frac{1}{b-a}\int_a^b g(s)ds\right|^q dt\right)^{1/q} \le \frac{1}{2^{2/q}}(b-a)^{2/q}\left\|g-\frac{1}{b-a}\int_a^b g(s)ds\right\|_{[a,b],\infty},$$

which proves the second branch in the second inequality in (4.7). □

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