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WEIGHTED GENERALIZATIONS OF OPIAL'S INEQUALITIES FOR TWO FUNCTIONS AND APPLICATIONS

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Abstract. In this paper we establish some weighted generalizations of Opial's inequalities for two functions. Applications related to Grüss' type weighted inequalities are also given.

1. Introduction

We recall the following Opial type inequalities:

Theorem 1. *Assume that* $u:[a,b]\subset\mathbb{R}\to\mathbb{R}$ *is an absolutely continuous function on the interval* $[a,b]$ *and such that* $u'\in L_2[a,b]$.

(i) *If* $u(a)=u(b)=0$, *then*

$$\int_a^b |u(t)\,u'(t)|\,dt\le\frac{1}{4}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.1}$$

with equality if and only if

$$u(t)=\begin{cases} c(t-a) & \text{if } a\le t\le \frac{a+b}{2},\\ c(b-t) & \text{if } \frac{a+b}{2}<t\le b,\end{cases}$$

where c *is an arbitrary constant.*

(ii) *If* $u(a)=0$, *then*

$$\int_a^b |u(t)\,u'(t)|\,dt\le\frac{1}{2}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.2}$$

with equality if and only if $u(t)=c(t-a)$ *for some constant* c.

The inequality (1.1) was obtained by Olech in [11] in which he gave a simplified proof of an inequality originally due in a slightly less general form to Zdzislaw Opial [12].

Embedded in Olech's proof is the half-interval form of Opial's inequality, also discovered by Beesack [1], which is satisfied by those u vanishing only at a.

For various proofs of the above inequalities, see [7]-[10] and [14].

In the recent paper [4] we obtained the following generalization of Opial's inequalities for two functions:

Theorem 2. *Assume that* $f,\ g:[a,b]\to\mathbb{C}$ *are absolutely continuous on* $[a,b]$ *with* $f',\ g'\in L_2[a,b]$.

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(i) *If* $g(a)=0,$ *then*

$$\int_a^b |f'(t)\,g(t)|\,dt \le \left(\int_a^b (t-a)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b (b-t)\,|g'(t)|^2\,dt\right)^{1/2} \le \frac{1}{2}\int_a^b \left[(t-a)\,|f'(t)|^2+(b-t)\,|g'(t)|^2\right]dt. \tag{1.3}$$

(ii) *If* $g(b)=0,$ *then*

$$\int_a^b |f'(t)\,g(t)|\,dt \le \left(\int_a^b (b-t)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b (t-a)\,|g'(t)|^2\,dt\right)^{1/2} \le \frac{1}{2}\int_a^b \left[(b-t)\,|f'(t)|^2+(t-a)\,|g'(t)|^2\right]dt. \tag{1.4}$$

(iii) *If* $g(a)=g(b)=0,$ *then*

$$\begin{aligned}\int_a^b |f'(t)\,g(t)|\,dt &\le \left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,dt-\int_a^b \left|t-\frac{a+b}{2}\right| |f'(t)|^2\,dt\right)^{1/2} \\ &\quad\times\left(\int_a^b \left|\frac{a+b}{2}-t\right| |g'(t)|^2\,dt\right)^{1/2} \\ &\le \frac{1}{4}(b-a)\int_a^b |f'(t)|^2\,dt+\frac{1}{2}\int_a^b \left|\frac{a+b}{2}-t\right|\left(|g'(t)|^2-|f'(t)|^2\right)dt.\end{aligned} \tag{1.5}$$

By taking $g=f$ we obtain the following refinement of Opial's inequalities from Theorem 1:

Corollary 1. *Assume that* $f:[a,b]\to\mathbb{C}$ *is absolutely continuous on* $[a,b]$ *with* $f'\in L_2[a,b]$.

(i) *If either* $f(a)=0$ *or* $f(b)=0,$ *then*

$$\int_a^b |f'(t)\,f(t)|\,dt \le \left(\int_a^b (t-a)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b (b-t)\,|f'(t)|^2\,dt\right)^{1/2} \le \frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,dt. \tag{1.6}$$

(ii) *If* $f(a)=f(b)=0,$ *then*

$$\begin{aligned}\int_a^b |f'(t)\,f(t)|\,dt &\le \left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,dt-\int_a^b \left|t-\frac{a+b}{2}\right| |f'(t)|^2\,dt\right)^{1/2} \\ &\quad\times\left(\int_a^b \left|\frac{a+b}{2}-t\right| |f'(t)|^2\,dt\right)^{1/2} \le \frac{1}{4}(b-a)\int_a^b |f'(t)|^2\,dt.\end{aligned} \tag{1.7}$$

In this paper we establish some weighted generalizations of Opial's inequalities for two functions. Applications related to Grüss' type weighted inequalities are given as well.

2. The Main Results

We have:

Theorem 3. *Assume that $f,\ g:[a,b]\to\mathbb{C}$ are absolutely continuous on $[a,b]$ and $w:[a,b]\to[0,\infty)$ is integrable with $f'\sqrt{w},\ g'\in L_2[a,b]$.*

(i) *If $g(a)=0$, then*

$$\begin{aligned}(2.1)\quad &\int_a^b |f'(t)\,g(t)|\,w(t)\,dt\\ &\le\left(\int_a^b (t-a)\,|f'(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_a^b\left(\int_t^b w(s)\,ds\right)|g'(t)|^2\,dt\right)^{1/2}\\ &\le\frac{1}{2}\int_a^b\left[(t-a)\,|f'(t)|^2\,w(t)+\left(\int_t^b w(s)\,ds\right)|g'(t)|^2\right]dt.\end{aligned}$$

(ii) *If $g(b)=0$, then*

$$\begin{aligned}(2.2)\quad &\int_a^b |f'(t)\,g(t)|\,w(t)\,dt\\ &\le\left(\int_a^b (b-t)\,|f'(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_a^b\left(\int_a^t w(s)\,ds\right)|g'(t)|^2\,dt\right)^{1/2}\\ &\le\frac{1}{2}\int_a^b\left[(b-t)\,|f'(t)|^2\,w(t)+\left(\int_a^t w(s)\,ds\right)|g'(t)|^2\right]dt.\end{aligned}$$

Proof. (i) Since $g(a)=0$, then $g(t)=\int_a^t g'(s)\,ds$ for $t\in[a,b]$. We have

$$\begin{aligned}\int_a^b |f'(t)\,g(t)|\,w(t)\,dt &= \int_a^b |f'(t)|\,|g(t)|\,w(t)\,dt\\ &=\int_a^b (t-a)^{1/2}\,|f'(t)|\,(t-a)^{-1/2}\,|g(t)|\,w(t)\,dt\\ &=\int_a^b (t-a)^{1/2}\,|f'(t)|\,(t-a)^{-1/2}\left|\int_a^t g'(s)\,ds\right| w(t)\,dt =: A.\end{aligned}$$

Using the weighted Cauchy-Bunyakovsky-Schwarz (CBS) inequality, we have

$$(2.3)\qquad A\le\left(\int_a^b\left[(t-a)^{1/2}\,|f'(t)|\right]^2 w(t)\,dt\right)^{1/2}$$

$$\times\left(\int_a^b\left[(t-a)^{-1/2}\left|\int_a^t g'(s)\,ds\right|\right]^2 w(t)\,dt\right)^{1/2}$$

$$=\left(\int_a^b (t-a)\,|f'(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_a^b (t-a)^{-1}\left|\int_a^t g'(s)\,ds\right|^2 w(t)\,dt\right)^{1/2} =: B.$$

By (CBS) inequality we also have

$$(t-a)^{-1}\left|\int_a^t g'(s)\,ds\right|^2 \le \int_a^t |g'(s)|^2\,ds,$$

which gives

$$B \le \left(\int_a^b (t-a)\,|f'(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_a^b\left(\int_a^t |g'(s)|^2\,ds\right)w(t)\,dt\right)^{1/2}. \tag{2.4}$$

Using integration by parts, we have

$$\begin{aligned}
&\int_a^b\left(\int_a^t |g'(s)|^2\,ds\right)w(t)\,dt\\
&=\int_a^b\left(\int_a^t |g'(s)|^2\,ds\right)d\left(\int_a^t w(s)\,ds\right)\\
&=\left(\int_a^t |g'(s)|^2\,ds\right)\left(\int_a^t w(s)\,ds\right)\Big|_a^b-\int_a^b\left(\int_a^t w(s)\,ds\right)|g'(t)|^2\,dt\\
&=\left(\int_a^b |g'(s)|^2\,ds\right)\left(\int_a^b w(s)\,ds\right)-\int_a^b\left(\int_a^t w(s)\,ds\right)|g'(t)|^2\,dt\\
&=\int_a^b\left(\int_t^b w(s)\,ds\right)|g'(t)|^2\,dt
\end{aligned}$$

and by (2.3) we get the first inequality in (2.1).

The last part follows by the elementary inequality

$$\sqrt{\alpha\beta}\le\frac{1}{2}(\alpha+\beta),\ \ \alpha,\beta\ge 0. \tag{2.5}$$

(ii) Since $g(b)=0$, then $g(t)=-\int_t^b g'(s)\,ds$ for $t\in[a,b]$. We have

$$\begin{aligned}
\int_a^b |f'(t)\,g(t)|\,w(t)\,dt&=\int_a^b |f'(t)|\,|g(t)|\,w(t)\,dt\\
&=\int_a^b (b-t)^{1/2}\,|f'(t)|\,(b-t)^{-1/2}\,|g(t)|\,w(t)\,dt\\
&=\int_a^b (b-t)^{1/2}\,|f'(t)|\,(b-t)^{-1/2}\left|\int_t^b g'(s)\,ds\right|w(t)\,dt=:C.
\end{aligned}$$

Using the weighted (CBS) inequality we also have

$$\begin{aligned}
C&\le\left(\int_a^b\left[(b-t)^{1/2}\,|f'(t)|\right]^2 w(t)\,dt\right)^{1/2}\\
&\quad\times\left(\int_a^b\left[(b-t)^{-1/2}\left|\int_t^b g'(s)\,ds\right|\right]^2 w(t)\,dt\right)^{1/2}\\
&=\left(\int_a^b (b-t)\,|f'(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_a^b (b-t)^{-1}\left|\int_t^b g'(s)\,ds\right|^2 w(t)\,dt\right)^{1/2}=:D.
\end{aligned} \tag{2.6}$$

By (CBS) inequality we also have

$$(b-t)^{-1}\left|\int_t^b g'(s)\,ds\right|^2 \le \int_t^b |g'(s)|^2\,ds,$$

which gives

$$D \le \left(\int_a^b (b-t)\,|f'(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_a^b\left(\int_t^b |g'(s)|^2\,ds\right)w(t)\,dt\right)^{1/2}. \tag{2.7}$$

Using integration by parts, we have

$$\begin{aligned}
&\int_a^b\left(\int_t^b |g'(s)|^2\,ds\right)w(t)\,dt\\
&=\int_a^b\left(\int_t^b |g'(s)|^2\,ds\right)d\left(\int_a^t w(s)\,ds\right)\\
&=\left(\int_t^b |g'(s)|^2\,ds\right)\left(\int_a^t w(s)\,ds\right)\Bigg|_a^b+\int_a^b |g'(t)|^2\left(\int_a^t w(s)\,ds\right)dt\\
&=\int_a^b |g'(t)|^2\left(\int_a^t w(s)\,ds\right)dt
\end{aligned}$$

and by (2.6) and (2.7) we obtain (2.2). □

We have:

Corollary 2. *Assume that* $f,\ g:[a,b]\to\mathbb{C}$ *are absolutely continuous on* $[a,b]$ *and* $w:[a,b]\to[0,\infty)$ *is integrable with* $f'\sqrt{w},\ g'\in L_2[a,b]$.

(i) *If* $g(a)=0$ *and* w *is nonincreasing on* $[a,b]$, *then*

$$\begin{aligned}
\int_a^b |f'(t)\,g(t)|\,w(t)\,dt
&\le\left(\int_a^b (t-a)\,|f'(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_a^b (b-t)\,|g'(t)|^2\,w(t)\,dt\right)^{1/2}\\
&\le\frac{1}{2}\int_a^b\left[(t-a)\,|f'(t)|^2+(b-t)\,|g'(t)|^2\right]w(t)\,dt.
\end{aligned} \tag{2.8}$$

(ii) *If* $g(b)=0$ *and* w *is nondecreasing on* $[a,b]$, *then*

$$\begin{aligned}
\int_a^b |f'(t)\,g(t)|\,w(t)\,dt
&\le\left(\int_a^b (b-t)\,|f'(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_a^b (t-a)\,|g'(t)|^2\,w(t)\,dt\right)^{1/2}\\
&\le\frac{1}{2}\int_a^b\left[(b-t)\,|f'(t)|^2+(t-a)\,|g'(t)|^2\right]w(t)\,dt.
\end{aligned} \tag{2.9}$$

Remark 1. *Assume that* $f:[a,b]\to\mathbb{C}$ *is absolutely continuous on* $[a,b]$ *and* $w:[a,b]\to[0,\infty)$ *is integrable with* $f'\sqrt{w},\ f'\in L_2[a,b]$.

(i) *If* $f(a)=0$ *and* w *is nonincreasing on* $[a,b]$, *then*

$$\int_a^b |f'(t)f(t)|w(t)\,dt \le \left(\int_a^b (t-a)|f'(t)|^2 w(t)\,dt\right)^{1/2}\left(\int_a^b (b-t)|f'(t)|^2 w(t)\,dt\right)^{1/2} \le \frac{1}{2}(b-a)\int_a^b |f'(t)|^2 w(t)\,dt. \tag{2.10}$$

(ii) *If* $f(b)=0$ *and* w *is nondecreasing on* $[a,b]$, *then*

$$\int_a^b |f'(t)f(t)|w(t)\,dt \le \left(\int_a^b (b-t)|f'(t)|^2 w(t)\,dt\right)^{1/2}\left(\int_a^b (t-a)|f'(t)|^2 w(t)\,dt\right)^{1/2} \le \frac{1}{2}(b-a)\int_a^b |f'(t)|^2 w(t)\,dt. \tag{2.11}$$

For $w\equiv 1$ *we get from (2.10) and (2.11) the first two inequalities in Theorem 2.*

We have:

Theorem 4. *Assume that* $f,\ g:[a,b]\to\mathbb{C}$ *are absolutely continuous on* $[a,b]$ *and* $w:[a,b]\to[0,\infty)$ *is integrable with* $f'\sqrt{w},\ g'\in L_2[a,b]$. *If* $g(b)=g(a)=0$, *then*

$$\int_a^b |f'(t)g(t)|w(t)\,dt \le \frac{1}{2}(b-a)^{1/2}\left(\int_a^b w(s)\,ds\right)^{1/2}\left(\int_a^b |f'(t)|^2 w(t)\,dt\right)^{1/2}\left(\int_a^b |g'(t)|^2\,dt\right)^{1/2} \tag{2.12}$$

and

$$\int_a^b |f'(t)g(t)|w(t)\,dt \le \left(\int_a^b K(t)|f'(t)|^2 w(t)\,dt\right)^{1/2}\left(\int_a^b \left|\int_{\frac{a+b}{2}}^t w(s)\,ds\right||g'(t)|^2\,dt\right)^{1/2} \le \frac{1}{2}\int_a^b \left(K(t)|f'(t)|^2 w(t)+\left|\int_{\frac{a+b}{2}}^t w(s)\,ds\right||g'(t)|^2\right)dt, \tag{2.13}$$

where

$$K(t):=\frac{1}{2}(b-a)-\left|t-\frac{a+b}{2}\right|,\ t\in[a,b].$$

Proof. If we add the first inequalities in (2.1) and (2.2), then we get

$$
\begin{aligned}
(2.14)\quad & 2\int_a^b |f'(t)\,g(t)|\,w(t)\,dt \\
&\leq \left(\int_a^b (t-a)\,|f'(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_a^b\left(\int_t^b w(s)\,ds\right)|g'(t)|^2\,dt\right)^{1/2} \\
&\quad + \left(\int_a^b (b-t)\,|f'(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_a^b\left(\int_a^t w(s)\,ds\right)|g'(t)|^2\,dt\right)^{1/2}.
\end{aligned}
$$

By the elementary (CBS) inequality

$$
(2.15)\qquad \alpha\beta+\gamma\delta \leq \left(\alpha^2+\gamma^2\right)^{1/2}\left(\beta^2+\delta^2\right)^{1/2},\ \alpha,\ \beta,\ \gamma,\ \delta\geq 0,
$$

we have

$$
\begin{aligned}
(2.16)\quad & \left(\int_a^b (t-a)\,|f'(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_a^b\left(\int_t^b w(s)\,ds\right)|g'(t)|^2\,dt\right)^{1/2} \\
&\quad + \left(\int_a^b (b-t)\,|f'(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_a^b\left(\int_a^t w(s)\,ds\right)|g'(t)|^2\,dt\right)^{1/2} \\
&\leq \left(\int_a^b (t-a)\,|f'(t)|^2\,w(t)\,dt+\int_a^b (b-t)\,|f'(t)|^2\,w(t)\,dt\right)^{1/2} \\
&\quad \times\left(\int_a^b\left(\int_t^b w(s)\,ds\right)|g'(t)|^2\,dt+\int_a^b\left(\int_a^t w(s)\,ds\right)|g'(t)|^2\,dt\right)^{1/2} \\
&= (b-a)^{1/2}\left(\int_a^b |f'(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_a^b w(s)\,ds\right)^{1/2}\left(\int_a^b |g'(t)|^2\,dt\right)^{1/2}.
\end{aligned}
$$

By making use of (2.14) and (2.16) we get (2.12).

Now, if we write the inequality (2.1) on the interval $\left[a, \frac{a+b}{2}\right]$ and the inequality (2.2) on the interval $\left[\frac{a+b}{2}\right]$, then we have

$$
\begin{aligned}
(2.17)\quad & \int_a^{\frac{a+b}{2}} |f'(t)\,g(t)|\,w(t)\,dt \\
&\leq \left(\int_a^{\frac{a+b}{2}} (t-a)\,|f'(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_a^{\frac{a+b}{2}}\left(\int_t^{\frac{a+b}{2}} w(s)\,ds\right)|g'(t)|^2\,dt\right)^{1/2}
\end{aligned}
$$

and

$$
\begin{aligned}
(2.18)\quad & \int_{\frac{a+b}{2}}^b |f'(t)\,g(t)|\,w(t)\,dt \\
&\leq \left(\int_{\frac{a+b}{2}}^b (b-t)\,|f'(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_{\frac{a+b}{2}}^b\left(\int_{\frac{a+b}{2}}^t w(s)\,ds\right)|g'(t)|^2\,dt\right)^{1/2}.
\end{aligned}
$$

If we add the inequalities (2.17) and (2.18) and use (2.15), then we get

$$\begin{aligned}
&\int_a^b |f'(t) g(t)| w(t) dt \\
&\leq \left(\int_a^{\frac{a+b}{2}} (t-a) |f'(t)|^2 w(t) dt\right)^{1/2} \left(\int_a^{\frac{a+b}{2}} \left(\int_t^{\frac{a+b}{2}} w(s) ds\right) |g'(t)|^2 dt\right)^{1/2} \\
&+ \left(\int_{\frac{a+b}{2}}^b (b-t) |f'(t)|^2 w(t) dt\right)^{1/2} \left(\int_{\frac{a+b}{2}}^b \left(\int_{\frac{a+b}{2}}^t w(s) ds\right) |g'(t)|^2 dt\right)^{1/2} \\
&\leq \left(\int_a^b K(t) |f'(t)|^2 w(t) dt\right)^{1/2} \left(\int_a^b \left|\int_{\frac{a+b}{2}}^t w(s) ds\right| |g'(t)|^2 dt\right)^{1/2}.
\end{aligned}$$

By the (2.5) inequality we also have

$$\begin{aligned}
&\left(\int_a^b K(t) |f'(t)|^2 w(t) dt\right)^{1/2} \left(\int_a^b \left|\int_{\frac{a+b}{2}}^t w(s) ds\right| |g'(t)|^2 dt\right)^{1/2} \\
&\leq \frac{1}{2} \int_a^b \left(K(t) |f'(t)|^2 w(t) + \left|\int_{\frac{a+b}{2}}^t w(s) ds\right| |g'(t)|^2\right) dt,
\end{aligned}$$

and the inequality (2.13) is proved. □

Corollary 3. *Assume that $f : [a,b] \to \mathbb{C}$ are absolutely continuous on $[a,b]$ and $w : [a,b] \to [0,\infty)$ is integrable with $f'\sqrt{w}$, $f' \in L_2[a,b]$. If $f(b) = f(a) = 0$, then*

$$\begin{aligned}
&\int_a^b |f'(t) f(t)| w(t) dt \\
&\leq \frac{1}{2} (b-a)^{1/2} \left(\int_a^b w(s) ds\right)^{1/2} \left(\int_a^b |f'(t)|^2 w(t) dt\right)^{1/2} \left(\int_a^b |f'(t)|^2 dt\right)^{1/2}
\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}
&\int_a^b |f'(t) f(t)| w(t) dt \\
&\leq \left(\int_a^b K(t) |f'(t)|^2 w(t) dt\right)^{1/2} \left(\int_a^b \left|\int_{\frac{a+b}{2}}^t w(s) ds\right| |f'(t)|^2 dt\right)^{1/2} \\
&\leq \frac{1}{2} \int_a^b \left(K(t) w(t) + \left|\int_{\frac{a+b}{2}}^t w(s) ds\right|\right) |f'(t)|^2 dt.
\end{aligned} \tag{2.20}$$

Remark 2. *Since*

$$K(t) = \frac{1}{2}(b-a) - \left|t - \frac{a+b}{2}\right|,$$

then by (2.20) we get

$$\begin{aligned}
(2.21)\quad & \int_a^b |f'(t) f(t)| w(t) \, dt \\
& \leq \left(\frac{1}{2}(b-a) \int_a^b |f'(t)|^2 w(t) \, dt - \int_a^b \left| t - \frac{a+b}{2} \right| |f'(t)|^2 w(t) \, dt \right)^{1/2} \\
& \quad \times \left(\int_a^b \left| \int_{\frac{a+b}{2}}^t w(s) \, ds \right| |f'(t)|^2 \, dt \right)^{1/2} \\
& \leq \frac{1}{4}(b-a) \int_a^b w(t) |f'(t)|^2 \, dt + \int_a^b \left(\left| \int_{\frac{a+b}{2}}^t w(s) \, ds \right| - \left| t - \frac{a+b}{2} \right| w(t) \right) |f'(t)|^2 \, dt \\
& \leq \frac{1}{4}(b-a) \int_a^b w(t) |f'(t)|^2 \, dt \\
& \quad + \int_a^b \left(\|w\|_{[a,b],\infty} - w(t) \right) \left| t - \frac{a+b}{2} \right| |f'(t)|^2 \, dt,
\end{aligned}$$

provided $w \in L_\infty [a,b]$.

We observe that, if we take $w \equiv 1$ *in (2.20), then we get the inequality (1.7).*

3. Some Inequalities for the Weighted Čebyšev Functional

Consider now the *weighted Čebyšev functional*

$$\begin{aligned}
(3.1)\quad C_w(f,g) := & \frac{1}{\int_a^b w(t) \, dt} \int_a^b w(t) f(t) g(t) \, dt \\
& - \frac{1}{\int_a^b w(t) \, dt} \int_a^b w(t) f(t) \, dt \frac{1}{\int_a^b w(t) \, dt} \int_a^b w(t) g(t) \, dt
\end{aligned}$$

where $f,\ g,\ w : [a,b] \to \mathbb{R}$ and $w(t) \geq 0$ for a.e. $t \in [a,b]$ are measurable functions such that the involved integrals exist and $\int_a^b w(t) \, dt > 0$.

In [3], Cerone and Dragomir obtained, among others, the following inequalities:

$$\begin{aligned}
(3.2)\quad & |C_w(f,g)| \\
& \leq \frac{1}{2}(M-m) \frac{1}{\int_a^b w(t) \, dt} \int_a^b w(t) \left| g(t) - \frac{1}{\int_a^b w(s) \, ds} \int_a^b w(s) g(s) \, ds \right| dt \\
& \leq \frac{1}{2}(M-m) \left[\frac{1}{\int_a^b w(t) \, dt} \int_a^b w(t) \left| g(t) - \frac{1}{\int_a^b w(s) \, ds} \int_a^b w(s) g(s) \, ds \right|^p dt \right]^{\frac{1}{p}} \\
& \leq \frac{1}{2}(M-m) \operatorname*{essup}_{t \in [a,b]} \left| g(t) - \frac{1}{\int_a^b w(s) \, ds} \int_a^b w(s) g(s) \, ds \right|
\end{aligned}$$

for $p > 1$, provided $-\infty < m \leq f(t) \leq M < \infty$ for a.e. $t \in [a,b]$ and the corresponding integrals are finite. The constant $\frac{1}{2}$ is sharp in all the inequalities in (3.2) in the sense that it cannot be replaced by a smaller constant.

In addition, if $-\infty < n \leq g(t) \leq N < \infty$ for a.e. $t \in [a,b]$, then the following refinement of the celebrated Grüss inequality is obtained:

$$
\begin{aligned}
(3.3)\quad & |C_w(f,g)| \\
& \leq \frac{1}{2}(M-m)\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)\,g(s)\,ds\right|dt \\
& \leq \frac{1}{2}(M-m)\left[\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)\,g(s)\,ds\right|^2 dt\right]^{\frac{1}{2}} \\
& \leq \frac{1}{4}(M-m)(N-n).
\end{aligned}
$$

Here, the constants $\frac{1}{2}$ and $\frac{1}{4}$ are also sharp in the sense mentioned above.

If we write Theorem 3 for the function $f = \int_a h(t)\,dt$, where $h : [a,b] \to \mathbb{C}$ is integrable on $[a,b]$, then we have:

Lemma 1. *Assume that $g : [a,b] \to \mathbb{C}$ is absolutely continuous on $[a,b]$, $h : [a,b] \to \mathbb{C}$ is integrable on $[a,b]$ and $w : [a,b] \to [0,\infty)$ is integrable with $h\sqrt{w}$, $g' \in L_2[a,b]$.*

(i) *If $g(a) = 0$, then*

$$
\begin{aligned}
(3.4)\quad & \int_a^b |h(t)\,g(t)|\,w(t)\,dt \\
& \leq \left(\int_a^b (t-a)\,|h(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_a^b\left(\int_t^b w(s)\,ds\right)|g'(t)|^2\,dt\right)^{1/2}.
\end{aligned}
$$

(ii) *If $g(b) = 0$, then*

$$
\begin{aligned}
(3.5)\quad & \int_a^b |f'(t)\,g(t)|\,w(t)\,dt \\
& \leq \left(\int_a^b (b-t)\,|f'(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_a^b\left(\int_a^t w(s)\,ds\right)|g'(t)|^2\,dt\right)^{1/2}.
\end{aligned}
$$

We have:

Corollary 4. *Assume that $g : [a,b] \to \mathbb{C}$ is absolutely continuous on $[a,b]$, $h : [a,b] \to \mathbb{C}$ is integrable on $[a,b]$ and $w : [a,b] \to [0,\infty)$ is integrable with $h\sqrt{w}$, $g' \in L_2[a,b]$.*

(i) *If $g(a) = 0$ and w is nonincreasing on $[a,b]$, then*

$$
\begin{aligned}
(3.6)\quad & \int_a^b |h(t)\,g(t)|\,w(t)\,dt \\
& \leq \left(\int_a^b (t-a)\,|h(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_a^b (b-t)\,|g'(t)|^2\,w(t)\,dt\right)^{1/2}.
\end{aligned}
$$

(ii) *If* $g(b)=0$ *and* w *is nondecreasing on* $[a,b]$, *then*

$$\int_a^b |h(t)\,g(t)|\,w(t)\,dt \leq \left(\int_a^b (b-t)\,|h(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_a^b (t-a)\,|g'(t)|^2\,w(t)\,dt\right)^{1/2}. \tag{3.7}$$

We have the following inequality for the weighted Čebyšev functional.

Theorem 5. *Assume that* $g:[a,b]\to\mathbb{C}$ *is absolutely continuous on* $[a,b]$, $f:[a,b]\to\mathbb{C}$ *is integrable on* $[a,b]$ *and* $w:[a,b]\to[0,\infty)$ *is integrable with* $f\sqrt{w}$, $g'\in L_2[a,b]$.

(i) *If* w *is nonincreasing on* $[a,b]$, *then*

$$\begin{aligned}
|C_w(f,g)| \tag{3.8} \\
&\leq \left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b (t-a)\left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^2 w(t)\,dt\right)^{1/2} \\
&\qquad\times\left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b (b-t)\,|g'(t)|^2\,w(t)\,dt\right)^{1/2} \\
&\leq (b-a)^{1/2}\left[\frac{1}{\int_a^b w(s)\,ds}\int_a^b |f(t)|^2\,w(t)\,dt-\left|\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^2\right]^{1/2} \\
&\qquad\times\left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b (b-t)\,|g'(t)|^2\,w(t)\,dt\right)^{1/2}.
\end{aligned}$$

(ii) *If* w *is nondecreasing on* $[a,b]$, *then*

$$\begin{aligned}
|C_w(f,g)| \tag{3.9} \\
&\leq \left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b (b-t)\left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^2 w(t)\,dt\right)^{1/2} \\
&\qquad\times\left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b (t-a)\,|g'(t)|^2\,w(t)\,dt\right)^{1/2} \\
&\leq (b-a)^{1/2}\left[\frac{1}{\int_a^b w(s)\,ds}\int_a^b |f(t)|^2\,w(t)\,dt-\left|\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^2\right]^{1/2} \\
&\qquad\times\left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b (t-a)\,|g'(t)|^2\,w(t)\,dt\right)^{1/2}.
\end{aligned}$$

Proof. We use the following Sonin type identity

$$C_w(f,g) = \frac{1}{\int_a^b w(s)\,ds}\int_a^b \left(f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right)(g(t)-\gamma)\,w(t)\,dt, \tag{3.10}$$

for $\gamma \in \mathbb{C}$, which can be proved directly on calculating the integral from the right hand side.

Using the inequality (3.6) for $\gamma = g(a)$, we have

$$\begin{aligned}
&|C_w(f,g)| \\
&\leq \frac{1}{\int_a^b w(s)\,ds}\int_a^b \left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right| |g(t)-g(a)|\,w(t)\,dt \\
&\leq \frac{1}{\int_a^b w(s)\,ds}\left(\int_a^b (t-a)\left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^2 w(t)\,dt\right)^{1/2} \\
&\qquad\qquad \times\left(\int_a^b (b-t)\,|g'(t)|^2\,w(t)\,dt\right)^{1/2}
\end{aligned}$$

that proves the first inequality in (3.8).

Since

$$\begin{aligned}
&\frac{1}{\int_a^b w(s)\,ds}\int_a^b (t-a)\left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^2 w(t)\,dt \\
&\leq (b-a)\frac{1}{\int_a^b w(s)\,ds}\int_a^b \left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^2 w(t)\,dt \\
&= (b-a)\left[\frac{1}{\int_a^b w(s)\,ds}\int_a^b |f(t)|^2\,w(t)\,dt - \left|\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^2\right],
\end{aligned}$$

hence the second part of (3.8) follows.

Using the inequality (3.7) for $\gamma = g(b)$, we have

$$\begin{aligned}
&|C_w(f,g)| \\
&\leq \frac{1}{\int_a^b w(s)\,ds}\int_a^b \left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right| |g(t)-g(b)|\,w(t)\,dt \\
&\leq \frac{1}{\int_a^b w(s)\,ds}\left(\int_a^b (b-t)\left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^2 w(t)\,dt\right)^{1/2} \\
&\qquad\qquad \times\left(\int_a^b (t-a)\,|g'(t)|^2\,w(t)\,dt\right)^{1/2}
\end{aligned}$$

that proves the first part of (3.9).

The second part follows in the same way as above. □

We also have:

Theorem 6. *Assume that* $g:[a,b]\to\mathbb{C}$ *is absolutely continuous on* $[a,b]$, $h:[a,b]\to\mathbb{C}$ *is integrable on* $[a,b]$ *and* $w:[a,b]\to[0,\infty)$ *is integrable with* $h\sqrt{w}$, $g'\in L_2[a,b]$. *Then*

$$
\begin{aligned}
&(3.11)\quad |C_w(h,f)| \\
&\le \frac{1}{\int_a^b w(s)\,ds}\left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,dt-\int_a^b\left|t-\frac{a+b}{2}\right||f'(t)|^2\,dt\right)^{1/2} \\
&\times\left(\int_a^b\left|\frac{a+b}{2}-t\right|\left|h(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b h(s)w(s)\,ds\right|^2 w^2(t)\,dt\right)^{1/2} \\
&\le \frac{1}{\left(\int_a^b w(s)\,ds\right)^{1/2}}\left\|\left|\frac{a+b}{2}-\ell\right|^{1/2}w^{1/2}\right\|_\infty \\
&\times\left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,dt-\int_a^b\left|t-\frac{a+b}{2}\right||f'(t)|^2\,dt\right)^{1/2} \\
&\times\left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b |h(t)|^2 w(t)\,dt-\left|\frac{1}{\int_a^b w(s)\,ds}\int_a^b h(s)w(s)\,ds\right|^2\right)^{1/2} \\
&\le \frac{\sqrt{2}}{2}\left(\frac{(b-a)\|w\|_{[a,b],\infty}}{\int_a^b w(s)\,ds}\right)^{1/2} \\
&\times\left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,dt-\int_a^b\left|t-\frac{a+b}{2}\right||f'(t)|^2\,dt\right)^{1/2} \\
&\times\left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b |h(t)|^2 w(t)\,dt-\left|\frac{1}{\int_a^b w(s)\,ds}\int_a^b h(s)w(s)\,ds\right|^2\right)^{1/2}.
\end{aligned}
$$

Proof. Integrating by parts, we have

$$
\begin{aligned}
&\frac{1}{\int_a^b w(s)\,ds}\int_a^b\left(\int_a^x h(t)w(t)\,dt-\frac{\int_a^x w(s)\,ds}{\int_a^b w(s)\,ds}\int_a^b h(s)w(s)\,ds\right)f'(x)\,dx \\
&=\frac{1}{\int_a^b w(s)\,ds}\left[\left(\int_a^x h(t)w(t)\,dt-\frac{\int_a^x w(s)\,ds}{\int_a^b w(s)\,ds}\int_a^b h(s)w(s)\,ds\right)f(x)\Big|_a^b\right. \\
&\left.-\int_a^b f(x)\left(h(x)w(x)-\frac{w(x)}{\int_a^b w(s)\,ds}\int_a^b h(s)w(s)\,ds\right)dx\right. \\
&=-\frac{1}{\int_a^b w(s)\,ds}\int_a^b h(x)f(x)w(x)\,dx \\
&+\frac{1}{\int_a^b w(s)\,ds}\int_a^b h(s)w(s)\,ds\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(x)w(x)\,dx,
\end{aligned}
$$

which gives that

$$C_w(h,f) = \frac{1}{\int_a^b w(s)\,ds} \times \int_a^b \left(\frac{\int_a^x w(s)\,ds}{\int_a^b w(s)\,ds} \int_a^b h(s)\,w(s)\,ds - \int_a^x h(t)\,w(t)\,dt \right) f'(x)\,dx.$$

Consider the function

$$g(x) := \frac{\int_a^x w(s)\,ds}{\int_a^b w(s)\,ds} \int_a^b h(s)\,w(s)\,ds - \int_a^x h(t)\,w(t)\,dt, \ x \in [a,b].$$

We observe that $g(a) = g(b) = 0$ and g is absolutely continuous on $[a,b]$.

If we use the inequality (1.5), we get

$$\begin{aligned}
&|C_w(h,f)| \\
&\leq \frac{1}{\int_a^b w(s)\,ds} \int_a^b \left| f'(t) \left(\frac{\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds} \int_a^b h(s)\,w(s)\,ds - \int_a^t h(s)\,w(s)\,ds \right) \right| dt \\
&\quad \leq \frac{1}{\int_a^b w(s)\,ds} \left(\frac{1}{2}(b-a) \int_a^b |f'(t)|^2\,dt - \int_a^b \left| t - \frac{a+b}{2} \right| |f'(t)|^2\,dt \right)^{1/2} \\
&\quad \times \left(\int_a^b \left| \frac{a+b}{2} - t \right| \left| \frac{w(t)}{\int_a^b w(s)\,ds} \int_a^b h(s)\,w(s)\,ds - h(t)\,w(t) \right|^2 dt \right)^{1/2} \\
&\quad = \frac{1}{\int_a^b w(s)\,ds} \left(\frac{1}{2}(b-a) \int_a^b |f'(t)|^2\,dt - \int_a^b \left| t - \frac{a+b}{2} \right| |f'(t)|^2\,dt \right)^{1/2} \\
&\quad\quad \times \left(\int_a^b \left| \frac{a+b}{2} - t \right| \left| h(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b h(s)\,w(s)\,ds \right|^2 w^2(t)\,dt \right)^{1/2},
\end{aligned}$$

which proves the first inequality in (3.11).

Since

$$\begin{aligned}
&\frac{1}{\int_a^b w(s)\,ds} \int_a^b \left| \frac{a+b}{2} - t \right| \left| h(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b h(s)\,w(s)\,ds \right|^2 w^2(t)\,dt \\
&\qquad \leq \operatorname*{essup}_{t\in[a,b]} \left[\left| \frac{a+b}{2} - t \right| w(t) \right] \\
&\quad \times \frac{1}{\int_a^b w(s)\,ds} \int_a^b \left| h(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b h(s)\,w(s)\,ds \right|^2 w(t)\,dt \\
&\qquad = \operatorname*{essup}_{t\in[a,b]} \left[\left| \frac{a+b}{2} - t \right| w(t) \right] \\
&\quad \times \left(\frac{1}{\int_a^b w(s)\,ds} \int_a^b |h(t)|^2\,w(t)\,dt - \left| \frac{1}{\int_a^b w(s)\,ds} \int_a^b h(s)\,w(s)\,ds \right|^2 \right),
\end{aligned}$$

hence the second part of the inequality (3.11) is proved. The last part is obvious. □

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