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WEIGHTED GENERALIZATIONS OF OPIAL'S INEQUALITIES FOR p-NORMS AND TWO FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper we establish some weighted generalizations of Opial's inequalities in terms of p-norms and for two functions. Applications related to Grüss' type weighted inequalities are also given.

1. Introduction

We recall the following Opial type inequalities:

Theorem 1. *Assume that $u:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is an absolutely continuous function on the interval $[a,b]$ and such that $u'\in L_2[a,b]$.*

(i) *If $u(a)=u(b)=0$, then*

$$\int_a^b |u(t)\,u'(t)|\,dt\le\frac{1}{4}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.1}$$

with equality if and only if

$$u(t)=\begin{cases} c(t-a) & \text{if } a\le t\le\frac{a+b}{2},\\ c(b-t) & \text{if } \frac{a+b}{2}<t\le b,\end{cases}$$

where c is an arbitrary constant.

(ii) *If $u(a)=0$, then*

$$\int_a^b |u(t)\,u'(t)|\,dt\le\frac{1}{2}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.2}$$

with equality if and only if $u(t)=c(t-a)$ for some constant c.

The inequality (1.1) was obtained by Olech in [12] in which he gave a simplified proof of an inequality originally due in a slightly less general form to Zdzislaw Opial [13].

Embedded in Olech's proof is the half-interval form of Opial's inequality, also discovered by Beesack [1], which is satisfied by those u vanishing only at a.

For various proofs of the above inequalities, see [8]-[11] and [15].

In [5] we obtained the following generalizations of Opial inequalities for p-norms of two functions:

Theorem 2. *Assume that $f,\ g:[a,b]\to\mathbb{C}$ are absolutely continuous on $[a,b]$ with $f'\in L_p[a,b]$ and $g'\in L_q[a,b]$ for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.*

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(i) *If* $g(a)=0,$ *then*

$$\int_a^b |f'(t)\, g(t)|\, dt \le \left(\int_a^b (t-a)\, |f'(t)|^p\, dt\right)^{1/p} \left(\int_a^b (b-t)\, |g'(t)|^q\, dt\right)^{1/q} \le \int_a^b \left[\frac{1}{p}(t-a)\,|f'(t)|^p + \frac{1}{q}(b-t)\,|g'(t)|^q\right] dt. \tag{1.3}$$

(ii) *If* $g(b)=0,$ *then*

$$\int_a^b |f'(t)\, g(t)|\, dt \le \left(\int_a^b (b-t)\, |f'(t)|^p\, dt\right)^{1/p} \left(\int_a^b (t-a)\, |g'(t)|^q\, dt\right)^{1/q} \le \int_a^b \left[\frac{1}{p}(b-t)\,|f'(t)|^p\, dt + \frac{1}{q}(t-a)\,|g'(t)|^q\right] dt. \tag{1.4}$$

(iii) *If* $g(a)=g(b)=0,$ *then*

$$\begin{aligned}\int_a^b |f'(t)\, g(t)|\, dt &\le \left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^p\, dt - \int_a^b \left|\frac{a+b}{2}-t\right| |f'(t)|^p\, dt\right)^{1/p} \\ &\quad\times \left(\int_a^b \left|\frac{a+b}{2}-t\right| |g'(t)|^q\, dt\right)^{1/q} \\ &\le \frac{1}{2p}(b-a)\int_a^b |f'(t)|^p\, dt + \int_a^b \left|\frac{a+b}{2}-t\right| \left[\frac{1}{q}|g'(t)|^q - \frac{1}{p}|f'(t)|^p\right] dt.\end{aligned} \tag{1.5}$$

In particular, we have:

Corollary 1. *Assume that* $f:[a,b]\to\mathbb{C}$ *is absolutely continuous on* $[a,b]$ *and* $f'\in L_p[a,b]\cap L_q[a,b]$ *for* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1.$

(i) *If* $f(a)=0,$ *then*

$$\int_a^b |f'(t)\, f(t)|\, dt \le \left(\int_a^b (t-a)\, |f'(t)|^p\, dt\right)^{1/p} \left(\int_a^b (b-t)\, |f'(t)|^q\, dt\right)^{1/q} \le \int_a^b \left[\frac{1}{p}(t-a)\,|f'(t)|^p + \frac{1}{q}(b-t)\,|f'(t)|^q\right] dt. \tag{1.6}$$

(ii) *If* $f(b)=0,$ *then*

$$\int_a^b |f'(t)\, f(t)|\, dt \le \left(\int_a^b (b-t)\, |f'(t)|^p\, dt\right)^{1/p} \left(\int_a^b (t-a)\, |f'(t)|^q\, dt\right)^{1/q} \le \int_a^b \left[\frac{1}{p}(b-t)\,|f'(t)|^p\, dt + \frac{1}{q}(t-a)\,|f'(t)|^q\right] dt. \tag{1.7}$$

(iii) *If* $f(a) = f(b) = 0$, *then*

$$\begin{aligned}
(1.8)\quad & \int_a^b |f'(t) f(t)| \, dt \\
&\le \left(\frac{1}{2}(b-a) \int_a^b |f'(t)|^p \, dt - \int_a^b \left| \frac{a+b}{2} - t \right| |f'(t)|^p \, dt \right)^{1/p} \\
&\qquad \times \left(\int_a^b \left| \frac{a+b}{2} - t \right| |f'(t)|^q \, dt \right)^{1/q} \\
&\le \frac{1}{2p}(b-a) \int_a^b |f'(t)|^p \, dt + \int_a^b \left| \frac{a+b}{2} - t \right| \left[\frac{1}{q} |f'(t)|^q - \frac{1}{p} |f'(t)|^p \right] dt.
\end{aligned}$$

In this paper we establish some weighted generalizations of Opial's inequalities in terms of p-norms and for two functions. Applications related to Grüss' type weighted inequalities are also given.

2. The Main Results

We have:

Theorem 3. *Assume that* $f, g : [a,b] \to \mathbb{C}$ *are absolutely continuous on* $[a,b]$, $w : [a,b] \to [0,\infty)$ *is integrable,* $f' w^{1/p} \in L_p[a,b]$ *and* $g' \in L_q[a,b]$ *for* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

(i) *If* $g(a) = 0$, *then*

$$\begin{aligned}
(2.1)\quad & \int_a^b |f'(t) g(t)| \, w(t) \, dt \\
&\le \left(\int_a^b (t-a) |f'(t)|^p w(t) \, dt \right)^{1/p} \left(\int_a^b \left(\int_t^b w(s) \, ds \right) |g'(t)|^q \, dt \right)^{1/q} \\
&\le \int_a^b \left[\frac{1}{p}(t-a) |f'(t)|^p w(t) + \frac{1}{q} \left(\int_t^b w(s) \, ds \right) |g'(t)|^q \right] dt.
\end{aligned}$$

(ii) *If* $g(b) = 0$, *then*

$$\begin{aligned}
(2.2)\quad & \int_a^b |f'(t) g(t)| \, w(t) \, dt \\
&\le \left(\int_a^b (b-t) |f'(t)|^p w(t) \, dt \right)^{1/p} \left(\int_a^b \left(\int_a^t w(s) \, ds \right) |g'(t)|^q \, dt \right)^{1/q} \\
&\le \int_a^b \left[\frac{1}{p}(b-t) |f'(t)|^p w(t) + \frac{1}{q} \left(\int_a^t w(s) \, ds \right) |g'(t)|^q \right] dt.
\end{aligned}$$

Proof. (i) Since $g(a)=0$, then $g(t)=\int_a^t g'(s)\,ds$ for $t\in[a,b]$. We have

$$\begin{aligned}\int_a^b |f'(t)\,g(t)|\,w(t)\,dt &= \int_a^b |f'(t)|\,|g(t)|\,w(t)\,dt \\ &= \int_a^b (t-a)^{1/p}|f'(t)|\,(t-a)^{-1/p}|g(t)|\,w(t)\,dt \\ &= \int_a^b (t-a)^{1/p}|f'(t)|\,(t-a)^{-1/p}\left|\int_a^t g'(s)\,ds\right| w(t)\,dt =: A.\end{aligned}$$

Using the weighted Hölder's inequality for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\begin{aligned}A &\le \left(\int_a^b \left[(t-a)^{1/p}|f'(t)|\right]^p w(t)\,dt\right)^{1/p} \\ &\qquad\times\left(\int_a^b \left[(t-a)^{-1/p}\left|\int_a^t g'(s)\,ds\right|\right]^q w(t)\,dt\right)^{1/q} \\ &= \left(\int_a^b (t-a)|f'(t)|^p w(t)\,dt\right)^{1/p}\left(\int_a^b \left[(t-a)^{-1/p}\left|\int_a^t g'(s)\,ds\right|\right]^q w(t)\,dt\right)^{1/q} \\ &=: B.\end{aligned} \tag{2.3}$$

By Hölder's inequality for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ we also have

$$(t-a)^{-1/p}\left|\int_a^t g'(s)\,ds\right| \le \left(\int_a^t |g'(s)|^q\,ds\right)^{1/q}$$

that implies

$$\left[(t-a)^{-1/p}\left|\int_a^t g'(s)\,ds\right|\right]^q \le \int_a^t |g'(s)|^q\,ds,$$

which gives

$$B \le \left(\int_a^b (t-a)|f'(t)|^p w(t)\,dt\right)^{1/p}\left(\int_a^b \left(\int_a^t |g'(s)|^q\,ds\right) w(t)\,dt\right)^{1/q}. \tag{2.4}$$

Using integration by parts, we have

$$\begin{aligned}&\int_a^b \left(\int_a^t |g'(s)|^q\,ds\right) w(t)\,dt \\ &= \int_a^b \left(\int_a^t |g'(s)|^q\,ds\right) d\left(\int_a^t w(s)\,ds\right) \\ &= \left(\int_a^t |g'(s)|^q\,ds\right)\left(\int_a^t w(s)\,ds\right)\Big|_a^b - \int_a^b \left(\int_a^t w(s)\,ds\right)|g'(t)|^q\,dt \\ &= \left(\int_a^b |g'(s)|^q\,ds\right)\left(\int_a^b w(s)\,ds\right) - \int_a^b \left(\int_a^t w(s)\,ds\right)|g'(t)|^q\,dt \\ &= \int_a^b \left(\int_t^b w(s)\,ds\right)|g'(t)|^q\,dt\end{aligned}$$

and by (2.3) we get the first inequality in (2.1).

The last part follows by the elementary *Young's inequality*

$$\alpha^{1/p}\beta^{1/q} \leq \frac{1}{p}\alpha + \frac{1}{q}\beta, \ \alpha, \ \beta \geq 0. \tag{2.5}$$

(ii) Since $g(b) = 0$, then $g(t) = -\int_t^b g'(s)\,ds$ for $t \in [a,b]$. We have

$$\begin{aligned}
\int_a^b |f'(t)\,g(t)|\,w(t)\,dt &= \int_a^b |f'(t)|\,|g(t)|\,w(t)\,dt \\
&= \int_a^b (b-t)^{1/p}\,|f'(t)|\,(b-t)^{-1/p}\,|g(t)|\,w(t)\,dt \\
&= \int_a^b (b-t)^{1/p}\,|f'(t)|\,(b-t)^{-1/p}\left|\int_t^b g'(s)\,ds\right| w(t)\,dt =: C.
\end{aligned}$$

Using Hölder's inequality for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we also have

$$\begin{aligned}
C &\leq \left(\int_a^b \left[(b-t)^{1/p}\,|f'(t)|\right]^p w(t)\,dt\right)^{1/p} \\
&\qquad \times \left(\int_a^b \left[(b-t)^{-1/p}\left|\int_t^b g'(s)\,ds\right|\right]^q w(t)\,dt\right)^{1/q} \\
&= \left(\int_a^b (b-t)\,|f'(t)|^p\,w(t)\,dt\right)^{1/p} \left(\left[(b-t)^{-1/p}\left|\int_t^b g'(s)\,ds\right|\right]^q w(t)\,dt\right)^{1/q} \\
&=: D.
\end{aligned} \tag{2.6}$$

By Hölder's inequality for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we also have

$$(b-t)^{-1/p}\left|\int_t^b g'(s)\,ds\right| \leq \left(\int_t^b |g'(s)|^q\,ds\right)^{1/q},$$

which gives

$$D \leq \left(\int_a^b (b-t)\,|f'(t)|^p\,w(t)\,dt\right)^{1/p} \left(\int_a^b \left(\int_t^b |g'(s)|^q\,ds\right) w(t)\,dt\right)^{1/2}. \tag{2.7}$$

Using integration by parts, we have

$$\begin{aligned}
&\int_a^b \left(\int_t^b |g'(s)|^q\,ds\right) w(t)\,dt \\
&= \int_a^b \left(\int_t^b |g'(s)|^q\,ds\right) d\left(\int_a^t w(s)\,ds\right) \\
&= \left(\int_t^b |g'(s)|^q\,ds\right)\left(\int_a^t w(s)\,ds\right)\Bigg|_a^b + \int_a^b |g'(t)|^q \left(\int_a^t w(s)\,ds\right) dt \\
&= \int_a^b \left(\int_a^t w(s)\,ds\right) |g'(t)|^q\,dt
\end{aligned}$$

and by (2.6) and (2.7) we obtain (2.2). □

Corollary 2. *Assume that f, $g:[a,b]\to\mathbb{C}$ are absolutely continuous on $[a,b]$, $w:[a,b]\to[0,\infty)$ is integrable, $f'w^{1/p}\in L_p[a,b]$ and $g'\in L_q[a,b]$ for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.*

(i) *If $g(a)=0$ and w is nonincreasing on $[a,b]$, then*

$$\begin{aligned}\int_a^b |f'(t)g(t)|\,w(t)\,dt &\le \left(\int_a^b (t-a)\,|f'(t)|^p\,w(t)\,dt\right)^{1/p}\left(\int_a^b (b-t)\,|g'(t)|^q\,w(t)\,dt\right)^{1/q}\\ &\le \int_a^b\left[\frac{1}{p}(t-a)\,|f'(t)|^p+\frac{1}{q}(b-t)\,|g'(t)|^q\right]w(t)\,dt.\end{aligned}\tag{2.8}$$

(ii) *If $g(b)=0$ and w is nondecreasing on $[a,b]$, then*

$$\begin{aligned}\int_a^b |f'(t)g(t)|\,w(t)\,dt &\le \left(\int_a^b (b-t)\,|f'(t)|^p\,w(t)\,dt\right)^{1/p}\left(\int_a^b (t-a)\,|g'(t)|^q\,w(t)\,dt\right)^{1/q}\\ &\le \int_a^b\left[\frac{1}{p}(b-t)\,|f'(t)|^p+\frac{1}{q}(t-a)\,|g'(t)|^q\right]w(t)\,dt.\end{aligned}\tag{2.9}$$

Remark 1. *Assume that $f:[a,b]\to\mathbb{C}$ are absolutely continuous on $[a,b]$, $w:[a,b]\to[0,\infty)$ is integrable, $f'w^{1/p}\in L_p[a,b]$ and $f'\in L_q[a,b]$ for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.*

(i) *If $f(a)=0$ and w is nonincreasing on $[a,b]$, then*

$$\begin{aligned}\int_a^b |f'(t)f(t)|\,w(t)\,dt &\le \left(\int_a^b (t-a)\,|f'(t)|^p\,w(t)\,dt\right)^{1/p}\left(\int_a^b (b-t)\,|f'(t)|^q\,w(t)\,dt\right)^{1/q}\\ &\le \int_a^b\left[\frac{1}{p}(t-a)\,|f'(t)|^p+\frac{1}{q}(b-t)\,|f'(t)|^q\right]w(t)\,dt.\end{aligned}\tag{2.10}$$

(ii) *If $f(b)=0$ and w is nondecreasing on $[a,b]$, then*

$$\begin{aligned}\int_a^b |f'(t)f(t)|\,w(t)\,dt &\le \left(\int_a^b (b-t)\,|f'(t)|^p\,w(t)\,dt\right)^{1/p}\left(\int_a^b (t-a)\,|f'(t)|^q\,w(t)\,dt\right)^{1/q}\\ &\le \int_a^b\left[\frac{1}{p}(b-t)\,|f'(t)|^p+\frac{1}{q}(t-a)\,|f'(t)|^q\right]w(t)\,dt.\end{aligned}\tag{2.11}$$

We have:

Theorem 4. *Assume that* $f,\ g : [a,b] \to \mathbb{C}$ *are absolutely continuous on* $[a,b]$, $w : [a,b] \to [0,\infty)$ *is integrable,* $f' w^{1/p} \in L_p[a,b]$ *and* $g' \in L_q[a,b]$ *for* $p,\ q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$. *If* $g(a)=g(b)=0$, *then*

$$\begin{aligned} &\int_a^b |f'(t)\,g(t)|\,w(t)\,dt \\ &\le \frac{1}{2}(b-a)^{1/p}\left(\int_a^b w(s)\,ds\right)^{1/q}\left(\int_a^b |f'(t)|^p w(t)\,dt\right)^{1/p}\left(\int_a^b |g'(t)|^q\,dt\right)^{1/q} \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} &\int_a^b |f'(t)\,g(t)|\,w(t)\,dt \\ &\le \left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^p w(t)\,dt - \int_a^b \left|t-\frac{a+b}{2}\right| |f'(t)|^p w(t)\,dt\right)^{1/p} \\ &\qquad\times\left(\int_a^b \left|\int_t^{\frac{a+b}{2}} w(s)\,ds\right| |g'(t)|^q\,dt\right)^{1/q}. \end{aligned} \tag{2.13}$$

Proof. If we add the inequalities (2.1) and (2.2) we get

$$\begin{aligned} &2\int_a^b |f'(t)\,g(t)|\,w(t)\,dt \\ &\le \left(\int_a^b (t-a)\,|f'(t)|^p w(t)\,dt\right)^{1/p}\left(\int_a^b\left(\int_t^b w(s)\,ds\right)|g'(t)|^q\,dt\right)^{1/q} \\ &\quad+\left(\int_a^b (b-t)\,|f'(t)|^p w(t)\,dt\right)^{1/p}\left(\int_a^b\left(\int_a^t w(s)\,ds\right)|g'(t)|^q\,dt\right)^{1/q}. \end{aligned} \tag{2.14}$$

If we use the elementary Hölder inequality for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

$$\alpha\beta+\gamma\delta \le (\alpha^p+\gamma^p)^{1/p}(\beta^q+\delta^q)^{1/q},\ \ \alpha,\ \beta,\ \gamma,\ \delta\ge 0, \tag{2.15}$$

we have

$$\begin{aligned} &\left(\int_a^b (t-a)\,|f'(t)|^p w(t)\,dt\right)^{1/p}\left(\int_a^b\left(\int_t^b w(s)\,ds\right)|g'(t)|^q\,dt\right)^{1/q} \\ &\quad+\left(\int_a^b (b-t)\,|f'(t)|^p w(t)\,dt\right)^{1/p}\left(\int_a^b\left(\int_a^t w(s)\,ds\right)|g'(t)|^q\,dt\right)^{1/q} \end{aligned} \tag{2.16}$$

$$\leq \left(\int_a^b (t-a)\left|f'(t)\right|^p w(t)\,dt + \int_a^b (b-t)\left|f'(t)\right|^p w(t)\,dt\right)^{1/p}$$
$$\times \left(\int_a^b \left(\int_t^b w(s)\,ds\right)\left|g'(t)\right|^q dt + \int_a^b \left(\int_a^t w(s)\,ds\right)\left|g'(t)\right|^q dt\right)^{1/q}$$
$$= (b-a)^{1/p}\left(\int_a^b \left|f'(t)\right|^p w(t)\,dt\right)^{1/p}\left(\int_a^b w(s)\,ds\right)^{1/q}\left(\int_a^b \left|g'(t)\right|^q dt\right)^{1/q}.$$

By making use of (2.14) and (2.16) we get (2.12).

If we use the inequality (2.1) on the interval $\left[a, \frac{a+b}{2}\right]$, then we have

$$\int_a^{\frac{a+b}{2}} \left|f'(t)\,g(t)\right| w(t)\,dt \tag{2.17}$$
$$\leq \left(\int_a^{\frac{a+b}{2}} (t-a)\left|f'(t)\right|^p w(t)\,dt\right)^{1/p}\left(\int_a^{\frac{a+b}{2}} \left(\int_t^{\frac{a+b}{2}} w(s)\,ds\right)\left|g'(t)\right|^q dt\right)^{1/q}$$

while if we use the inequality (2.2) on the interval $\left[\frac{a+b}{2}, b\right]$, then we have

$$\int_{\frac{a+b}{2}}^b \left|f'(t)\,g(t)\right| w(t)\,dt \tag{2.18}$$
$$\leq \left(\int_{\frac{a+b}{2}}^b (b-t)\left|f'(t)\right|^p w(t)\,dt\right)^{1/p}\left(\int_{\frac{a+b}{2}}^b \left(\int_{\frac{a+b}{2}}^t w(s)\,ds\right)\left|g'(t)\right|^q dt\right)^{1/q}.$$

If we add these two inequalities, then we get by (2.15) that

$$\int_a^b \left|f'(t)\,g(t)\right| w(t)\,dt$$
$$\leq \left(\int_a^{\frac{a+b}{2}} (t-a)\left|f'(t)\right|^p w(t)\,dt\right)^{1/p}\left(\int_a^{\frac{a+b}{2}} \left(\int_t^{\frac{a+b}{2}} w(s)\,ds\right)\left|g'(t)\right|^q dt\right)^{1/q}$$
$$+\left(\int_{\frac{a+b}{2}}^b (b-t)\left|f'(t)\right|^p w(t)\,dt\right)^{1/p}\left(\int_{\frac{a+b}{2}}^b \left(\int_{\frac{a+b}{2}}^t w(s)\,ds\right)\left|g'(t)\right|^q dt\right)^{1/q}$$
$$\leq \left(\int_a^b K(t)\left|f'(t)\right|^p w(t)\,dt\right)^{1/p}\left(\int_a^b \left|\int_t^{\frac{a+b}{2}} w(s)\,ds\right|\left|g'(t)\right|^q dt\right)^{1/q}$$

where

$$K(t) := \begin{cases} t-a, \text{ if } t \in \left[a, \frac{a+b}{2}\right] \\ \\ b-t, \text{ if } t \in \left[\frac{a+b}{2}, b\right] \end{cases} = \frac{1}{2}(b-a) - \left|t - \frac{a+b}{2}\right|.$$

This proves (2.13). □

Corollary 3. *Assume that $f : [a,b] \to \mathbb{C}$ is absolutely continuous on $[a,b]$, $w : [a,b] \to [0,\infty)$ is integrable, $f'w^{1/p} \in L_p[a,b]$ and $f' \in L_q[a,b]$ for $p,\ q > 1$ with*

$\frac{1}{p}+\frac{1}{q}=1$. *If* $f(a)=f(b)=0,$ *then*

$$\begin{aligned}(2.19)\quad &\int_a^b |f'(t)\,f(t)|\,w(t)\,dt \\ &\le \frac{1}{2}(b-a)^{1/p}\left(\int_a^b w(s)\,ds\right)^{1/q}\left(\int_a^b |f'(t)|^p\,w(t)\,dt\right)^{1/p}\left(\int_a^b |f'(t)|^q\,dt\right)^{1/q}\end{aligned}$$

and

$$\begin{aligned}(2.20)\quad &\int_a^b |f'(t)\,f(t)|\,w(t)\,dt \\ &\le \left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^p\,w(t)\,dt-\int_a^b \left|t-\frac{a+b}{2}\right||f'(t)|^p\,w(t)\,dt\right)^{1/p} \\ &\qquad\times\left(\int_a^b \left|\int_t^{\frac{a+b}{2}} w(s)\,ds\right||f'(t)|^q\,dt\right)^{1/q}.\end{aligned}$$

3. Some Inequalities for the Weighted Čebyšev Functional

Consider now the *weighted Čebyšev functional*

$$\begin{aligned}(3.1)\quad C_w(f,g) := &\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\,f(t)\,g(t)\,dt \\ &-\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\,f(t)\,dt\,\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\,g(t)\,dt\end{aligned}$$

where $f,\ g,\ w:[a,b]\to\mathbb{R}$ and $w(t)\ge 0$ for a.e. $t\in[a,b]$ are measurable functions such that the involved integrals exist and $\int_a^b w(t)\,dt>0$.

In [3], Cerone and Dragomir obtained, among others, the following inequalities:

$$\begin{aligned}(3.2)\quad &|C_w(f,g)| \\ &\le \frac{1}{2}(M-m)\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)\,g(s)\,ds\right|dt \\ &\le \frac{1}{2}(M-m)\left[\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)\,g(s)\,ds\right|^p dt\right]^{\frac{1}{p}} \\ &\le \frac{1}{2}(M-m)\operatorname*{essup}_{t\in[a,b]}\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)\,g(s)\,ds\right|\end{aligned}$$

for $p>1$, provided $-\infty<m\le f(t)\le M<\infty$ for a.e. $t\in[a,b]$ and the corresponding integrals are finite. The constant $\frac{1}{2}$ is sharp in all the inequalities in (3.2) in the sense that it cannot be replaced by a smaller constant.

In addition, if $-\infty < n \leq g(t) \leq N < \infty$ for a.e. $t \in [a,b]$, then the following refinement of the celebrated Grüss inequality is obtained:

$$
\begin{aligned}
(3.3)\quad & |C_w(f,g)| \\
&\leq \frac{1}{2}(M-m)\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)g(s)\,ds\right|dt \\
&\leq \frac{1}{2}(M-m)\left[\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)g(s)\,ds\right|^2 dt\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}(M-m)(N-n).
\end{aligned}
$$

Here, the constants $\frac{1}{2}$ and $\frac{1}{4}$ are also sharp in the sense mentioned above.

If we write Corollary 2 for the function $f = \int_a h(t)\,dt$, where $h : [a,b] \to \mathbb{C}$ is integrable on $[a,b]$, then we have:

Lemma 1. *Assume that $g : [a,b] \to \mathbb{C}$ is absolutely continuous on $[a,b]$, $h : [a,b] \to \mathbb{C}$ is integrable on $[a,b]$ and $w : [a,b] \to [0,\infty)$ is integrable, $hw^{1/p} \in L_p[a,b]$ and $g' \in L_q[a,b]$ for $p,\ q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$.*

(i) *If $g(a)=0$ and w is nonincreasing on $[a,b]$, then*

$$
\begin{aligned}
(3.4)\quad & \int_a^b |h(t)g(t)|\,w(t)\,dt \\
&\leq \left(\int_a^b (t-a)|h(t)|^p w(t)\,dt\right)^{1/p}\left(\int_a^b (b-t)|g'(t)|^q w(t)\,dt\right)^{1/q}.
\end{aligned}
$$

(1) *If $g(b)=0$ and w is nondecreasing on $[a,b]$, then*

$$
\begin{aligned}
(3.5)\quad & \int_a^b |h(t)g(t)|\,w(t)\,dt \\
&\leq \left(\int_a^b (b-t)|h(t)|^p w(t)\,dt\right)^{1/p}\left(\int_a^b (t-a)|g'(t)|^q w(t)\,dt\right)^{1/q}.
\end{aligned}
$$

We have the following inequality for the weighted Čebyšev functional.

Theorem 5. *Assume that $g : [a,b] \to \mathbb{C}$ is absolutely continuous on $[a,b]$, $f : [a,b] \to \mathbb{C}$ is integrable on $[a,b]$ and $w : [a,b] \to [0,\infty)$ is integrable with $fw^{1/p} \in L_p[a,b]$, $g' \in L_q[a,b]$, where for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.*

(i) *If w is nonincreasing on $[a,b]$, then*

$$
\begin{aligned}
(3.6)\quad & \left|C_w(f,g)\right| \\
&\leq \left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b (t-a)\left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^p w(t)\,dt\right)^{1/p} \\
&\qquad \times \left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b (b-t)\left|g'(t)\right|^q w(t)\,dt\right)^{1/q} \\
&\leq (b-a)^{1/p}\left[\frac{1}{\int_a^b w(s)\,ds}\int_a^b \left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^p w(t)\,dt\right]^{1/p} \\
&\qquad \times \left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b (b-t)\left|g'(t)\right|^q w(t)\,dt\right)^{1/q}.
\end{aligned}
$$

(ii) *If w is nondecreasing on $[a,b]$, then*

$$
\begin{aligned}
(3.7)\quad & \left|C_w(f,g)\right| \\
&\leq \left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b (b-t)\left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^p w(t)\,dt\right)^{1/p} \\
&\qquad \times \left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b (t-a)\left|g'(t)\right|^q w(t)\,dt\right)^{1/q} \\
&\leq (b-a)^{1/p}\left[\frac{1}{\int_a^b w(s)\,ds}\int_a^b \left|f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^p w(t)\,dt\right]^{1/p} \\
&\qquad \times \left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b (t-a)\left|g'(t)\right|^q w(t)\,dt\right)^{1/q}.
\end{aligned}
$$

Proof. We use the following *Sonin type identity*

$$
\begin{aligned}
(3.8)\quad & C_w(f,g) \\
&= \frac{1}{\int_a^b w(s)\,ds}\int_a^b \left(f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right)(g(t)-\gamma)\,w(t)\,dt,
\end{aligned}
$$

for $\gamma \in \mathbb{C}$, which can be proved directly on calculating the integral from the right hand side.

Using the inequality (3.4) for $\gamma = g(a)$, we have

$$\begin{aligned}
&|C_w(f,g)| \\
&\leq \frac{1}{\int_a^b w(s)\,ds}\int_a^b \left|f(t) - \frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right| |g(t) - g(a)|\,w(t)\,dt \\
&\leq \frac{1}{\int_a^b w(s)\,ds}\left(\int_a^b (t-a)\left|f(t) - \frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^p w(t)\,dt\right)^{1/p} \\
&\qquad\qquad \times \left(\int_a^b (b-t)\,|g'(t)|^q\,w(t)\,dt\right)^{1/q}
\end{aligned}$$

that proves the first inequality in (3.6).

Since

$$\begin{aligned}
&\frac{1}{\int_a^b w(s)\,ds}\int_a^b (t-a)\left|f(t) - \frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^p w(t)\,dt \\
&\leq (b-a)\,\frac{1}{\int_a^b w(s)\,ds}\int_a^b \left|f(t) - \frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^p w(t)\,dt,
\end{aligned}$$

hence the second part of (3.6) follows.

Using the inequality (3.5) for $\gamma = g(b)$, we have

$$\begin{aligned}
&|C_w(f,g)| \\
&\leq \frac{1}{\int_a^b w(s)\,ds}\int_a^b \left|f(t) - \frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right| |g(t) - g(b)|\,w(t)\,dt \\
&\leq \frac{1}{\int_a^b w(s)\,ds}\left(\int_a^b (b-t)\left|f(t) - \frac{1}{\int_a^b w(s)\,ds}\int_a^b f(s)\,w(s)\,ds\right|^p w(t)\,dt\right)^{1/p} \\
&\qquad\qquad \times \left(\int_a^b (t-a)\,|g'(t)|^q\,w(t)\,dt\right)^{1/q}
\end{aligned}$$

that proves the first part of (3.7).

The second part follows in the same way as above. □

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