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SOME WEIGHTED VERSIONS OF STEKLOFF AND ALMANSI INEQUALITIES WITH APPLICATIONS

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Abstract. In this paper we establish some weighted versions of Stekloff and Almansi inequalities. Applications for bounding the weighted *Čebyšev functional* are also given.

1. Introduction

It is well known that, see for instance [5], or [9], if $u \in C^1([a,b],\mathbb{R})$, namely u is continuous on $[a,b]$ and has a derivative that is continuous on (a,b) and satisfies $u(a)=u(b)=0$, then the following *Wirtinger type inequality* is valid

$$\int_a^b u^2(t)\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.1}$$

with the equality holding if and only if $u(t)=K\sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant $K\in\mathbb{R}$.

If $u\in C^1([a,b],\mathbb{R})$ satisfies the condition $u(a)=0$, then also

$$\int_a^b u^2(t)\,dt \le \frac{4(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.2}$$

and the equality holds if and only if $u(t)=L\sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant $L\in\mathbb{R}$.

For some related Wirtinger type integral inequalities see [1], [3], [5] and [8]-[11].

In 1901, W. Stekloff, [13], proved that, if $u\in C^1([a,b],\mathbb{R})$ and $\int_a^b u(t)\,dt=0$, then

$$\int_a^b u^2(x)\,dx \le \frac{(b-a)^2}{\pi^2}\int_a^b [u'(x)]^2\,dx. \tag{1.3}$$

In addition, if $u(a)=u(b)$, then, as proved by E. Almansi in 1905, [1], the inequality (1.3) can be improved as follows

$$\int_a^b u^2(x)\,dx \le \frac{(b-a)^2}{4\pi^2}\int_a^b [u'(x)]^2\,dx. \tag{1.4}$$

We can state the following result for complex functions $h:[a,b]\to\mathbb{C}$.

Theorem 1. *If $h\in C^1([a,b],\mathbb{C})$ and $\int_a^b h(t)\,dt=0$, then*

$$\int_a^b |h(x)|^2\,dx \le \frac{(b-a)^2}{\pi^2}\int_a^b |h'(x)|^2\,dx. \tag{1.5}$$

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In addition, if $h(a) = h(b)$, then

$$\int_a^b |h(x)|^2 \, dx \le \frac{(b-a)^2}{4\pi^2} \int_a^b |h'(x)|^2 \, dx. \tag{1.6}$$

The proof follows by (1.3) and (1.4) applied for $u = \operatorname{Re} h$ and $u = \operatorname{Im} h$ and by adding the corresponding inequalities.

In the recent paper [6] we obtained the following simple weighted version of Wirtinger's inequality

Theorem 2. *Assume that $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ and $f \in C^1([a,b],\mathbb{C})$ is a function with complex values and $f(a) = f(b) = 0$, then*

$$\int_a^b |f(t)|^2 w(t) \, dt \le \frac{1}{\pi^2} \left(\int_a^b w(s) \, ds \right)^2 \int_a^b \frac{|f'(t)|^2}{w(t)} dt. \tag{1.7}$$

The equality holds in (3.14) iff

$$f(t) = K \sin \left[\frac{\pi \int_a^t w(s) \, ds}{\int_a^b w(s) \, ds} \right], \ K \in \mathbb{C}.$$

If $f(a) = 0$, then

$$\int_a^b |f(t)|^2 w(t) \, dt \le \frac{4}{\pi^2} \left(\int_a^b w(s) \, ds \right)^2 \int_a^b \frac{|f'(t)|^2}{w(t)} dt \tag{1.8}$$

with equality iff

$$f(t) = K \sin \left[\frac{\pi \int_a^t w(s) \, ds}{2 \int_a^b w(s) \, ds} \right], \ K \in \mathbb{C}.$$

Motivated by the above results, we establish in this paper some weighted versions of Stekloff and Almansi inequalities (1.5) and (1.6) above. Applications for bounding the weighted *Čebyšev functional* are also given.

2. Some Related Inequalities

If we assume that $g \in C^1([a,b],\mathbb{C})$ and take

$$h(t) = \tilde{g}(t) := \frac{1}{2} [g(a+b-t) - g(t)], \ t \in [a,b],$$

then $\int_a^b h(t) \, dt = 0$,

$$\begin{aligned}
\int_a^b |h(t)|^2 \, dt &= \frac{1}{4} \int_a^b |g(a+b-t) - g(t)|^2 \, dt \\
&= \frac{1}{4} \left[\int_a^b |g(a+b-t)|^2 - 2 \operatorname{Re} \left(g(a+b-t) \overline{g(t)} \right) + |g(t)|^2 \right] dt \\
&= \frac{1}{2} \left[\int_a^b |g(t)|^2 \, dt - \int_a^b \operatorname{Re} \left(g(a+b-t) \overline{g(t)} \right) dt \right]
\end{aligned}$$

and

$$\int_a^b |g'(t)|^2 dt = \frac{1}{4}\int_a^b |g'(a+b-t)+g'(t)|^2 dt$$
$$= \frac{1}{2}\left[\int_a^b |g'(t)|^2 dt + \int_a^b \operatorname{Re}\left(g'(a+b-t)\overline{g'(t)}\right) dt\right]$$

and by (1.5) we get

$$0 \le \int_a^b |g(t)|^2 dt - \int_a^b \operatorname{Re}\left(g(a+b-t)\overline{g(t)}\right) dt \tag{2.1}$$
$$\le \frac{(b-a)^2}{\pi^2}\left[\int_a^b |g'(t)|^2 + \int_a^b \operatorname{Re}\left(g'(a+b-t)\overline{g'(t)}\right) dt\right].$$

If we assume that $g \in C^1([a,b],\mathbb{C})$ with $\int_a^b g(t)\,dt = 0$, and if we take

$$h(t) = \breve{g}(t) := \frac{1}{2}[g(a+b-t)+g(t)]$$

we have $h(a) = h(b)$ and by (1.6) we have

$$0 \le \frac{1}{2}\left[\int_a^b |g(t)|^2 dt + \int_a^b \operatorname{Re}\left(g(a+b-t)\overline{g(t)}\right) dt\right] \tag{2.2}$$
$$\le \frac{(b-a)^2}{16\pi^2}\int_a^b |g'(t)-g'(a+b-t)|^2 dt$$
$$= \frac{(b-a)^2}{8\pi^2}\left[\int_a^b |g'(t)|^2 - \int_a^b \operatorname{Re}\left(g'(a+b-t)\overline{g'(t)}\right) dt\right].$$

If g' is Lipschitzian with the constant K, namely $|g'(t)-g'(s)| \le K|t-s|$, then

$$\frac{1}{2}|g'(t)-g'(a+b-t)| \le K\left|t-\frac{a+b}{2}\right|,\ t\in[a,b].$$

By the inequality (2.2) we have

$$0 \le \int_a^b |g(t)|^2 dt + \int_a^b \operatorname{Re}\left(g(a+b-t)\overline{g(t)}\right) dt$$
$$\le \frac{(b-a)^2}{2\pi^2}\int_a^b \left|\frac{g'(t)-g'(a+b-t)}{2}\right|^2 dt \le \frac{(b-a)^2}{2\pi^2}K^2\int_a^b \left|t-\frac{a+b}{2}\right|^2 dt$$

and since

$$\int_a^b \left|t-\frac{a+b}{2}\right|^2 dt = \frac{(b-a)^3}{12},$$

hence we obtain the inequality

$$0 \le \int_a^b |g(t)|^2 dt + \int_a^b \operatorname{Re}\left(g(a+b-t)\overline{g(t)}\right) dt \le \frac{(b-a)^3}{24\pi^2}K^2. \tag{2.3}$$

If we assume that $g \in C^1([a,b],\mathbb{C})$ and take $h = g - \frac{1}{b-a}\int_a^b g(s)\,ds$ in (1.5), then we get

$$\int_a^b \left|g(x) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 dx \le \frac{(b-a)^2}{\pi^2}\int_a^b |g'(x)|^2 dx,$$

which is equivalent to

$$(2.4)\qquad 0\leq\frac{1}{b-a}\int_a^b|g(x)|^2\,dx-\left|\frac{1}{b-a}\int_a^b g(x)\,dx\right|^2\leq\frac{b-a}{\pi^2}\int_a^b|g'(x)|^2\,dx.$$

If $g\in C^1([a,b],\mathbb{C})$ with $g(a)=g(b)$, then we have a better inequality than (2.4), namely

$$(2.5)\qquad 0\leq\frac{1}{b-a}\int_a^b|g(x)|^2\,dx-\left|\frac{1}{b-a}\int_a^b g(x)\,dx\right|^2\leq\frac{b-a}{4\pi^2}\int_a^b|g'(x)|^2\,dx.$$

Moreover, if we write the inequality (2.5) for $\breve{g}$ that is symmetrical on $[a,b]$, then we get

$$(2.6)\qquad 0\leq\frac{1}{b-a}\int_a^b|\breve{g}(x)|^2\,dx-\left|\frac{1}{b-a}\int_a^b g(x)\,dx\right|^2\leq\frac{b-a}{4\pi^2}\int_a^b\left|\widetilde{g'}(x)\right|^2\,dx.$$

In addition, if g' is Lipschitzian with the constant $K>0$, then we get from (2.6) that

$$(2.7)\qquad 0\leq\frac{1}{b-a}\int_a^b|\breve{g}(x)|^2\,dx-\left|\frac{1}{b-a}\int_a^b g(x)\,dx\right|^2\leq\frac{1}{48\pi^2}K^2(b-a)^4.$$

Finally if g is *symmetrical* on $[a,b]$, namely $g(a+b-t)=g(t)$ for any $t\in[a,b]$, g' is Lipschitzian with the constant $K>0$, then we get from (2.7) that

$$(2.8)\qquad 0\leq\frac{1}{b-a}\int_a^b|g(x)|^2\,dx-\left|\frac{1}{b-a}\int_a^b g(x)\,dx\right|^2\leq\frac{1}{48\pi^2}K^2(b-a)^4.$$

3. Composite Inequalities

We have:

Theorem 3. *Let $g:[a,b]\to[g(a),g(b)]$ be a continuous strictly increasing function that is of class C^1 on (a,b).*

(i) *If $f\in C^1([a,b],\mathbb{C})$ with $\frac{f'}{\sqrt{g'(t)}}\in L_2[a,b]$ and $\int_a^b f(t)\,g'(t)\,dt=0$, then*

$$(3.1)\qquad \int_a^b|f(t)|^2\,g'(t)\,dt\leq\frac{[g(b)-g(a)]^2}{\pi^2}\int_a^b\frac{|f'(t)|^2}{g'(t)}dt.$$

(ii) *In addition, if $f(a)=f(b)$, then we have the better inequality*

$$(3.2)\qquad \int_a^b|f(t)|^2\,g'(t)\,dt\leq\frac{[g(b)-g(a)]^2}{4\pi^2}\int_a^b\frac{|f'(t)|^2}{g'(t)}dt.$$

Proof. (i) We write the inequality (1.5) for the function $h=f\circ g^{-1}$ on the interval $[g(a),g(b)]$ to get

$$(3.3)\qquad \int_{g(a)}^{g(b)}\left|\left(f\circ g^{-1}\right)(z)\right|^2dz\leq\frac{(g(b)-g(a))^2}{\pi^2}\int_{g(a)}^{g(b)}\left|\left(f\circ g^{-1}\right)'(z)\right|^2dz,$$

provided

$$\int_{g(a)}^{g(b)}f\circ g^{-1}(z)\,dz=0.$$

If $f : [c,d] \to \mathbb{C}$ is absolutely continuous on $[c,d]$, then $f \circ g^{-1} : [g(c), g(d)] \to \mathbb{C}$ is absolutely continuous on $[g(c), g(d)]$ and using the chain rule and the derivative of inverse functions we have

$$(f \circ g^{-1})'(z) = (f' \circ g^{-1})(z)(g^{-1})'(z) = \frac{(f' \circ g^{-1})(z)}{(g' \circ g^{-1})(z)} \tag{3.4}$$

for almost every (a.e.) $z \in [g(c), g(d)]$.

Using the inequality (3.3) we then get

$$\int_{g(a)}^{g(b)} \left|(f \circ g^{-1})(z)\right|^2 dz \le \frac{(g(b) - g(a))^2}{\pi^2} \int_{g(a)}^{g(b)} \left|\frac{(f' \circ g^{-1})(z)}{(g' \circ g^{-1})(z)}\right|^2 dz, \tag{3.5}$$

provided $\int_{g(a)}^{g(b)} f \circ g^{-1}(z)\, dz = 0$.

Observe also that, by the change of variable $t = g^{-1}(z)$, $z \in [g(a), g(b)]$, we have $z = g(t)$ that gives $dz = g'(t)\, dt$,

$$\int_{g(a)}^{g(b)} f \circ g^{-1}(z)\, dx = \int_a^b f(t)\, h'(t)\, dt,$$

and

$$\int_{g(a)}^{g(b)} \left|(f \circ g^{-1})(z)\right|^2 dz = \int_a^b |f(t)|^2 g'(t)\, dt. \tag{3.6}$$

We also have

$$\int_{g(a)}^{g(b)} \left|\frac{(f' \circ g^{-1})(z)}{(g' \circ g^{-1})(z)}\right|^2 dz = \int_a^b \left|\frac{f'(t)}{g'(t)}\right|^2 g'(t)\, dt = \int_a^b \frac{|f'(t)|^2}{g'(t)} dt.$$

By making use of (3.5) we get (3.1).

(ii) The inequality (3.2) follows by (3.2) in a similar way. □

a). If we take $g : [a,b] \subset (0,\infty) \to \mathbb{R}$, $g(t) = \ln t$ and assume that $f \in C^1([a,b], \mathbb{C})$ is a function with complex values and

$$\int_a^b \frac{f(t)}{t} dt = 0, \tag{3.7}$$

then by (3.1) we get

$$\int_a^b \frac{|f(t)|^2}{t} dt \le \frac{\left[\ln\left(\frac{b}{a}\right)\right]^2}{\pi^2} \int_a^b |f'(t)|^2 t dt. \tag{3.8}$$

In addition, if $f(a) = f(b)$, then

$$\int_a^b \frac{|f(t)|^2}{t} dt \le \frac{\left[\ln\left(\frac{b}{a}\right)\right]^2}{4\pi^2} \int_a^b |f'(t)|^2 t dt. \tag{3.9}$$

b). If we take $g : [a,b] \subset \mathbb{R} \to (0,\infty)$, $g(t) = \exp t$ and assume that $f \in C^1([a,b], \mathbb{C})$ is a function with complex values and

$$\int_a^b f(t) \exp t dt = 0,$$

then by (3.1) we get

$$\int_a^b |f(t)|^2 \exp t dt \le \frac{(\exp b - \exp a)^2}{\pi^2} \int_a^b |f'(t)|^2 \exp(-t)\, dt. \tag{3.10}$$

In addition, if $f(a) = f(b)$, then

$$\int_a^b |f(t)|^2 \exp t dt \le \frac{(\exp b - \exp a)^2}{4\pi^2} \int_a^b |f'(t)|^2 \exp(-t) \, dt. \tag{3.11}$$

c). If we take $g : [a,b] \subset (0,\infty) \to \mathbb{R}$, $g(t) = t^r$, $r > 0$ and assume that $f \in C^1([a,b],\mathbb{C})$ is a function with complex values and

$$\int_a^b f(t) t^r dt = 0,$$

then by (3.1) we get

$$\int_a^b |f(t)|^2 t^{r-1} dt \le \frac{(b^r - a^r)^2}{r^2\pi^2} \int_a^b |f'(t)|^2 t^{1-r} dt. \tag{3.12}$$

In addition, if $f(a) = f(b)$, then

$$\int_a^b |f(t)|^2 t^{r-1} dt \le \frac{(b^r - a^r)^2}{4r^2\pi^2} \int_a^b |f'(t)|^2 t^{1-r} dt. \tag{3.13}$$

If $w : [a,b] \to \mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W : [a,b] \to [0,\infty)$, $W(x) := \int_a^x w(s) \, ds$ is strictly increasing and differentiable on (a,b). We have $W'(x) = w(x)$ for any $x \in (a,b)$.

Corollary 1. *Assume that $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ and $f \in C^1([a,b],\mathbb{C})$.*

(i) *If $\frac{f'}{\sqrt{w}} \in L_2[a,b]$ and $\int_a^b f(t) w(t) \, dt = 0$, then*

$$\int_a^b |f(t)|^2 w(t) \, dt \le \frac{1}{\pi^2} \left(\int_a^b w(s) \, ds \right)^2 \int_a^b \frac{|f'(t)|^2}{w(t)} dt. \tag{3.14}$$

(ii) *In addition, if $f(a) = f(b)$, then we have the better inequality*

$$\int_a^b |f(t)|^2 w(t) \, dt \le \frac{1}{4\pi^2} \left(\int_a^b w(s) \, ds \right)^2 \int_a^b \frac{|f'(t)|^2}{w(t)} dt. \tag{3.15}$$

4. Some Inequalities for the Weighted Čebyšev Functional

Consider now the *weighted Čebyšev functional*

$$\begin{aligned} C_w(f,g) := & \frac{1}{\int_a^b w(t) \, dt} \int_a^b w(t) f(t) g(t) \, dt \\ & - \frac{1}{\int_a^b w(t) \, dt} \int_a^b w(t) f(t) \, dt \frac{1}{\int_a^b w(t) \, dt} \int_a^b w(t) g(t) \, dt \end{aligned} \tag{4.1}$$

where f, g, $w : [a,b] \to \mathbb{R}$ and $w(t) \ge 0$ for a.e. $t \in [a,b]$ are measurable functions such that the involved integrals exist and $\int_a^b w(t) \, dt > 0$.

In [4], Cerone and Dragomir obtained, among others, the following inequalities:

$$\begin{aligned}(4.2)\quad & \left|C_w(f,g)\right| \\ & \leq \frac{1}{2}(M-m)\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)\,g(s)\,ds\right|dt \\ & \leq \frac{1}{2}(M-m)\left[\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)\,g(s)\,ds\right|^p dt\right]^{\frac{1}{p}} \\ & \leq \frac{1}{2}(M-m)\operatorname*{essup}_{t\in[a,b]}\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)\,g(s)\,ds\right|\end{aligned}$$

for $p>1$, provided $-\infty<m\leq f(t)\leq M<\infty$ for a.e. $t\in[a,b]$ and the corresponding integrals are finite. The constant $\frac{1}{2}$ is sharp in all the inequalities in (4.2) in the sense that it cannot be replaced by a smaller constant.

In addition, if $-\infty<n\leq g(t)\leq N<\infty$ for a.e. $t\in[a,b]$, then the following refinement of the celebrated Grüss inequality is obtained:

$$\begin{aligned}(4.3)\quad & \left|C_w(f,g)\right| \\ & \leq \frac{1}{2}(M-m)\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)\,g(s)\,ds\right|dt \\ & \leq \frac{1}{2}(M-m)\left[\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)\,g(s)\,ds\right|^2 dt\right]^{\frac{1}{2}} \\ & \leq \frac{1}{4}(M-m)(N-n).\end{aligned}$$

Here, the constants $\frac{1}{2}$ and $\frac{1}{4}$ are also sharp in the sense mentioned above.

We have:

Lemma 1. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ and $h\in C^1([a,b],\mathbb{C})$ with $\frac{h'}{\sqrt{w}}\in L_2[a,b]$. Then*

$$\begin{aligned}(4.4)\quad & 0\leq C_w\left(h,\bar{h}\right) \\ & =\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\left|h(t)\right|^2 dt-\left|\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\,h(t)\,dt\right|^2 \\ & \leq \frac{1}{\pi^2}\int_a^b w(t)\,dt\int_a^b \frac{\left|h'(t)\right|^2}{w(t)}dt.\end{aligned}$$

In addition, if $h(a)=h(b)$, then

$$0\leq C_w\left(h,\bar{h}\right)\leq \frac{1}{4\pi^2}\int_a^b w(t)\,dt\int_a^b \frac{\left|h'(t)\right|^2}{w(t)}dt. \tag{4.5}$$

Proof. Let

$$f(t):=h(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)\,h(s)\,ds,$$

then

$$\int_a^b f(t)\,w(t)\,dt=\int_a^b\left(h(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)\,h(s)\,ds\right)w(t)\,dt=0.$$

From (3.14) we have

$$\frac{1}{\int_a^b w(s)\,ds}\int_a^b\left|h(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)\,h(s)\,ds\right|^2 w(t)\,dt \le \frac{1}{\pi^2}\int_a^b w(t)\,dt\int_a^b \frac{|h'(t)|^2}{w(t)}dt \tag{4.6}$$

and since

$$\begin{aligned}&\frac{1}{\int_a^b w(s)\,ds}\int_a^b\left|h(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)\,h(s)\,ds\right|^2 w(t)\,dt\\ &=\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\,|h(t)|^2\,dt-\left|\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\,h(t)\,dt\right|^2,\end{aligned}$$

hence the inequality (4.4) is proved.

The inequality (4.5) follows by (3.15) in a similar way and we omit the details. □

We have the following Grüss' type inequality:

Theorem 4. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ and $f,\ g\in C^1([a,b],\mathbb{C})$ with $\frac{f'}{\sqrt{w}},\ \frac{g'}{\sqrt{w}}\in L_2[a,b]$. Then*

$$|C_w(f,g)|\le\left[C_w\left(f,\bar{f}\right)\right]^{1/2}\left[C_w(g,\bar{g})\right]^{1/2} \tag{4.7}$$

$$\le\begin{cases}\frac{1}{\pi^2}\int_a^b w(t)\,dt\left(\int_a^b\frac{|f'(t)|^2}{w(t)}dt\right)^{1/2}\left(\int_a^b\frac{|g'(t)|^2}{w(t)}dt\right)^{1/2}, \\[2ex] \frac{1}{2\pi^2}\int_a^b w(t)\,dt\left(\int_a^b\frac{|f'(t)|^2}{w(t)}dt\right)^{1/2}\left(\int_a^b\frac{|g'(t)|^2}{w(t)}dt\right)^{1/2} \\ \text{if either } f(a)=f(b)\ \text{ or } g(a)=g(b), \\[2ex] \frac{1}{4\pi^2}\int_a^b w(t)\,dt\left(\int_a^b\frac{|f'(t)|^2}{w(t)}dt\right)^{1/2}\left(\int_a^b\frac{|g'(t)|^2}{w(t)}dt\right)^{1/2} \\ \text{if } f(a)=f(b)\ \text{ and } g(a)=g(b).\end{cases}$$

Proof. The first inequality follows by the equality

$$\begin{aligned}C_w(f,g)=\frac{1}{\int_a^b w(s)\,ds}\int_a^b&\left(f(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)\,f(s)\,ds\right)\\ &\times\left(g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)\,g(s)\,ds\right)w(t)\,dt\end{aligned}$$

and by the Cauchy-Bunyakovsky-Schwarz inequality.

The rest follows by Lemma 1 □

For $w\equiv 1$ we consider the unweighted *Čebyšev functional*

$$C(f,g):=\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt.$$

We have the following particular result:

Corollary 2. *Assume that $f,\ g\in C^1([a,b],\mathbb{C})$ with $f',\ g'\in L_2[a,b]$. Then*

$$|C(f,g)|\le\left[C\left(f,\bar f\right)\right]^{1/2}\left[C(g,\bar g)\right]^{1/2} \tag{4.8}$$
$$\le\begin{cases}\frac{1}{\pi^2}(b-a)\left(\int_a^b|f'(t)|^2dt\right)^{1/2}\left(\int_a^b|g'(t)|^2dt\right)^{1/2},\\[2ex] \frac{1}{2\pi^2}(b-a)\left(\int_a^b|f'(t)|^2dt\right)^{1/2}\left(\int_a^b|g'(t)|^2dt\right)^{1/2}\\ \text{if either } f(a)=f(b)\ \text{ or } g(a)=g(b),\\[2ex] \frac{1}{4\pi^2}(b-a)\left(\int_a^b|f'(t)|^2dt\right)^{1/2}\left(\int_a^b|g'(t)|^2dt\right)^{1/2}\\ \text{if } f(a)=f(b)\ \text{ and } g(a)=g(b).\end{cases}$$

Remark 1. *The first inequality in (4.8) in the case of real functions was obtained by Lupaş in 1973,* [10].

We observe that

$$\begin{aligned}C\left(\breve f,g\right)&=\frac{1}{b-a}\int_a^b\breve f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b\breve f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt\\&=\frac{1}{b-a}\int_a^b\frac{f(t)+f(a+b-t)}{2}g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt\\&=\frac{1}{b-a}\int_a^b f(t)\,\frac{g(t)+g(a+b-t)}{2}dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt\\&=C(f,\breve g)\end{aligned}$$

and

$$\begin{aligned}C\left(\breve f,\breve g\right)&=\frac{1}{b-a}\int_a^b\breve f(t)\,\breve g(t)\,dt-\frac{1}{b-a}\int_a^b\breve f(t)\,dt\frac{1}{b-a}\int_a^b\breve g(t)\,dt\\&=\frac{1}{b-a}\int_a^b\breve f(t)\,\frac{g(t)+g(a+b-t)}{2}dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt\\&=C\left(\breve f,g\right).\end{aligned}$$

Proposition 1. *Assume that $f,\ g\in C^1([a,b],\mathbb{C})$.*

(i) *If f' is Lipschitzian with the constant K and $g'\in L_2[a,b]$, then*

$$\left|C\left(\breve f,g\right)\right|\le\frac{\sqrt3}{12\pi^2}K(b-a)^{5/2}\left(\int_a^b|g'(t)|^2dt\right)^{1/2}. \tag{4.9}$$

(ii) *If f' is Lipschitzian with the constant K and g' is Lipschitzian with the constant $L>0$, then*

$$\left|C\left(\breve f,g\right)\right|\le\frac{1}{48\pi^2}KL(b-a)^4. \tag{4.10}$$

The inequality (4.9) follows by the second inequality in (4.8) for the functions $\breve{f}$ and g while the inequality (4.10) follows by the third inequality in (4.8) for the functions $\breve{f}$ and $\breve{g}$.

Corollary 3. *Assume that* $f,\, g \in C^1([a,b],\mathbb{C})$.

(i) *If f is symmetrical on $[a,b]$, f' is Lipschitzian with the constant K and $g' \in L_2[a,b]$, then*

$$|C(f,g)| \leq \frac{\sqrt{3}}{12\pi^2} K (b-a)^{5/2} \left(\int_a^b |g'(t)|^2 \, dt \right)^{1/2}. \tag{4.11}$$

(ii) *If f and g are symmetrical on $[a,b]$, f' is Lipschitzian with the constant K and g' is Lipschitzian with the constant $L > 0$, then*

$$|C(f,g)| \leq \frac{1}{48\pi^2} KL (b-a)^4. \tag{4.12}$$

References

[1] E. Almansi, Sopra una delle esperienze di Plateau. (Italian) *Ann. Mat. Pura Appl.* **12** (1905), No. 3, 1-17.

[2] M. W. Alomari, On Beesack–Wirtinger Inequality, *Results Math.*, Online First 2017 Springer International Publishing DOI 10.1007/s00025-016-0644-6.

[3] P. R. Beesack, Extensions of Wirtinger's inequality. *Trans. R. Soc. Can.* **53**, 21–30 (1959)

[4] P. Cerone and S. S. Dragomir, A refinement of the Grüss inequality and applications, *Tamkang J. Math.*, **38**(1) (2007), 37-49. Preprint *RGMIA Res. Rep. Coll.,* **5**(2) (2002), Article 14. [Online: `http://rgmia.vu.edu.au/v5n2.html`].

[5] J. B. Diaz and F. T. Metcalf, Variations on Wirtinger's inequality, in: *Inequalities* Academic Press, New York, 1967, pp. 79–103.

[6] S. S. Dragomir, Integral inequalities related to Wirtinger's result, Preprint *RGMIA Res. Rep. Coll.* , **21** (2018), Art. 59, `[Online http://rgmia.org/papers/v21/v21a59.pdf]`.

[7] L. Fejér, Über die Fourierreihen, II, (In Hungarian) *Math. Naturwiss, Anz. Ungar. Akad. Wiss.*, **24** (1906), 369-390.

[8] R. Giova, An estimate for the best constant in the L_p-Wirtinger inequality with weights, *J. Func. Spaces Appl.*,Volume **6**, Number 1 (2008), 1-16.

[9] J. Jaroš, On an integral inequality of the Wirtinger type, *Appl. Math. Letters,* **24** (2011) 1389–1392.

[10] A. Lupaş, The best constant in an integral inequality, *Mathematica* (Cluj, Romania), **15**(**38**)(2) (1973), 219-222.

[11] T. Ricciardi, A sharp weighted Wirtinger inequality, *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat.* (8), **8** (1) (2005), 259–267.

[12] C. F. Lee, C. C. Yeh, C. H. Hong and R. P. Agarwal, Lyapunov and Wirtinger inequalities, *Appl. Math. Lett.* **17** (2004) 847–853.

[13] W. Stekloff, Problème de refroidissement d'une barre hétérogène. (French) *Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys.* **3** (1901), No. 3, 281–313.

[14] C. A. Swanson, Wirtinger's inequality, *SIAM J. Math. Anal.* **9** (1978) 484–491.

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