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GENERALIZATIONS OF OPIAL'S INEQUALITIES FOR RIEMANN-STIELTJES INTEGRAL WITH APPLICATIONS

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ABSTRACT. In this paper we establish some generalizations of Opial's inequalities for Riemann-Stieltjes integral and for two functions. Applications related to trapezoid and Grüss' type inequalities are also given.

1. INTRODUCTION

We recall the following Opial type inequalities:

Theorem 1. *Assume that $u:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is an absolutely continuous function on the interval $[a,b]$ and such that $u'\in L_2[a,b]$.*

(i) *If $u(a)=u(b)=0$, then*

$$\int_a^b |u(t)\,u'(t)|\,dt\leq\frac{1}{4}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.1}$$

with equality if and only if

$$u(t)=\begin{cases} c(t-a) & \text{if } a\leq t\leq\frac{a+b}{2},\\ c(b-t) & \text{if } \frac{a+b}{2}<t\leq b,\end{cases}$$

where c is an arbitrary constant.

(ii) *If $u(a)=0$, then*

$$\int_a^b |u(t)\,u'(t)|\,dt\leq\frac{1}{2}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.2}$$

with equality if and only if $u(t)=c(t-a)$ for some constant c.

The inequality (1.1) was obtained by Olech in [15] in which he gave a simplified proof of an inequality originally due in a slightly less general form to Zdzislaw Opial [16].

Embedded in Olech's proof is the half-interval form of Opial's inequality, also discovered by Beesack [2], which is satisfied by those u vanishing only at a.

For various proofs of the above inequalities, see [11]-[14] and [18].

In the recent paper [7] we obtained the following generalization of Opial's inequalities for two functions:

Theorem 2. *Assume that $f,\ g:[a,b]\to\mathbb{C}$ are absolutely continuous on $[a,b]$ with $f',\ g'\in L_2[a,b]$.*

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(i) *If $g(a)=0$, then*

$$\int_a^b |f'(t)\,g(t)|\,dt \le \left(\int_a^b (t-a)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b (b-t)\,|g'(t)|^2\,dt\right)^{1/2} \le \frac{1}{2}\int_a^b \left[(t-a)\,|f'(t)|^2+(b-t)\,|g'(t)|^2\right]dt. \tag{1.3}$$

(ii) *If $g(b)=0$, then*

$$\int_a^b |f'(t)\,g(t)|\,dt \le \left(\int_a^b (b-t)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b (t-a)\,|g'(t)|^2\,dt\right)^{1/2} \le \frac{1}{2}\int_a^b \left[(b-t)\,|f'(t)|^2+(t-a)\,|g'(t)|^2\right]dt. \tag{1.4}$$

(iii) *If $g(a)=g(b)=0$, then*

$$\begin{aligned}\int_a^b |f'(t)\,g(t)|\,dt &\le \left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,dt-\int_a^b \left|t-\frac{a+b}{2}\right||f'(t)|^2\,dt\right)^{1/2}\\ &\quad\times\left(\int_a^b \left|\frac{a+b}{2}-t\right||g'(t)|^2\,dt\right)^{1/2}\\ &\le \frac{1}{4}(b-a)\int_a^b |f'(t)|^2\,dt+\frac{1}{2}\int_a^b \left|\frac{a+b}{2}-t\right|\left(|g'(t)|^2-|f'(t)|^2\right)dt.\end{aligned} \tag{1.5}$$

By taking $g=f$ we obtain the following refinement of Opial's inequalities from Theorem 1:

Corollary 1. *Assume that $f:[a,b]\to\mathbb{C}$ is absolutely continuous on $[a,b]$ with $f'\in L_2[a,b]$.*

(i) *If either $f(a)=0$ or $f(b)=0$, then*

$$\int_a^b |f'(t)\,f(t)|\,dt \le \left(\int_a^b (t-a)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b (b-t)\,|f'(t)|^2\,dt\right)^{1/2} \le \frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,dt. \tag{1.6}$$

(ii) *If $f(a)=f(b)=0$, then*

$$\begin{aligned}\int_a^b |f'(t)\,f(t)|\,dt &\le \left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,dt-\int_a^b \left|t-\frac{a+b}{2}\right||f'(t)|^2\,dt\right)^{1/2}\\ &\quad\times\left(\int_a^b \left|\frac{a+b}{2}-t\right||f'(t)|^2\,dt\right)^{1/2} \le \frac{1}{4}(b-a)\int_a^b |f'(t)|^2\,dt.\end{aligned} \tag{1.7}$$

Motivated by the above results, in this paper we establish some generalizations of Opial's inequalities for Riemann-Stieltjes integral and for two functions. Applications related to trapezoid and Grüss' type inequalities are also given.

2. The Main Results

We have:

Theorem 3. *Assume that $f : [a,b] \to \mathbb{C}$ is in $C^1[a,b]$, namely is continuous on $[a,b]$, differentiable on (a,b) with the derivative f' continuous on (a,b), g is absolutely continuous on $[a,b]$ and u is monotonic nondecreasing on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b |f'(t)\, g(t)|\, du(t)$ exists and*

(i) *If $g(a) = 0$, then*

$$\begin{aligned} &\int_a^b |f'(t)\, g(t)|\, du(t) \\ &\leq \left(\int_a^b (t-a) |f'(t)|^2 du(t) \right)^{1/2} \left(\int_a^b [u(b) - u(t)] |g'(t)|^2 dt \right)^{1/2} \\ &\leq \frac{1}{2} \left\{ \int_a^b (t-a) |f'(t)|^2 du(t) + \int_a^b [u(b) - u(t)] |g'(t)|^2 dt \right\}. \end{aligned} \tag{2.1}$$

(ii) *If $g(b) = 0$, then*

$$\begin{aligned} &\int_a^b |f'(t)\, g(t)|\, du(t) \\ &\leq \left(\int_a^b (b-t) |f'(t)|^2 du(t) \right)^{1/2} \left(\int_a^b [u(t) - u(a)] |g'(t)|^2 dt \right)^{1/2} \\ &\leq \frac{1}{2} \left\{ \int_a^b (b-t) |f'(t)|^2 du(t) + \int_a^b [u(t) - u(a)] |g'(t)|^2 dt \right\}. \end{aligned} \tag{2.2}$$

Proof. (i) Since $g(a) = 0$, then $g(t) = \int_a^t g'(s)\, ds$ and

$$\begin{aligned} \int_a^b |f'(t)\, g(t)|\, du(t) &= \int_a^b |f'(t)| \left| \int_a^t g'(s)\, ds \right| du(t) \\ &= \int_a^b (t-a)^{1/2} |f'(t)| \frac{1}{(t-a)^{1/2}} \left| \int_a^t g'(s)\, ds \right| du(t) =: I. \end{aligned}$$

By Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$\frac{1}{(t-a)^{1/2}} \left| \int_a^t g'(s)\, ds \right| \leq \left(\int_a^t |g'(s)|^2 ds \right)^{1/2}, \ t \in (a,b]$$

and then

$$I \leq \int_a^b (t-a)^{1/2} |f'(t)| \left(\int_a^t |g'(s)|^2 ds \right)^{1/2} du(t). \tag{2.3}$$

We use the following Cauchy-Bunyakovsky-Schwarz integral inequality for the Riemann-Stieltjes integral for monotonic nondecreasing integrators $u:[a,b]\to\mathbb{R}$ and continuous integrands $h,\ z:[a,b]\to\mathbb{C}$

$$\left|\int_a^b h(t)\,z(t)\,du(t)\right|^2 \le \int_a^b |h(t)|^2\,du(t)\int_a^b |z(t)|^2\,du(t). \tag{2.4}$$

Therefore,

$$\begin{aligned}&\int_a^b (t-a)^{1/2}\,|f'(t)|\left(\int_a^t |g'(s)|^2\,ds\right)^{1/2} du(t)\\ &\le \left(\int_a^b (t-a)\,|f'(t)|^2\,du(t)\right)^{1/2}\left(\int_a^b\left(\int_a^t |g'(s)|^2\,ds\right)du(t)\right)^{1/2}.\end{aligned} \tag{2.5}$$

Using the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\begin{aligned}\int_a^b\left(\int_a^t |g'(s)|^2\,ds\right)du(t) &= \left(\int_a^b |g'(s)|^2\,ds\right)u(b)-\int_a^b u(t)\,|g'(t)|^2\,dt\\ &= \int_a^b [u(b)-u(t)]\,|g'(t)|^2\,dt.\end{aligned}$$

By (2.3) and (2.5) we then get (2.1).

The second part follows by the elementary inequality

$$\sqrt{\alpha\beta}\le\frac{1}{2}(\alpha+\beta)\ \text{ for }\alpha,\ \beta\ge 0. \tag{2.6}$$

(ii) Since $g(a)=0$, then $g(t)=-\int_t^b g'(s)\,ds$ and

$$\begin{aligned}\int_a^b |f'(t)\,g(t)|\,du(t) &= \int_a^b |f'(t)|\left|\int_t^b g'(s)\,ds\right|du(t)\\ &= \int_a^b (b-t)^{1/2}\,|f'(t)|\,\frac{1}{(b-t)^{1/2}}\left|\int_t^b g'(s)\,ds\right|du(t)=:J.\end{aligned}$$

By Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$\frac{1}{(b-t)^{1/2}}\left|\int_t^b g'(s)\,ds\right|\le\left(\int_t^b |g'(s)|^2\,ds\right)^{1/2},\ t\in(a,b]$$

and then

$$J\le\int_a^b (b-t)^{1/2}\,|f'(t)|\left(\int_t^b |g'(s)|^2\,ds\right)^{1/2}du(t). \tag{2.7}$$

By (2.4) we then get

$$\begin{aligned}&\int_a^b (b-t)^{1/2}\,|f'(t)|\left(\int_t^b |g'(s)|^2\,ds\right)^{1/2} du(t)\\ &\le\left(\int_a^b (b-t)\,|f'(t)|^2\,du(t)\right)^{1/2}\left(\int_a^b\left(\int_t^b |g'(s)|^2\,ds\right)du(t)\right).\end{aligned} \tag{2.8}$$

Using the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\int_a^b \left(\int_t^b \left| g'(s) \right|^2 ds \right) du(t) = -u(a) \left(\int_a^b \left| g'(s) \right|^2 ds \right) + \int_a^b u(t) \left| g'(t) \right|^2 dt$$
$$= \int_a^b \left[u(t) - u(a) \right] \left| g'(t) \right|^2 dt$$

and by (2.7) and (2.8) we get (2.2). □

Remark 1. *Since u is monotonic nondecreasing on $[a,b]$, then from (2.1) we get*

$$\begin{aligned} (2.9) \qquad & \int_a^b \left| f'(t) g(t) \right| du(t) \\ & \leq \left[u(b) - u(a) \right]^{1/2} \left(\int_a^b (t-a) \left| f'(t) \right|^2 du(t) \right)^{1/2} \left(\int_a^b \left| g'(t) \right|^2 dt \right)^{1/2} \\ & \leq \frac{1}{2} \left[u(b) - u(a) \right]^{1/2} \left\{ \int_a^b (t-a) \left| f'(t) \right|^2 du(t) + \int_a^b \left| g'(t) \right|^2 dt \right\}, \end{aligned}$$

provided $g(a) = 0$ and from (2.2) we get

$$\begin{aligned} (2.10) \qquad & \int_a^b \left| f'(t) g(t) \right| du(t) \\ & \leq \left[u(b) - u(a) \right]^{1/2} \left(\int_a^b (b-t) \left| f'(t) \right|^2 du(t) \right)^{1/2} \left(\int_a^b \left| g'(t) \right|^2 dt \right)^{1/2} \\ & \leq \frac{1}{2} \left[u(b) - u(a) \right]^{1/2} \left\{ \int_a^b (b-t) \left| f'(t) \right|^2 du(t) + \int_a^b \left| g'(t) \right|^2 dt \right\}, \end{aligned}$$

provided $g(b) = 0$.

Corollary 2. *Assume that $f : [a,b] \to \mathbb{C}$ is in $C^1[a,b]$ and u is monotonic nondecreasing on $[a,b]$. Then the Riemann-Stieltjes integral $\int_a^b \left| f'(t) f(t) \right| du(t)$ exists and*

(i) *If $f(a) = 0$, then*

$$\begin{aligned} (2.11) \qquad & \int_a^b \left| f'(t) f(t) \right| du(t) \\ & \leq \left(\int_a^b (t-a) \left| f'(t) \right|^2 du(t) \right)^{1/2} \left(\int_a^b \left[u(b) - u(t) \right] \left| f'(t) \right|^2 dt \right)^{1/2} \\ & \leq \frac{1}{2} \left\{ \int_a^b (t-a) \left| f'(t) \right|^2 du(t) + \int_a^b \left[u(b) - u(t) \right] \left| f'(t) \right|^2 dt \right\}. \end{aligned}$$

(ii) *If* $f(b)=0$, *then*

$$\begin{aligned}(2.12)\qquad &\int_a^b |f'(t)\,f(t)|\,du(t)\\ &\leq \left(\int_a^b (b-t)\,|f'(t)|^2\,du(t)\right)^{1/2}\left(\int_a^b [u(t)-u(a)]\,|f'(t)|^2\,dt\right)^{1/2}\\ &\leq \frac{1}{2}\left\{\int_a^b (b-t)\,|f'(t)|^2\,du(t)+\int_a^b [u(t)-u(a)]\,|f'(t)|^2\,dt\right\}.\end{aligned}$$

Remark 2. *If* $w:[a,b]\to[0,\infty)$ *is continuous and we take* $u(t)=\int_a^t w(s)\,ds$ *in (2.11) and (2.12), then we get the weighted integral inequalities*

$$\begin{aligned}(2.13)\qquad &\int_a^b |f'(t)\,f(t)|\,w(t)\,dt\\ &\leq \left(\int_a^b (t-a)\,|f'(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_a^b |f'(t)|^2\left(\int_t^b w(s)\,ds\right)dt\right)^{1/2}\\ &\leq \frac{1}{2}\left\{\int_a^b (t-a)\,|f'(t)|^2\,w(t)\,dt+\int_a^b |f'(t)|^2\left(\int_t^b w(s)\,ds\right)dt\right\}\end{aligned}$$

provided $f(a)=0$, *and*

$$\begin{aligned}(2.14)\qquad &\int_a^b |f'(t)\,f(t)|\,w(t)\,dt\\ &\leq \left(\int_a^b (b-t)\,|f'(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_a^b |f'(t)|^2\left(\int_a^t w(s)\,ds\right)dt\right)^{1/2}\\ &\leq \frac{1}{2}\left\{\int_a^b (b-t)\,|f'(t)|^2\,w(t)\,dt+\int_a^b |f'(t)|^2\left(\int_a^t w(s)\,ds\right)dt\right\}\end{aligned}$$

provided $f(b)=0$.

If w *is nonincreasing on* $[a,b]$ *and* $f(a)=0$, *then by (2.13) we get*

$$\begin{aligned}(2.15)\qquad &\int_a^b |f'(t)\,f(t)|\,w(t)\,dt\\ &\leq \left(\int_a^b (t-a)\,|f'(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_a^b (b-t)\,|f'(t)|^2\,w(t)\,dt\right)^{1/2}\\ &\leq \frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,w(t)\,dt,\end{aligned}$$

while if w is nondecreasing on $[a,b]$ and $f(b)=0$, then by (2.14) we get

$$\begin{aligned}(2.16)\qquad &\int_a^b |f'(t)\,f(t)|\,w(t)\,dt\\ &\le \left(\int_a^b (b-t)\,|f'(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_a^b (t-a)\,|f'(t)|^2\,w(t)\,dt\right)^{1/2}\\ &\le \frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,w(t)\,dt.\end{aligned}$$

Furthermore, if $w\equiv 1$, then by (2.13) and (2.14) we get the inequalities from the first part of Theorem 2.

Corollary 3. *Assume that $f:[a,b]\to\mathbb{C}$ is in $C^1[a,b]$, u is monotonic nondecreasing on $[a,b]$ and Lipschitzian with the constant $L>0$, namely $|u(s)-u(t)|\le L|t-s|$ for any t, $s\in[a,b]$ and g is absolutely continuous on $[a,b]$.*

(i) *If $g(a)=0$, then*

$$\begin{aligned}(2.17)\qquad &\int_a^b |f'(t)\,g(t)|\,du(t)\\ &\le L\left(\int_a^b (t-a)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b (b-t)\,|g'(t)|^2\,dt\right)^{1/2}\\ &\le \frac{1}{2}L\int_a^b \left[(t-a)\,|f'(t)|^2+(b-t)\,|g'(t)|^2\right]dt.\end{aligned}$$

(ii) *If $g(b)=0$, then*

$$\begin{aligned}(2.18)\qquad &\int_a^b |f'(t)\,g(t)|\,du(t)\\ &\le L\left(\int_a^b (b-t)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b (t-a)\,|g'(t)|^2\,dt\right)^{1/2}\\ &\le \frac{1}{2}L\int_a^b \left[(b-t)\,|f'(t)|^2+(t-a)\,|g'(t)|^2\right]dt.\end{aligned}$$

Proof. Using Riemann-Stieltjes integral inequality properties, we have

$$\int_a^b (t-a)\,|f'(t)|^2\,du(t)\le L\int_a^b (t-a)\,|f'(t)|^2\,dt.$$

Also

$$\int_a^b [u(b)-u(t)]\,|g'(t)|^2\,dt\le L\int_a^b (b-t)\,|g'(t)|^2\,dt.$$

By using the inequality (2.1) we obtain (2.17).

The inequality (2.18) follows in a similar way from (2.2). □

Remark 3. *Assume that $f:[a,b]\to\mathbb{C}$ is in $C^1[a,b]$ and g is absolutely continuous on $[a,b]$. If $w:[a,b]\to[0,\infty)$ is continuous, then the function $u(t)=\int_a^t w(s)\,ds$*

is Lipschitzian with the constant $\|w\|_{[a,b],\infty} = \max_{t\in[a,b]} w(t)$ *and by (2.17) we get*

$$\begin{aligned}(2.19)\qquad &\int_a^b |f'(t)\, g(t)|\, w(t)\, dt \\ &\leq \|w\|_{[a,b],\infty} \left(\int_a^b (t-a)\,|f'(t)|^2\, dt\right)^{1/2} \left(\int_a^b (b-t)\,|g'(t)|^2\, dt\right)^{1/2} \\ &\leq \frac{1}{2}\,\|w\|_{[a,b],\infty} \int_a^b \left[(t-a)\,|f'(t)|^2 + (b-t)\,|g'(t)|^2\right] dt,\end{aligned}$$

provided $g(a) = 0$, *while from (2.18) we get*

$$\begin{aligned}(2.20)\qquad &\int_a^b |f'(t)\, g(t)|\, w(t)\, dt \\ &\leq \|w\|_{[a,b],\infty} \left(\int_a^b (b-t)\,|f'(t)|^2\, dt\right)^{1/2} \left(\int_a^b (t-a)\,|g'(t)|^2\, dt\right)^{1/2} \\ &\leq \frac{1}{2}\,\|w\|_{[a,b],\infty} \int_a^b \left[(b-t)\,|f'(t)|^2 + (t-a)\,|g'(t)|^2\right] dt,\end{aligned}$$

provided $g(b) = 0$.

In particular, if either $f(a) = 0$ *or* $f(b) = 0$, *then we have*

$$\begin{aligned}(2.21)\qquad &\int_a^b |f'(t)\, f(t)|\, w(t)\, dt \\ &\leq \|w\|_{[a,b],\infty} \left(\int_a^b (t-a)\,|f'(t)|^2\, dt\right)^{1/2} \left(\int_a^b (b-t)\,|f'(t)|^2\, dt\right)^{1/2} \\ &\leq \frac{1}{2}\,(b-a)\,\|w\|_{[a,b],\infty} \int_a^b |f'(t)|^2\, dt.\end{aligned}$$

3. Related Results

We have the following inequalities as well:

Theorem 4. *Assume that* $f : [a,b] \to \mathbb{C}$ *is in* $C^1[a,b]$, g *is absolutely continuous on* $[a,b]$ *with* $g(a) = g(b) = 0$ *and* u *is monotonic nondecreasing on* $[a,b]$. *Then*

$$\begin{aligned}(3.1)\quad &\int_a^b |f'(t)\, g(t)|\, du(t) \\ &\leq \frac{1}{2}\,(b-a)^{1/2}\,[u(b)-u(a)]^{1/2} \left(\int_a^b |f'(t)|^2\, du(t)\right)^{1/2} \left(\int_a^b |g'(t)|^2\, dt\right)^{1/2} \\ &\leq \frac{1}{4}\,(b-a)^{1/2}\,[u(b)-u(a)]^{1/2} \left[\int_a^b |f'(t)|^2\, du(t) + \int_a^b |g'(t)|^2\, dt\right]\end{aligned}$$

and

$$\begin{aligned}(3.2)\qquad &\int_a^b |f'(t)\,g(t)|\,du(t)\\ &\leq \left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,du(t)-\int_a^b \left|t-\frac{a+b}{2}\right||f'(t)|^2\,du(t)\right)^{1/2}\\ &\times\left(\int_a^b \left|u(t)-u\left(\frac{a+b}{2}\right)\right||g'(t)|^2\,dt\right)^{1/2}\\ &\leq \frac{1}{4}(b-a)\int_a^b |f'(t)|^2\,du(t)-\frac{1}{2}\int_a^b \left|t-\frac{a+b}{2}\right||f'(t)|^2\,du(t)\\ &+\frac{1}{2}\int_a^b \left|u(t)-u\left(\frac{a+b}{2}\right)\right||g'(t)|^2\,dt.\end{aligned}$$

Proof. If $g(a)=g(b)=0$, then by adding the first parts of the inequalities (2.1) and (2.2), we get

$$\begin{aligned}(3.3)\qquad &2\int_a^b |f'(t)\,g(t)|\,du(t)\\ &\leq \left(\int_a^b (t-a)|f'(t)|^2\,du(t)\right)^{1/2}\left(\int_a^b [u(b)-u(t)]|g'(t)|^2\,dt\right)^{1/2}\\ &+\left(\int_a^b (b-t)|f'(t)|^2\,du(t)\right)^{1/2}\left(\int_a^b [u(t)-u(a)]|g'(t)|^2\,dt\right)^{1/2}.\end{aligned}$$

If we use the elementary (CBS) inequality

$$(3.4)\qquad \alpha\beta+\gamma\delta\leq\left(\alpha^2+\gamma^2\right)^{1/2}\left(\beta^2+\delta^2\right)^{1/2},\ \alpha,\ \beta,\ \gamma,\ \delta\geq 0,$$

then we get

$$\begin{aligned}(3.5)\qquad &\left(\int_a^b (t-a)|f'(t)|^2\,du(t)\right)^{1/2}\left(\int_a^b [u(b)-u(t)]|g'(t)|^2\,dt\right)^{1/2}\\ &+\left(\int_a^b (b-t)|f'(t)|^2\,du(t)\right)^{1/2}\left(\int_a^b [u(t)-u(a)]|g'(t)|^2\,dt\right)^{1/2}\\ &\leq\left(\int_a^b (t-a)|f'(t)|^2\,du(t)+\int_a^b (b-t)|f'(t)|^2\,du(t)\right)^{1/2}\\ &\times\left(\int_a^b [u(b)-u(t)]|g'(t)|^2\,dt+\int_a^b [u(t)-u(a)]|g'(t)|^2\,dt\right)^{1/2}\\ &=(b-a)^{1/2}[u(b)-u(a)]^{1/2}\left(\int_a^b |f'(t)|^2\,du(t)\right)^{1/2}\left(\int_a^b |g'(t)|^2\,dt\right)^{1/2}.\end{aligned}$$

By making use of (3.3) and (3.5) we get the first part of (3.1). The second part follows by the inequality (2.6).

Further on, if we write the first part of the inequality (2.1) on the interval $\left[a, \frac{a+b}{2}\right]$, we get

$$\int_a^{\frac{a+b}{2}} \left|f'(t)\, g(t)\right| du(t) \tag{3.6}$$
$$\leq \left(\int_a^{\frac{a+b}{2}} (t-a)\left|f'(t)\right|^2 du(t)\right)^{1/2} \left(\int_a^{\frac{a+b}{2}} \left[u\left(\frac{a+b}{2}\right) - u(t)\right] \left|g'(t)\right|^2 dt\right)^{1/2},$$

while from the first part of (2.2) written on $\left[\frac{a+b}{2}, b\right]$ we have

$$\int_{\frac{a+b}{2}}^b \left|f'(t)\, g(t)\right| du(t) \tag{3.7}$$
$$\leq \left(\int_{\frac{a+b}{2}}^b (b-t)\left|f'(t)\right|^2 du(t)\right)^{1/2} \left(\int_{\frac{a+b}{2}}^b \left[u(t) - u\left(\frac{a+b}{2}\right)\right] \left|g'(t)\right|^2 dt\right)^{1/2}.$$

If we add (3.6) with (3.7), then we get

$$\int_a^b \left|f'(t)\, g(t)\right| du(t) \tag{3.8}$$
$$\leq \left(\int_a^{\frac{a+b}{2}} (t-a)\left|f'(t)\right|^2 du(t)\right)^{1/2} \left(\int_a^{\frac{a+b}{2}} \left[u\left(\frac{a+b}{2}\right) - u(t)\right] \left|g'(t)\right|^2 dt\right)^{1/2}$$
$$+ \left(\int_{\frac{a+b}{2}}^b (b-t)\left|f'(t)\right|^2 du(t)\right)^{1/2} \left(\int_{\frac{a+b}{2}}^b \left[u(t) - u\left(\frac{a+b}{2}\right)\right] \left|g'(t)\right|^2 dt\right)^{1/2}.$$

Moreover, if we use the elementary (CBS) inequality (3.4), then we have

(3.9)
$$\left(\int_a^{\frac{a+b}{2}} (t-a)\left|f'(t)\right|^2 du(t)\right)^{1/2} \left(\int_a^{\frac{a+b}{2}} \left[u\left(\frac{a+b}{2}\right) - u(t)\right] \left|g'(t)\right|^2 dt\right)^{1/2}$$
$$+ \left(\int_{\frac{a+b}{2}}^b (b-t)\left|f'(t)\right|^2 du(t)\right)^{1/2} \left(\int_{\frac{a+b}{2}}^b \left[u(t) - u\left(\frac{a+b}{2}\right)\right] \left|g'(t)\right|^2 dt\right)^{1/2}$$
$$\leq \left(\int_a^{\frac{a+b}{2}} (t-a)\left|f'(t)\right|^2 du(t) + \int_{\frac{a+b}{2}}^b (b-t)\left|f'(t)\right|^2 du(t)\right)^{1/2}$$
$$\times \left(\int_a^{\frac{a+b}{2}} \left[u\left(\frac{a+b}{2}\right) - u(t)\right] \left|g'(t)\right|^2 dt + \int_{\frac{a+b}{2}}^b \left[u(t) - u\left(\frac{a+b}{2}\right)\right] \left|g'(t)\right|^2 dt\right)^{1/2}$$
$$= \left(\int_a^b K(t)\left|f'(t)\right|^2 du(t)\right)^{1/2} \left(\int_a^b \left|u(t) - u\left(\frac{a+b}{2}\right)\right| \left|g'(t)\right|^2 dt\right)^{1/2},$$

where

$$K(t) := \begin{cases} t-a \text{ if } t \in \left[a, \frac{a+b}{2}\right], \\ \\ b-t \text{ if } t \in \left(\frac{a+b}{2}, b\right]. \end{cases}$$

Since

$$K(t)=\frac{1}{2}(b-a)-\left|t-\frac{a+b}{2}\right|,\ t\in[a,b],$$

hence by (3.8) and (3.9) we get the first inequality in (3.2). The second part follows by (2.6). □

Remark 4. *Assume that $f:[a,b]\to\mathbb{C}$ is in $C^1[a,b]$ with $f(a)=f(b)=0$ and u is monotonic nondecreasing on $[a,b]$. Then*

$$\begin{aligned}(3.10)\quad &\int_a^b |f'(t)f(t)|\,du(t)\\ &\le\frac{1}{2}(b-a)^{1/2}[u(b)-u(a)]^{1/2}\left(\int_a^b |f'(t)|^2\,du(t)\right)^{1/2}\left(\int_a^b |f'(t)|^2\,dt\right)^{1/2}\\ &\le\frac{1}{4}(b-a)^{1/2}[u(b)-u(a)]^{1/2}\left[\int_a^b |f'(t)|^2\,du(t)+\int_a^b |f'(t)|^2\,dt\right]\end{aligned}$$

and

$$\begin{aligned}(3.11)\quad &\int_a^b |f'(t)f(t)|\,du(t)\\ &\le\left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,du(t)-\int_a^b\left|t-\frac{a+b}{2}\right||f'(t)|^2\,du(t)\right)^{1/2}\\ &\qquad\times\left(\int_a^b\left|u(t)-u\left(\frac{a+b}{2}\right)\right||f'(t)|^2\,dt\right)^{1/2}\\ &\le\frac{1}{4}(b-a)\int_a^b |f'(t)|^2\,du(t)-\frac{1}{2}\int_a^b\left|t-\frac{a+b}{2}\right||f'(t)|^2\,du(t)\\ &\qquad+\frac{1}{2}\int_a^b\left|u(t)-u\left(\frac{a+b}{2}\right)\right||f'(t)|^2\,dt.\end{aligned}$$

Corollary 4. *Assume that $f:[a,b]\to\mathbb{C}$ is in $C^1[a,b]$, g is absolutely continuous on $[a,b]$ with $g(a)=g(b)=0$ and u is monotonic nondecreasing and L-Lipschitzian on $[a,b]$. Then*

$$\begin{aligned}(3.12)\quad &\int_a^b |f'(t)g(t)|\,du(t)\\ &\le\frac{1}{2}(b-a)L\left(\int_a^b |f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b |g'(t)|^2\,dt\right)^{1/2}\\ &\le\frac{1}{4}(b-a)L\int_a^b\left(|f'(t)|^2+|g'(t)|^2\right)dt\end{aligned}$$

and

$$\begin{aligned}(3.13)\quad &\int_a^b |f'(t)\,g(t)|\,du(t)\\ &\leq L\left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,dt-\int_a^b\left|t-\frac{a+b}{2}\right||f'(t)|^2\,dt\right)^{1/2}\\ &\qquad\times\left(\int_a^b\left|t-\frac{a+b}{2}\right||g'(t)|^2\,dt\right)^{1/2}\\ &\leq \frac{1}{2}L\left[\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,dt-\int_a^b\left|t-\frac{a+b}{2}\right|\left[|f'(t)|^2-|g'(t)|^2\right]dt\right].\end{aligned}$$

In particular, if $f(a)=f(b)=0$, then

$$(3.14)\qquad \int_a^b |f'(t)\,f(t)|\,du(t)\leq\frac{1}{2}(b-a)\,L\int_a^b |f'(t)|^2\,dt$$

and

$$\begin{aligned}(3.15)\quad &\int_a^b |f'(t)\,f(t)|\,du(t)\\ &\leq L\left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,dt-\int_a^b\left|t-\frac{a+b}{2}\right||f'(t)|^2\,dt\right)^{1/2}\\ &\qquad\times\left(\int_a^b\left|t-\frac{a+b}{2}\right||f'(t)|^2\,dt\right)^{1/2}\\ &\qquad\leq\frac{1}{4}(b-a)\,L\int_a^b |f'(t)|^2\,dt.\end{aligned}$$

Remark 5. *Assume that $f:[a,b]\to\mathbb{C}$ is in $C^1[a,b]$ and g is absolutely continuous on $[a,b]$ with $g(a)=g(b)=0$. If $w:[a,b]\to[0,\infty)$ is continuous and we take $u(t)=\int_a^t w(s)\,ds$ in (3.1) and (3.2), then we get*

$$\begin{aligned}(3.16)\quad &\int_a^b |f'(t)\,g(t)|\,w(t)\,dt\\ &\leq\frac{1}{2}(b-a)^{1/2}\left(\int_a^b w(s)\,ds\right)^{1/2}\left(\int_a^b |f'(t)|^2\,w(t)\,dt\right)^{1/2}\left(\int_a^b |g'(t)|^2\,dt\right)^{1/2}\\ &\qquad\leq\frac{1}{4}(b-a)^{1/2}\left(\int_a^b w(s)\,ds\right)^{1/2}\left[\int_a^b |f'(t)|^2\,w(t)\,dt+\int_a^b |g'(t)|^2\,dt\right]\end{aligned}$$

and

$$\begin{aligned}(3.17)\quad &\int_a^b |f'(t)\,g(t)|\,w(t)\,dt \\ &\leq \left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,w(t)\,dt - \int_a^b \left|t-\frac{a+b}{2}\right| |f'(t)|^2\,w(t)\,dt\right)^{1/2} \\ &\qquad \times \left(\int_a^b \left|\int_{\frac{a+b}{2}}^t w(s)\,ds\right| |g'(t)|^2\,dt\right)^{1/2} \\ &\leq \frac{1}{4}(b-a)\int_a^b |f'(t)|^2\,w(t)\,dt - \frac{1}{2}\int_a^b \left|t-\frac{a+b}{2}\right| |f'(t)|^2\,w(t)\,dt \\ &\qquad + \frac{1}{2}\int_a^b \left|\int_{\frac{a+b}{2}}^t w(s)\,ds\right| |g'(t)|^2\,dt.\end{aligned}$$

We also have, by taking $L = \|w\|_{[a,b],\infty}$, *in (3.12) and (3.13), that*

$$\begin{aligned}(3.18)\quad &\int_a^b |f'(t)\,g(t)|\,w(t)\,dt \\ &\leq \frac{1}{2}(b-a)\|w\|_{[a,b],\infty}\left(\int_a^b |f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b |g'(t)|^2\,dt\right)^{1/2} \\ &\leq \frac{1}{4}(b-a)\|w\|_{[a,b],\infty}\int_a^b \left(|f'(t)|^2+|g'(t)|^2\right)dt\end{aligned}$$

and

$$\begin{aligned}(3.19)\quad &\int_a^b |f'(t)\,g(t)|\,w(t)\,dt \\ &\leq \|w\|_{[a,b],\infty}\left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,dt - \int_a^b \left|t-\frac{a+b}{2}\right| |f'(t)|^2\,dt\right)^{1/2} \\ &\qquad \times \left(\int_a^b \left|t-\frac{a+b}{2}\right| |g'(t)|^2\,dt\right)^{1/2} \\ &\leq \frac{1}{2}\|w\|_{[a,b],\infty}\left[\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,dt - \int_a^b \left|t-\frac{a+b}{2}\right| \left[|f'(t)|^2-|g'(t)|^2\right]dt\right].\end{aligned}$$

By taking $w \equiv 1$ *in (3.19), we get the inequality (1.5) from Introduction.*

4. Some Inequalities for Functions of Bounded Variation

The following lemma was obtained by the author in 2007, [6] and is of interest in itself as well (see also [5]):

Lemma 1. *If $p:[a,b]\to\mathbb{C}$ is continuous on $[a,b]$ and $v:[a,b]\to\mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$\left|\int_a^b p(t)\,dv(t)\right| \le \int_a^b |p(t)|\,dV(t) \le \left(\int_a^b |p(t)|^p\,dV(t)\right)^{1/p}\left(\bigvee_a^b(v)\right)^{1/q} \le \max_{t\in[a,b]}|p(t)|\bigvee_a^b(v), \tag{4.1}$$

where $V(t):=\bigvee_a^t(v)$ is the total variation of v on $[a,t]$ with $t\in[a,b]$.

The function V is nondecreasing on $[a,b]$ with $V(a)=0$ and $V(b)=\bigvee_a^b(v)$. If we put $\overline{V}(t):=\bigvee_t^b(v)=\bigvee_a^b(v)-V(t)$, then $\overline{V}$is nonincreasing with $\overline{V}(a)=\bigvee_a^b(v)$ and $\overline{V}(b)=0$.

We have:

Proposition 1. *Assume that $h:[a,b]\to\mathbb{C}$ is continuous, g is absolutely continuous on $[a,b]$ and v is of bounded variation on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b h(t)\,g(t)\,dv(t)$ exists and:*

(i) *If $g(a)=0$, then*

$$\begin{aligned}\left|\int_a^b h(t)\,g(t)\,dv(t)\right| &\le \left(\int_a^b (t-a)\,|h(t)|^2\,dV(t)\right)^{1/2}\left(\int_a^b \overline{V}(t)\,|g'(t)|^2\,dt\right)^{1/2}\\ &\le \frac{1}{2}\left\{\int_a^b (t-a)\,|h(t)|^2\,dV(t)+\int_a^b \overline{V}(t)\,|g'(t)|^2\,dt\right\}.\end{aligned} \tag{4.2}$$

(ii) *If $g(b)=0$, then*

$$\begin{aligned}\left|\int_a^b h(t)\,g(t)\,dv(t)\right| &\le \left(\int_a^b (b-t)\,|h(t)|^2\,dV(t)\right)^{1/2}\left(\int_a^b V(t)\,|g'(t)|^2\,dt\right)^{1/2}\\ &\le \frac{1}{2}\left\{\int_a^b (b-t)\,|h(t)|^2\,dV(t)+\int_a^b V(t)\,|g'(t)|^2\,dt\right\}.\end{aligned} \tag{4.3}$$

Proof. Using the first inequality in (4.1), we get

$$\left|\int_a^b h(t)\,g(t)\,dv(t)\right| \le \int_a^b |h(t)\,g(t)|\,dV(t). \tag{4.4}$$

Using now Theorem 3 for $f = \int_a h$ and $u = V$, we get, for $g(a) = 0$, that

$$\begin{aligned}(4.5)\qquad &\int_a^b |h(t)\, g(t)|\, dV(t) \\ &\leq \left(\int_a^b (t-a)\,|h(t)|^2\, dV(t)\right)^{1/2} \left(\int_a^b [V(b) - V(t)]\,|g'(t)|^2\, dt\right)^{1/2} \\ &\leq \frac{1}{2}\left\{\int_a^b (t-a)\,|f'(t)|^2\, dV(t) + \int_a^b [V(b) - V(t)]\,|g'(t)|^2\, dt\right\}.\end{aligned}$$

If $g(b) = 0$, then

$$\begin{aligned}(4.6)\qquad &\int_a^b |h(t)\, g(t)|\, dV(t) \\ &\leq \left(\int_a^b (b-t)\,|f'(t)|^2\, dV(t)\right)^{1/2} \left(\int_a^b V(t)\,|g'(t)|^2\, dt\right)^{1/2} \\ &\leq \frac{1}{2}\left\{\int_a^b (b-t)\,|f'(t)|^2\, dV(t) + \int_a^b V(t)\,|g'(t)|^2\, dt\right\}.\end{aligned}$$

By utilising (4.4)-(4.6) we get the desired results (4.2) and (4.3). □

The case $h \equiv 1$ is of interest since in this case

$$\begin{aligned}\int_a^b (t-a)\, dV(t) &= (b-a)\, V(b) - \int_a^b V(t)\, dt \\ &= \int_a^b (V(b) - V(t))\, dt = \int_a^b \overline{V}(t)\, dt\end{aligned}$$

and

$$\int_a^b (b-t)\, dV(t) = \int_a^b V(t)\, dt.$$

We then can state:

Corollary 5. *Assume that g is absolutely continuous on $[a,b]$ and v is of bounded variation on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b g(t)\, dv(t)$ exists and:*

(i) *If $g(a) = 0$, then*

$$\begin{aligned}(4.7)\qquad \left|\int_a^b g(t)\, dv(t)\right| &\leq \left(\int_a^b \overline{V}(t)\, dt\right)^{1/2} \left(\int_a^b \overline{V}(t)\,|g'(t)|^2\, dt\right)^{1/2} \\ &\leq \frac{1}{2}\int_a^b \overline{V}(t)\left(1 + |g'(t)|^2\right) dt.\end{aligned}$$

(ii) *If $g(b) = 0$, then*

$$\begin{aligned}(4.8)\qquad \left|\int_a^b g(t)\, dv(t)\right| &\leq \left(\int_a^b V(t)\, dt\right)^{1/2} \left(\int_a^b V(t)\,|g'(t)|^2\, dt\right)^{1/2} \\ &\leq \frac{1}{2}\int_a^b V(t)\left(1 + |g'(t)|^2\right) dt.\end{aligned}$$

We also have:

Proposition 2. *Assume that $h:[a,b]\to\mathbb{C}$ is continuous, g is absolutely continuous on $[a,b]$ with $g(a)=g(b)=0$ and v is of bounded variation on $[a,b]$, then*

$$\begin{aligned}(4.9)\qquad &\left|\int_a^b h(t)\,g(t)\,dv(t)\right| \\ &\leq \frac{1}{2}(b-a)^{1/2}\left[\bigvee_a^b(v)\right]^{1/2}\left(\int_a^b|h(t)|^2\,dV(t)\right)^{1/2}\left(\int_a^b|g'(t)|^2\,dt\right)^{1/2} \\ &\leq \frac{1}{4}(b-a)^{1/2}\left[\bigvee_a^b(v)\right]^{1/2}\left[\int_a^b|h(t)|^2\,dV(t)+\int_a^b|g'(t)|^2\,dt\right]\end{aligned}$$

and

$$\begin{aligned}(4.10)\qquad &\left|\int_a^b h(t)\,g(t)\,dv(t)\right| \\ &\leq \left(\frac{1}{2}(b-a)\int_a^b|h(t)|^2\,dV(t)-\int_a^b\left|t-\frac{a+b}{2}\right||h(t)|^2\,dV(t)\right)^{1/2} \\ &\qquad\times\left(\int_a^b\left|V(t)-V\left(\frac{a+b}{2}\right)\right||g'(t)|^2\,dt\right)^{1/2} \\ &\leq \frac{1}{4}(b-a)\int_a^b|h(t)|^2\,dV(t)-\frac{1}{2}\int_a^b\left|t-\frac{a+b}{2}\right||h(t)|^2\,dV(t) \\ &\qquad+\frac{1}{2}\int_a^b\left|V(t)-V\left(\frac{a+b}{2}\right)\right||g'(t)|^2\,dt.\end{aligned}$$

The proof follows by Theorem 4 and Lemma 1.

Corollary 6. *Assume that g is absolutely continuous on $[a,b]$ with $g(a)=g(b)=0$ and v is of bounded variation on $[a,b]$, then*

$$(4.11)\qquad \left|\int_a^b g(t)\,dv(t)\right|\leq\frac{1}{2}(b-a)^{1/2}\left(\int_a^b|g'(t)|^2\,dt\right)^{1/2}\bigvee_a^b(v)$$

and

$$\begin{aligned}(4.12)\qquad &\left|\int_a^b g(t)\,dv(t)\right| \\ &\leq\left(\int_a^b\operatorname{sgn}\left(t-\frac{a+b}{2}\right)V(t)\,dt\right)^{1/2}\left(\int_a^b\left|V(t)-V\left(\frac{a+b}{2}\right)\right||g'(t)|^2\,dt\right)^{1/2}.\end{aligned}$$

Proof. The inequality (4.11) follows by (4.9) for $h\equiv 1$.

The same choice produces in (4.10)

$$\left|\int_a^b g(t)\,dv(t)\right| \le \left(\frac{1}{2}(b-a)\int_a^b dV(t) - \int_a^b \left|t-\frac{a+b}{2}\right| dV(t)\right)^{1/2} \times \left(\int_a^b \left|V(t) - V\left(\frac{a+b}{2}\right)\right| |g'(t)|^2\,dt\right)^{1/2}. \tag{4.13}$$

Since

$$K(t) := \begin{cases} t-a \text{ if } t \in \left[a, \frac{a+b}{2}\right], \\ b-t \text{ if } t \in \left(\frac{a+b}{2}, b\right], \end{cases}$$

hence, integrating by parts in the Riemann-Stieltjes integral, we have

$$\begin{aligned} &\frac{1}{2}(b-a)\int_a^b dV(t) - \int_a^b \left|t-\frac{a+b}{2}\right| dV(t) \\ &= \int_a^{\frac{a+b}{2}} (t-a)\,dV(t) + \int_{\frac{a+b}{2}}^b (b-t)\,dV(t) \\ &= \frac{1}{2}(b-a)V\left(\frac{a+b}{2}\right) - \int_a^{\frac{a+b}{2}} V(t)\,dt - \frac{1}{2}(b-a)V\left(\frac{a+b}{2}\right) + \int_{\frac{a+b}{2}}^b V(t)\,dt \\ &= \int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right) V(t)\,dt \end{aligned}$$

and by (4.13) we get (4.12). □

5. Applications for Trapezoid Inequality

We have the following equalities:

Lemma 2. *Let f, $v : [a,b] \to \mathbb{C}$ be such that one is continuous and the other is of bounded variation. Then*

$$\begin{aligned} T(f,v;[a,b]) &:= \int_a^b f(t)\,dv(t) \\ &\quad - f(b)\left[v(b) - \frac{1}{b-a}\int_a^b v(t)\,dt\right] - f(a)\left[\frac{1}{b-a}\int_a^b v(t)\,dt - v(a)\right] \\ &= \frac{f(b)-f(a)}{b-a}\int_a^b v(t)\,dt - \int_a^b v(t)\,df(t) \\ &= \int_a^b \left[f(t) - \frac{f(a)(b-t)+f(b)(t-a)}{b-a}\right] dv(t). \end{aligned} \tag{5.1}$$

Proof. Integrating by parts in the Riemann-Stieltjes integral, we have

$$\begin{aligned} &\int_a^b \left[f(t) - \frac{f(a)(b-t)+f(b)(t-a)}{b-a}\right] dv(t) \\ &= \int_a^b f(t)\,dv(t) - \int_a^b \frac{f(a)(b-t)+f(b)(t-a)}{b-a} dv(t) \end{aligned}$$

$$
\begin{aligned}
&= \int_a^b f(t)\,dv(t) - \frac{f(a)(b-t)+f(b)(t-a)}{b-a} v(t)\Big|_a^b \\
&+ \frac{f(b)-f(a)}{b-a}\int_a^b v(t)\,dt \\
&= \int_a^b f(t)\,dv(t) - f(b)\,v(b) + f(a)\,v(a) + \frac{f(b)-f(a)}{b-a}\int_a^b v(t)\,dt \\
&= \int_a^b f(t)\,dv(t) \\
&- f(b)\left[v(b) - \frac{1}{b-a}\int_a^b v(t)\,dt\right] - f(a)\left[\frac{1}{b-a}\int_a^b v(t)\,dt - v(a)\right].
\end{aligned}
$$

Integrating by parts again, we also have

$$
\begin{aligned}
&\int_a^b f(t)\,dv(t) - f(b)\,v(b) + f(a)\,v(a) + \frac{f(b)-f(a)}{b-a}\int_a^b v(t)\,dt \\
&= \frac{f(b)-f(a)}{b-a}\int_a^b v(t)\,dt - \int_a^b v(t)\,df(t).
\end{aligned}
$$

These prove the required identities. □

We have the following trapezoid type inequality:

Proposition 3. *Assume that f is absolutely continuous on $[a,b]$ and v is of bounded variation on $[a,b]$, then*

$$
|T(f,v;[a,b])| \le \frac{1}{2}(b-a)^{1/2}\left(\int_a^b \left|f'(t) - \frac{f(b)-f(a)}{b-a}\right|^2 dt\right)^{1/2} \bigvee_a^b (v) \tag{5.2}
$$

and

$$
\begin{aligned}
|T(f,v;[a,b])| \le &\left(\int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) V(t)\,dt\right)^{1/2} \\
&\times \left(\int_a^b \left|V(t) - V\left(\frac{a+b}{2}\right)\right|\left|f'(t) - \frac{f(b)-f(a)}{b-a}\right|^2 dt\right)^{1/2},
\end{aligned} \tag{5.3}
$$

where $V(t) := \bigvee_a^t (v)$ is the total variation of v on $[a,t]$ with $t \in [a,b]$.

Proof. Consider the function $g : [a,b] \to \mathbb{C}$ defined by

$$
g(t) := f(t) - \frac{f(a)(b-t)+f(b)(t-a)}{b-a},\ t \in [a,b].
$$

Then g is absolutely continuous on $[a,b]$, $g(a) = g(b) = 0$ and

$$
g'(t) = f'(t) - \frac{f(b)-f(a)}{b-a} \text{ for a.e. } t \in [a,b].
$$

By using (4.11), we then have

$$\left|\int_a^b \left[f(t) - \frac{f(a)(b-t)+f(b)(t-a)}{b-a}\right] dv(t)\right|$$
$$\leq \frac{1}{2}(b-a)^{1/2}\left(\int_a^b \left|f'(t) - \frac{f(b)-f(a)}{b-a}\right|^2 dt\right)^{1/2} \bigvee_a^b (v),$$

which by Lemma 2 produces the inequality (5.2).

The inequality (5.3) follows by (4.12). □

For a function $h : [a,b] \to \mathbb{C}$ we consider the *symmetrical transform* $\breve{h}$ defined by

$$\widehat{h}(t) := \frac{1}{2}[h(t) + h(a+b-t)],\ t \in [a,b]$$

and the *antisymmetrical transform* $\widetilde{h}$ defined by

$$\widetilde{h}(t) := \frac{1}{2}[h(t) - h(a+b-t)],\ t \in [a,b].$$

Proposition 4. *Assume that f is absolutely continuous on $[a,b]$ and v is of bounded variation on $[a,b]$, then*

$$B(f,v;[a,b]) := \int_a^b \widehat{f}(t)\,dv(t) - \frac{f(a)+f(b)}{2}[v(b)-v(a)] \tag{5.4}$$
$$= \int_a^b f(t)\,d\widetilde{v}(t) - \frac{f(a)+f(b)}{2}[v(b)-v(a)]$$

and we have the inequalities

$$|B(f,v;[a,b])| \leq \frac{1}{2}(b-a)^{1/2}\left(\int_a^b \left|\widetilde{f}'(t)\right|^2 dt\right)^{1/2} \bigvee_a^b (v) \tag{5.5}$$

and

$$|B(f,v;[a,b])| \tag{5.6}$$
$$\leq \left(\int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) V(t)\,dt\right)^{1/2} \left(\int_a^b \left|V(t) - V\left(\frac{a+b}{2}\right)\right| \left|\widetilde{f}'(t)\right|^2 dt\right)^{1/2}.$$

Proof. Consider the function $g : [a,b] \to \mathbb{C}$ defined by

$$g(t) := \widehat{f}(t) - \frac{f(a)+f(b)}{2},\ t \in [a,b].$$

Then g is absolutely continuous on $[a,b]$, $g(a) = g(b) = 0$,

$$g'(t) = \frac{f'(t) - f'(a+b-t)}{2} = \widetilde{f}'(t) \text{ for a.e. } t \in [a,b]$$

and

$$\int_a^b g(t)\,dv(t) = \int_a^b \left(\widehat{f}(t) - \frac{f(a)+f(b)}{2}\right) dv(t)$$
$$= \int_a^b \widehat{f}(t)\,dv(t) - \frac{f(a)+f(b)}{2}[v(b)-v(a)].$$

Using the change of variable formula for the Riemann-Stieltjes integral, see for instance [1, p. 144], we have

$$\begin{aligned}\int_a^b \widehat{f}(t)\,dv(t) &= \frac{1}{2}\int_a^b [f(t)+f(a+b-t)]\,dv(t)\\ &= \frac{1}{2}\left[\int_a^b f(t)\,dv(t)+\int_a^b f(a+b-t)\,dv(t)\right]\\ &= \frac{1}{2}\left[\int_a^b f(t)\,dv(t)+\int_b^a f(u)\,dv(a+b-u)\right]\\ &= \frac{1}{2}\left[\int_a^b f(t)\,dv(t)-\int_a^b f(u)\,dv(a+b-u)\right] = \int_a^b f(t)\,d\widetilde{v}(t),\end{aligned}$$

which proves the equality (5.4).

The inequalities (5.5) and (5.6) follows by Corollary 6. □

6. Some Grüss' Type Inequalities

For two *Lebesgue integrable* functions $f,\ g:[a,b]\to\mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt. \tag{6.1}$$

In 1935, Grüss [10] showed that

$$|C(f,g)| \le \frac{1}{4}(M-m)(N-n), \tag{6.2}$$

provided that there exists the real numbers $m,\ M,\ n,\ N$ such that

$$m \le f(t) \le M \quad \text{and} \quad n \le g(t) \le N \quad \text{for a.e. } t\in[a,b]. \tag{6.3}$$

The constant $\frac{1}{4}$ is best possible in (4.1) in the sense that it cannot be replaced by a smaller quantity.

Proposition 5. *Assume that $h:[a,b]\to\mathbb{C}$ is integrable on $[a,b]$ and $v:[a,b]\to\mathbb{C}$ of bounded variation on $[a,b]$. Then*

$$|C(h,v)| \le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b |h(t)|^2 - \left|\frac{1}{b-a}\int_a^b h(s)\,ds\right|^2\right]^{1/2}\bigvee_a^b(v) \tag{6.4}$$

and

$$
\begin{aligned}
(6.5)\quad |C(h,v)| &\leq \left(\frac{1}{b-a}\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right) V(t)\,dt\right)^{1/2} \\
&\times \left(\frac{1}{b-a}\int_a^b \left|V(t)-V\left(\frac{a+b}{2}\right)\right| \left|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right|^2 dt\right)^{1/2} \\
&\leq \max_{t\in[a,b]}\left|V(t)-V\left(\frac{a+b}{2}\right)\right| \left(\frac{1}{b-a}\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right) V(t)\,dt\right)^{1/2} \\
&\times \left[\frac{1}{b-a}\int_a^b |h(t)|^2 - \left|\frac{1}{b-a}\int_a^b h(s)\,ds\right|^2\right]^{1/2}.
\end{aligned}
$$

Proof. Using the integration by parts for the Riemann-Stieltjes integral, we have

$$
\begin{aligned}
&\int_a^b \left(\int_a^t h(s)\,ds - \frac{t-a}{b-a}\int_a^b h(s)\,ds\right) dv(t) \\
&= \left(\int_a^t h(s)\,ds - \frac{t-a}{b-a}\int_a^b h(s)\,ds\right) v(t)\Bigg|_a^b \\
&- \int_a^b v(t)\,d\left(\int_a^t h(s)\,ds - \frac{t-a}{b-a}\int_a^b h(s)\,ds\right) \\
&= -\int_a^b v(t)\,h(t)\,dt + \frac{1}{b-a}\int_a^b h(s)\,ds\int_a^b v(t)\,dt,
\end{aligned}
$$

which gives that

$$
(6.6)\qquad C(h,v) = \frac{1}{b-a}\int_a^b \left(\frac{t-a}{b-a}\int_a^b h(s)\,ds - \int_a^t h(s)\,ds\right) dv(t).
$$

Consider

$$
g(t) := \frac{t-a}{b-a}\int_a^b h(s)\,ds - \int_a^t h(s)\,ds,\ t\in[a,b],
$$

then g is absolutely continuous, $g(a)=g(b)=0$ and by (4.11) we get

$$
\begin{aligned}
(6.7)\qquad &\left|\int_a^b \left(\frac{t-a}{b-a}\int_a^b h(s)\,ds - \int_a^t h(s)\,ds\right) dv(t)\right| \\
&\leq \frac{1}{2}(b-a)^{1/2}\left(\int_a^b \left|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right|^2 dt\right)^{1/2} \bigvee_a^b (v) \\
&= \frac{1}{2}(b-a)\left[\frac{1}{b-a}\int_a^b |h(t)|^2 - \left|\frac{1}{b-a}\int_a^b h(s)\,ds\right|^2\right]^{1/2} \bigvee_a^b (v)
\end{aligned}
$$

which by the equality (6.6) is equivalent to (6.4).

Using (4.12) we get

$$\left|\int_a^b \left(\frac{t-a}{b-a}\int_a^b h(s)\,ds - \int_a^t h(s)\,ds\right) dv(t)\right|$$
$$\leq \left(\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right) V(t)\,dt\right)^{1/2}$$
$$\times \left(\int_a^b \left|V(t) - V\left(\frac{a+b}{2}\right)\right| \left|h(t) - \frac{1}{b-a}\int_a^b h(s)\,ds\right|^2 dt\right)^{1/2},$$

namely

$$\left|\frac{1}{b-a}\int_a^b \left(\frac{t-a}{b-a}\int_a^b h(s)\,ds - \int_a^t h(s)\,ds\right) dv(t)\right|$$
$$\leq \left(\frac{1}{b-a}\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right) V(t)\,dt\right)^{1/2}$$
$$\times \left(\frac{1}{b-a}\int_a^b \left|V(t) - V\left(\frac{a+b}{2}\right)\right| \left|h(t) - \frac{1}{b-a}\int_a^b h(s)\,ds\right|^2 dt\right)^{1/2},$$

and the first inequality in (6.5) is obtained.

The second part is obvious. □

Consider now the *weighted Čebyšev functional*

$$C_w(f,g) := \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) f(t) g(t)\,dt$$
$$- \frac{1}{\left(\int_a^b w(t)\,dt\right)^2}\int_a^b w(t) f(t)\,dt \int_a^b w(t) g(t)\,dt,$$

where f, g, $w : [a,b] \to \mathbb{R}$ and $w(t) \geq 0$ for a.e. $t \in [a,b]$ are measurable functions such that the involved integrals exist and $\int_a^b w(t)\,dt > 0$.

Proposition 6. *Assume that $h : [a,b] \to \mathbb{C}$ is integrable on $[a,b]$, $w : [a,b] \to \mathbb{R}$, $w(t) \geq 0$ for a.e. $t \in [a,b]$ is integrable with $\int_a^b w(t)\,dt > 0$ and $v : [a,b] \to \mathbb{C}$ is of*

bounded variation on $[a,b]$. *Then*

$$
\begin{aligned}
(6.8)\quad & |C_w(h,v)| \\
&\leq \frac{1}{2}\frac{(b-a)^{1/2}}{\int_a^b w(s)\,ds}\left(\int_a^b \left|\frac{1}{\int_a^b w(s)\,ds}\int_a^b h(s)\,w(s)\,ds - h(t)\right|^2 w^2(t)\,dt\right)^{1/2} \bigvee_a^b (v) \\
&\leq \frac{1}{2}\left(\frac{(b-a)\,\|w\|_{[a,b],\infty}}{\int_a^b w(s)\,ds}\right)^{1/2} \\
&\times \left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\,|h(t)|^2\,dt - \left|\frac{1}{\int_a^b w(s)\,ds}\int_a^b h(s)\,w(s)\,ds\right|^2\right)^{1/2} \bigvee_a^b (v)
\end{aligned}
$$

and

$$
\begin{aligned}
(6.9)\quad & |C_w(h,v)| \\
&\leq \frac{1}{\int_a^b w(s)\,ds}\left(\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)V(t)\,dt\right)^{1/2} \\
&\times \left(\int_a^b \left|V(t) - V\left(\frac{a+b}{2}\right)\right|\left|\frac{1}{\int_a^b w(s)\,ds}\int_a^b h(s)\,w(s)\,ds - h(t)\right|^2 w^2(t)\,dt\right)^{1/2} \\
&\leq \left(\frac{(b-a)\sup_{t\in[a,b]}\left\{\left|V(t) - V\left(\frac{a+b}{2}\right)\right| w(t)\right\}}{\int_a^b w(s)\,ds}\right)^{1/2} \\
&\times \left(\frac{1}{b-a}\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)V(t)\,dt\right)^{1/2} \\
&\times \left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\,|h(t)|^2\,dt - \left|\frac{1}{\int_a^b w(s)\,ds}\int_a^b h(s)\,w(s)\,ds\right|^2\right)^{1/2}.
\end{aligned}
$$

Proof. Consider

$$
g(t) := \frac{\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds}\int_a^b h(s)\,w(s)\,ds - \int_a^t h(s)\,w(s)\,ds,\ t\in[a,b],
$$

then g is absolutely continuous, $g(a) = g(b) = 0$,

$$
g'(t) := \left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b h(s)\,w(s)\,ds - h(t)\right)w(t) \text{ for a.e. } t\in[a,b]
$$

and, by using the integration by parts for the Riemann-Stieltjes integral,

$$\begin{aligned}
&\int_a^b \left(\frac{\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds} \int_a^b h(s)\,w(s)\,ds - \int_a^t h(s)\,w(s)\,ds \right) dv(t) \\
&= -\int_a^b v(t)\,d\left(\frac{\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds} \int_a^b h(s)\,w(s)\,ds - \int_a^t h(s)\,w(s)\,ds \right) \\
&= \int_a^b v(t)\,h(t)\,w(t)\,dt - \frac{1}{\int_a^b w(s)\,ds} \int_a^b h(s)\,w(s)\,ds \int_a^b v(t)\,w(t)\,dt,
\end{aligned}$$

which gives the identity

$$\begin{aligned}
&C_w(h,v) \\
&= \frac{1}{\int_a^b w(s)\,ds} \int_a^b \left(\frac{\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds} \int_a^b h(s)\,w(s)\,ds - \int_a^t h(s)\,w(s)\,ds \right) dv(t).
\end{aligned}$$

By using the inequality (4.11) we have

$$\begin{aligned}
&\left| \int_a^b \left(\frac{\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds} \int_a^b h(s)\,w(s)\,ds - \int_a^t h(s)\,w(s)\,ds \right) dv(t) \right| \\
&\leq \frac{1}{2}(b-a)^{1/2} \left(\int_a^b \left| \frac{1}{\int_a^b w(s)\,ds} \int_a^b h(s)\,w(s)\,ds - h(t) \right|^2 w^2(t)\,dt \right)^{1/2} \bigvee_a^b (v),
\end{aligned}$$

which by division with $\int_a^b w(s)\,ds > 0$ gives the first inequality in (6.8).

We also have

$$\begin{aligned}
&\frac{1}{\int_a^b w(s)\,ds} \int_a^b \left| \frac{1}{\int_a^b w(s)\,ds} \int_a^b h(s)\,w(s)\,ds - h(t) \right|^2 w^2(t)\,dt \\
&\leq \|w\|_{[a,b],\infty} \frac{1}{\int_a^b w(s)\,ds} \int_a^b \left| \frac{1}{\int_a^b w(s)\,ds} \int_a^b h(s)\,w(s)\,ds - h(t) \right|^2 w(t)\,dt \\
&= \|w\|_{[a,b],\infty} \left[\frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t)\,|h(t)|^2\,dt - \left| \frac{1}{\int_a^b w(s)\,ds} \int_a^b h(s)\,w(s)\,ds \right|^2 \right],
\end{aligned}$$

which proves the second part of (6.8).

Using (4.12) we also have

$$\begin{aligned}
&\left| \int_a^b \left(\frac{\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds} \int_a^b h(s)\,w(s)\,ds - \int_a^t h(s)\,w(s)\,ds \right) dv(t) \right| \\
&\leq \left(\int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2} \right) V(t)\,dt \right)^{1/2} \\
&\times \left(\int_a^b \left| V(t) - V\left(\frac{a+b}{2} \right) \right| \left| \frac{1}{\int_a^b w(s)\,ds} \int_a^b h(s)\,w(s)\,ds - h(t) \right|^2 w^2(t)\,dt \right)^{1/2},
\end{aligned}$$

which proves the first inequality in (6.9).

The second part is obvious. □

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