Received 25/06/18

GENERALIZATIONS OF OPIAL'S INEQUALITIES FOR RIEMANN-STIELTJES INTEGRAL, p-NORMS AND TWO FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper we establish some generalizations of Opial's inequalities for Riemann-Stieltjes integral in terms of p-norms and for two functions. Applications related to Grüss' type inequalities are also given.

1. Introduction

We recall the following Opial type inequalities:

Theorem 1. *Assume that $u:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is an absolutely continuous function on the interval $[a,b]$ and such that $u'\in L_2[a,b]$.*

(i) *If $u(a)=u(b)=0$, then*

$$\int_a^b |u(t)\,u'(t)|\,dt \le \frac{1}{4}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.1}$$

with equality if and only if

$$u(t)=\begin{cases} c(t-a) & \text{if } a\le t\le \frac{a+b}{2},\\ c(b-t) & \text{if } \frac{a+b}{2}<t\le b,\end{cases}$$

where c is an arbitrary constant.

(ii) *If $u(a)=0$, then*

$$\int_a^b |u(t)\,u'(t)|\,dt \le \frac{1}{2}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.2}$$

with equality if and only if $u(t)=c(t-a)$ for some constant c.

The inequality (1.1) was obtained by Olech in [14] in which he gave a simplified proof of an inequality originally due in a slightly less general form to Zdzislaw Opial [15].

Embedded in Olech's proof is the half-interval form of Opial's inequality, also discovered by Beesack [1], which is satisfied by those u vanishing only at a.

For various proofs of the above inequalities, see [10]-[13] and [17].

In [7] we obtained the following generalizations of Opial inequalities for p-norms of two functions:

Theorem 2. *Assume that $f,\ g:[a,b]\to\mathbb{C}$ are absolutely continuous on $[a,b]$ with $f'\in L_p[a,b]$ and $g'\in L_q[a,b]$ for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.*

1991 *Mathematics Subject Classification.* 26D15; 26D10.
Key words and phrases. Opial's inequality, Grüss' inequality.

(i) *If* $g(a)=0,$ *then*

$$\int_a^b |f'(t)\,g(t)|\,dt \le \left(\int_a^b (t-a)\,|f'(t)|^p\,dt\right)^{1/p}\left(\int_a^b (b-t)\,|g'(t)|^q\,dt\right)^{1/q} \le \int_a^b \left[\frac{1}{p}(t-a)\,|f'(t)|^p+\frac{1}{q}(b-t)\,|g'(t)|^q\right]dt. \tag{1.3}$$

(ii) *If* $g(b)=0,$ *then*

$$\int_a^b |f'(t)\,g(t)|\,dt \le \left(\int_a^b (b-t)\,|f'(t)|^p\,dt\right)^{1/p}\left(\int_a^b (t-a)\,|g'(t)|^q\,dt\right)^{1/q} \le \int_a^b \left[\frac{1}{p}(b-t)\,|f'(t)|^p\,dt+\frac{1}{q}(t-a)\,|g'(t)|^q\right]dt. \tag{1.4}$$

(iii) *If* $g(a)=g(b)=0,$ *then*

$$\begin{aligned}\int_a^b |f'(t)\,g(t)|\,dt &\le \left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^p\,dt-\int_a^b \left|\frac{a+b}{2}-t\right| |f'(t)|^p\,dt\right)^{1/p}\\ &\quad\times\left(\int_a^b \left|\frac{a+b}{2}-t\right| |g'(t)|^q\,dt\right)^{1/q}\\ &\le \frac{1}{2p}(b-a)\int_a^b |f'(t)|^p\,dt+\int_a^b \left|\frac{a+b}{2}-t\right|\left[\frac{1}{q}|g'(t)|^q-\frac{1}{p}|f'(t)|^p\right]dt.\end{aligned} \tag{1.5}$$

In particular, we have:

Corollary 1. *Assume that* $f:[a,b]\to\mathbb{C}$ *is absolutely continuous on* $[a,b]$ *and* $f'\in L_p[a,b]\cap L_q[a,b]$ *for* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1.$

(i) *If* $f(a)=0,$ *then*

$$\int_a^b |f'(t)\,f(t)|\,dt \le \left(\int_a^b (t-a)\,|f'(t)|^p\,dt\right)^{1/p}\left(\int_a^b (b-t)\,|f'(t)|^q\,dt\right)^{1/q} \le \int_a^b \left[\frac{1}{p}(t-a)\,|f'(t)|^p+\frac{1}{q}(b-t)\,|f'(t)|^q\right]dt. \tag{1.6}$$

(ii) *If* $f(b)=0,$ *then*

$$\int_a^b |f'(t)\,f(t)|\,dt \le \left(\int_a^b (b-t)\,|f'(t)|^p\,dt\right)^{1/p}\left(\int_a^b (t-a)\,|f'(t)|^q\,dt\right)^{1/q} \le \int_a^b \left[\frac{1}{p}(b-t)\,|f'(t)|^p\,dt+\frac{1}{q}(t-a)\,|f'(t)|^q\right]dt. \tag{1.7}$$

(iii) *If* $f(a) = f(b) = 0$, *then*

$$\begin{aligned}
(1.8) \quad & \int_a^b |f'(t) f(t)| \, dt \\
& \leq \left(\frac{1}{2}(b-a) \int_a^b |f'(t)|^p \, dt - \int_a^b \left| \frac{a+b}{2} - t \right| |f'(t)|^p \, dt \right)^{1/p} \\
& \qquad \times \left(\int_a^b \left| \frac{a+b}{2} - t \right| |f'(t)|^q \, dt \right)^{1/q} \\
& \leq \frac{1}{2p}(b-a) \int_a^b |f'(t)|^p \, dt + \int_a^b \left| \frac{a+b}{2} - t \right| \left[\frac{1}{q} |f'(t)|^q - \frac{1}{p} |f'(t)|^p \right] dt.
\end{aligned}$$

In this paper we establish some generalizations of Opial's inequalities for Riemann-Stieltjes integral in terms of p-norms and for two functions. Applications related to Grüss' type inequalities are also given.

2. The Main Results

We have:

Theorem 3. *Assume that* f, $g : [a,b] \to \mathbb{C}$ *are absolutely continuous on* $[a,b]$, f' *is continuous on* $[a,b]$, u *is monotonic nondecreasing on* $[a,b]$ *and* p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

(i) *If* $g(a) = 0$, *then*

$$\begin{aligned}
(2.1) \quad & \int_a^b |f'(t) g(t)| \, du(t) \\
& \leq \left(\int_a^b (t-a) |f'(t)|^p \, du(t) \right)^{1/p} \left(\int_a^b [u(b) - u(t)] |g'(t)|^q \, dt \right)^{1/q} \\
& \qquad \leq \frac{1}{p} \int_a^b (t-a) |f'(t)|^p \, du(t) + \frac{1}{q} \int_a^b [u(b) - u(t)] |g'(t)|^q \, dt.
\end{aligned}$$

(ii) *If* $g(b) = 0$, *then*

$$\begin{aligned}
(2.2) \quad & \int_a^b |f'(t) g(t)| \, du(t) \\
& \leq \left(\int_a^b (b-t) |f'(t)|^p \, du(t) \right)^{1/p} \left(\int_a^b [u(t) - u(a)] |g'(t)|^q \, dt \right)^{1/q} \\
& \qquad \leq \frac{1}{p} \int_a^b (b-t) |f'(t)|^p \, du(t) + \frac{1}{q} \int_a^b [u(t) - u(a)] |g'(t)|^q \, dt.
\end{aligned}$$

Proof. (i) Since $g(a) = 0$, then $g(t) = \int_a^t g'(s) \, ds$ for $t \in [a,b]$. We have

$$\int_a^b |f'(t) g(t)| \, du(t) = \int_a^b |f'(t)| |g(t)| \, du(t)$$

$$= \int_a^b (t-a)^{1/p} |f'(t)| (t-a)^{-1/p} |g(t)| \, du(t)$$
$$= \int_a^b (t-a)^{1/p} |f'(t)| (t-a)^{-1/p} \left| \int_a^t g'(s) \, ds \right| du(t) =: A.$$

Using the Hölder's inequality for the Riemann-Stieltjes integral of monotonic non-decreasing integrators and for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$A \leq \left(\int_a^b \left[(t-a)^{1/p} |f'(t)| \right]^p du(t) \right)^{1/p} \times \left(\int_a^b \left[(t-a)^{-1/p} \left| \int_a^t g'(s) \, ds \right| \right]^q du(t) \right)^{1/q} \tag{2.3}$$
$$= \left(\int_a^b (t-a) |f'(t)|^p \, du(t) \right)^{1/p} \left(\int_a^b \left[(t-a)^{-1/p} \left| \int_a^t g'(s) \, ds \right| \right]^q du(t) \right)^{1/q} =: B.$$

By Hölder's inequality for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we also have

$$(t-a)^{-1/p} \left| \int_a^t g'(s) \, ds \right| \leq \left(\int_a^t |g'(s)|^q \, ds \right)^{1/q}$$

that implies

$$\left[(t-a)^{-1/p} \left| \int_a^t g'(s) \, ds \right| \right]^q \leq \int_a^t |g'(s)|^q \, ds,$$

which gives

$$B \leq \left(\int_a^b (t-a) |f'(t)|^p \, du(t) \right)^{1/p} \left(\int_a^b \left(\int_a^t |g'(s)|^q \, ds \right) du(t) \right)^{1/q}. \tag{2.4}$$

Using integration by parts for the Riemann-Stieltjes integral, we have

$$\int_a^b \left(\int_a^t |g'(s)|^q \, ds \right) du(t) = \left(\int_a^t |g'(s)|^q \, ds \right) u(t) \bigg|_a^b - \int_a^b u(t) |g'(t)|^q \, dt$$
$$= u(b) \int_a^b |g'(s)|^q \, ds - \int_a^b u(t) |g'(t)|^q \, dt$$
$$= \int_a^b [u(b) - u(t)] |g'(t)|^q \, dt$$

and by (2.3) we get the first inequality in (2.1).

The last part follows by the elementary *Young's inequality*

$$\alpha^{1/p} \beta^{1/q} \leq \frac{1}{p} \alpha + \frac{1}{q} \beta, \ \ \alpha,\ \beta \geq 0. \tag{2.5}$$

(ii) Since $g(b) = 0$, then $g(t) = -\int_t^b g'(s)\,ds$ for $t \in [a,b]$. We have

$$\begin{aligned}\int_a^b |f'(t)\,g(t)|\,du(t) &= \int_a^b |f'(t)|\,|g(t)|\,du(t)\\ &= \int_a^b (b-t)^{1/p}\,|f'(t)|\,(b-t)^{-1/p}\,|g(t)|\,du(t)\\ &= \int_a^b (b-t)^{1/p}\,|f'(t)|\,(b-t)^{-1/p}\left|\int_t^b g'(s)\,ds\right| du(t) =: C.\end{aligned}$$

Using Hölder's inequality for Riemann-Stieltjes integral and for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we also have

$$\begin{aligned}C &\leq \left(\int_a^b \left[(b-t)^{1/p}\,|f'(t)|\right]^p du(t)\right)^{1/p}\\ &\quad\times\left(\int_a^b \left[(b-t)^{-1/p}\left|\int_t^b g'(s)\,ds\right|\right]^q du(t)\right)^{1/q}\\ &= \left(\int_a^b (b-t)\,|f'(t)|^p\,du(t)\right)^{1/p}\left(\left[(b-t)^{-1/p}\left|\int_t^b g'(s)\,ds\right|\right]^q du(t)\right)^{1/q}\\ &=: D.\end{aligned} \tag{2.6}$$

By Hölder's inequality for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we also have

$$(b-t)^{-1/p}\left|\int_t^b g'(s)\,ds\right| \leq \left(\int_t^b |g'(s)|^q\,ds\right)^{1/q},$$

which gives

$$D \leq \left(\int_a^b (b-t)\,|f'(t)|^p\,du(t)\right)^{1/p}\left(\int_a^b\left(\int_t^b |g'(s)|^q\,ds\right)du(t)\right)^{1/2}. \tag{2.7}$$

Using integration by parts, we have

$$\begin{aligned}\int_a^b\left(\int_t^b |g'(s)|^q\,ds\right)du(t) &= \left(\int_t^b |g'(s)|^q\,ds\right)u(t)\Big|_a^b + \int_a^b |g'(t)|^q\,u(t)\,dt\\ &= \int_a^b [u(t) - u(a)]\,|g'(t)|^q\,dt\end{aligned}$$

and by (2.6) and (2.7) we obtain (2.2). □

Remark 1. *If we take $u(t) = t$ in Theorem 3 we get the inequalities (1.3) and (1.4).*

Corollary 2. *Assume that $f : [a,b] \to \mathbb{C}$ is in $C^1[a,b]$, u is monotonic nondecreasing on $[a,b]$ and $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

(i) *If* $f(a)=0,$ *then*

$$\int_a^b |f'(t) f(t)| \, du(t) \le \left(\int_a^b (t-a) |f'(t)|^p \, du(t)\right)^{1/p} \left(\int_a^b [u(b)-u(t)] |f'(t)|^q \, dt\right)^{1/q} \le \frac{1}{p}\int_a^b (t-a) |f'(t)|^p \, du(t) + \frac{1}{q}\int_a^b [u(b)-u(t)] |f'(t)|^q \, dt. \tag{2.8}$$

(ii) *If* $f(b)=0,$ *then*

$$\int_a^b |f'(t) f(t)| \, du(t) \le \left(\int_a^b (b-t) |f'(t)|^p \, du(t)\right)^{1/p} \left(\int_a^b [u(t)-u(a)] |f'(t)|^q \, dt\right)^{1/q} \le \frac{1}{p}\int_a^b (b-t) |f'(t)|^p \, du(t) + \frac{1}{q}\int_a^b [u(t)-u(a)] |f'(t)|^q \, dt. \tag{2.9}$$

Corollary 3. *Assume that* $g:[a,b]\to\mathbb{C}$ *is absolutely continuous on* $[a,b]$, u *is monotonic nondecreasing on* $[a,b]$ *and* $p,$ $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1.$

(i) *If* $g(a)=0,$ *then*

$$\int_a^b |g(t)| \, du(t) \le \left(\int_a^b [u(b)-u(t)] \, dt\right)^{1/p} \left(\int_a^b [u(b)-u(t)] |g'(t)|^q \, dt\right)^{1/q}. \tag{2.10}$$

(ii) *If* $g(b)=0,$ *then*

$$\int_a^b |g(t)| \, du(t) \le \left(\int_a^b [u(t)-u(a)] \, dt\right)^{1/p} \left(\int_a^b [u(t)-u(a)] |g'(t)|^q \, dt\right)^{1/q}. \tag{2.11}$$

It follows by Theorem 3 for $f\equiv 1.$
We have:

Theorem 4. *Assume that* $f,$ $g:[a,b]\to\mathbb{C}$ *are absolutely continuous on* $[a,b]$, f' *is continuous on* $[a,b]$, u *is monotonic nondecreasing on* $[a,b]$ *and* $p,$ $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1.$ *If* $g(a)=g(b)=0,$ *then*

$$\int_a^b |f'(t) g(t)| \, du(t) \le \frac{1}{2}(b-a)^{1/p} [u(b)-u(a)]^{1/q} \left(\int_a^b |f'(t)|^p \, du(t)\right)^{1/p} \left(\int_a^b |g'(t)|^q \, dt\right)^{1/q} \tag{2.12}$$

and

$$\begin{aligned}(2.13)\quad & \int_a^b |f'(t)\, g(t)|\, du(t) \\ & \leq \left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^p\, du(t) - \int_a^b \left|t-\frac{a+b}{2}\right| |f'(t)|^p\, du(t)\right)^{1/p} \\ & \qquad \times \left(\int_a^b \left|u\left(\frac{a+b}{2}\right)-u(t)\right| |g'(t)|^q\, dt\right)^{1/q}.\end{aligned}$$

Proof. If we add the inequalities (2.1) and (2.2) we get

$$\begin{aligned}(2.14)\quad & 2\int_a^b |f'(t)\, g(t)|\, du(t) \\ & \leq \left(\int_a^b (t-a)\,|f'(t)|^p\, du(t)\right)^{1/p} \left(\int_a^b [u(b)-u(t)]\,|g'(t)|^q\, dt\right)^{1/q} \\ & \quad + \left(\int_a^b (b-t)\,|f'(t)|^p\, du(t)\right)^{1/p} \left(\int_a^b [u(t)-u(a)]\,|g'(t)|^q\, dt\right)^{1/q}.\end{aligned}$$

If we use the elementary Hölder inequality for $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

$$(2.15)\qquad \alpha\beta+\gamma\delta \leq (\alpha^p+\gamma^p)^{1/p}\,(\beta^q+\delta^q)^{1/q},\ \alpha,\ \beta,\ \gamma,\ \delta \geq 0,$$

we have

$$\begin{aligned}(2.16)\quad & \left(\int_a^b (t-a)\,|f'(t)|^p\, du(t)\right)^{1/p} \left(\int_a^b [u(b)]-u(t)\,|g'(t)|^q\, dt\right)^{1/q} \\ & \quad + \left(\int_a^b (b-t)\,|f'(t)|^p\, du(t)\right)^{1/p} \left(\int_a^b [u(t)-u(a)]\,|g'(t)|^q\, dt\right)^{1/q}\end{aligned}$$

$$\begin{aligned} & \leq \left(\int_a^b (t-a)\,|f'(t)|^p\, du(t) + \int_a^b (b-t)\,|f'(t)|^p\, du(t)\right)^{1/p} \\ & \qquad \times \left(\int_a^b [u(b)-u(t)]\,|g'(t)|^q\, dt + \int_a^b [u(t)-u(a)]\,|g'(t)|^q\, dt\right)^{1/q} \\ & = (b-a)^{1/p}\left(\int_a^b |f'(t)|^p\, du(t)\right)^{1/p} [u(b)-u(a)]^{1/q}\left(\int_a^b |g'(t)|^q\, dt\right)^{1/q}.\end{aligned}$$

By making use of (2.14) and (2.16) we get (2.12).

If we use the inequality (2.1) on the interval $\left[a, \frac{a+b}{2}\right]$, then we have

$$\int_a^{\frac{a+b}{2}} |f'(t) g(t)| \, du(t) \tag{2.17}$$
$$\leq \left(\int_a^{\frac{a+b}{2}} (t-a) |f'(t)|^p \, du(t) \right)^{1/p} \left(\int_a^{\frac{a+b}{2}} \left[u\left(\frac{a+b}{2}\right) - u(t) \right] |g'(t)|^q \, dt \right)^{1/q}$$

while if we use the inequality (2.2) on the interval $\left[\frac{a+b}{2}, b\right]$, then we have

$$\int_{\frac{a+b}{2}}^b |f'(t) g(t)| \, w(t) \, dt \tag{2.18}$$
$$\leq \left(\int_{\frac{a+b}{2}}^b (b-t) |f'(t)|^p \, du(t) \right)^{1/p} \left(\int_{\frac{a+b}{2}}^b \left[u(t) - u\left(\frac{a+b}{2}\right) \right] |g'(t)|^q \, dt \right)^{1/q}.$$

If we add these two inequalities, then we get by (2.15) that

$$\begin{aligned}
&\int_a^b |f'(t) g(t)| \, du(t) \\
&\leq \left(\int_a^{\frac{a+b}{2}} (t-a) |f'(t)|^p \, du(t) \right)^{1/p} \left(\int_a^{\frac{a+b}{2}} \left[u\left(\frac{a+b}{2}\right) - u(t) \right] |g'(t)|^q \, dt \right)^{1/q} \\
&+ \left(\int_{\frac{a+b}{2}}^b (b-t) |f'(t)|^p \, du(t) \right)^{1/p} \left(\int_{\frac{a+b}{2}}^b \left[u(t) - u\left(\frac{a+b}{2}\right) \right] |g'(t)|^q \, dt \right)^{1/q} \\
&\leq \left(\int_a^b K(t) |f'(t)|^p \, du(t) \right)^{1/p} \left(\int_a^b \left| u\left(\frac{a+b}{2}\right) - u(t) \right| |g'(t)|^q \, dt \right)^{1/q}
\end{aligned}$$

where

$$K(t) := \begin{cases} t-a, & \text{if } t \in \left[a, \frac{a+b}{2}\right] \\ b-t, & \text{if } t \in \left[\frac{a+b}{2}, b\right] \end{cases} = \frac{1}{2}(b-a) - \left| t - \frac{a+b}{2} \right|.$$

This proves (2.13). □

Remark 2. *If we take in (2.13) $u(t) = t$, then we get (1.5).*

Corollary 4. *Assume that $f : [a,b] \to \mathbb{C}$ is absolutely continuous on $[a,b]$, f' is continuous on $[a,b]$, u is monotonic nondecreasing on $[a,b]$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f(a) = f(b) = 0$, then*

$$\int_a^b |f'(t) f(t)| \, du(t) \tag{2.19}$$
$$\leq \frac{1}{2}(b-a)^{1/p} [u(b) - u(a)]^{1/q} \left(\int_a^b |f'(t)|^p \, du(t) \right)^{1/p} \left(\int_a^b |f'(t)|^q \, dt \right)^{1/q}$$

and

$$\int_a^b |f'(t) f(t)| \, du(t) \tag{2.20}$$
$$\leq \left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^p \, du(t) - \int_a^b \left|t - \frac{a+b}{2}\right| |f'(t)|^p \, du(t)\right)^{1/p}$$
$$\times \left(\int_a^b \left|u\left(\frac{a+b}{2}\right) - u(t)\right| |f'(t)|^q \, dt\right)^{1/q}.$$

We also have:

Corollary 5. *Assume that* $g : [a,b] \to \mathbb{C}$ *is absolutely continuous on* $[a,b]$, u *is monotonic nondecreasing on* $[a,b]$ *and* p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $g(a) = g(b) = 0$, *then*

$$\int_a^b |g(t)| \, du(t) \leq \frac{1}{2}(b-a)^{1/p} [u(b) - u(a)] \left(\int_a^b |g'(t)|^q \, dt\right)^{1/q} \tag{2.21}$$

and

$$\int_a^b |g(t)| \, du(t) \leq \left(\int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) u(t) \, dt\right)^{1/p} \tag{2.22}$$
$$\times \left(\int_a^b \left|u\left(\frac{a+b}{2}\right) - u(t)\right| |g'(t)|^q \, dt\right)^{1/q}.$$

Proof. The inequality (2.21) is obvious by (2.12) for $f \equiv 1$.

Observe that

$$\begin{aligned}
&\frac{1}{2}(b-a)\int_a^b du(t) - \int_a^b \left|t - \frac{a+b}{2}\right| du(t) \\
&= \int_a^{\frac{a+b}{2}} (t-a) \, du(t) + \int_{\frac{a+b}{2}}^b (b-t) \, du(t) \\
&= (t-a) u(t)\Big|_a^{\frac{a+b}{2}} - \int_a^{\frac{a+b}{2}} u(t) \, dt + (b-t) u(t)\Big|_{\frac{a+b}{2}}^b + \int_{\frac{a+b}{2}}^b u(t) \, dt \\
&= \int_{\frac{a+b}{2}}^b u(t) \, dt - \int_a^{\frac{a+b}{2}} u(t) \, dt = \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) u(t) \, dt
\end{aligned}$$

and by (2.13) for $f \equiv 1$ we get

$$\int_a^b |g(t)| \, du(t) \leq \left(\frac{1}{2}(b-a)\int_a^b du(t) - \int_a^b \left|t - \frac{a+b}{2}\right| du(t)\right)^{1/p}$$
$$\times \left(\int_a^b \left|u\left(\frac{a+b}{2}\right) - u(t)\right| |g'(t)|^q \, dt\right)^{1/q},$$

which proves (2.22). □

3. Some Inequalities for Functions of Bounded Variation

The following lemma was obtained by the author in 2007, [5] and is of interest in itself as well (see also [4]):

Lemma 1. *If $p:[a,b]\to\mathbb{C}$ is continuous on $[a,b]$ and $v:[a,b]\to\mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$\left|\int_a^b p(t)\,dv(t)\right| \leq \int_a^b |p(t)|\,dV(t) \tag{3.1}$$
$$\leq \left(\int_a^b |p(t)|^p\,dV(t)\right)^{1/p}\left(\bigvee_a^b(v)\right)^{1/q} \leq \max_{t\in[a,b]}|p(t)|\bigvee_a^b(v),$$

where $V(t):=\bigvee_a^t(v)$ is the total variation of v on $[a,t]$ with $t\in[a,b]$.

The function V is nondecreasing on $[a,b]$ with $V(a)=0$ and $V(b)=\bigvee_a^b(v)$. If we put $\overline{V}(t):=\bigvee_t^b(v)=\bigvee_a^b(v)-V(t)$, then $\overline{V}$is nonincreasing with $\overline{V}(a)=\bigvee_a^b(v)$ and $\overline{V}(b)=0$.

We have:

Proposition 1. *Assume that $h:[a,b]\to\mathbb{C}$ is continuous, g is absolutely continuous on $[a,b]$ and v is of bounded variation on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b h(t)g(t)\,dv(t)$ exists and for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have,*

(i) *If $g(a)=0$, then*

$$\left|\int_a^b h(t)g(t)\,dv(t)\right| \tag{3.2}$$
$$\leq \left(\int_a^b (t-a)|h(t)|^p\,dV(t)\right)^{1/p}\left(\int_a^b \overline{V}(t)|g'(t)|^q\,dt\right)^{1/q}.$$

(ii) *If $g(b)=0$, then*

$$\left|\int_a^b h(t)g(t)\,dv(t)\right| \tag{3.3}$$
$$\leq \left(\int_a^b (b-t)|h(t)|^p\,dV(t)\right)^{1/p}\left(\int_a^b V(t)|g'(t)|^q\,dt\right)^{1/q}.$$

Proof. Using the first inequality in (3.1), we get

$$\left|\int_a^b h(t)g(t)\,dv(t)\right| \leq \int_a^b |h(t)g(t)|\,dV(t). \tag{3.4}$$

Using now Theorem 3 for $f=\int_a h$ and $u=V$, we get, for $g(a)=0$, that

$$\int_a^b |h(t)g(t)|\,dV(t) \tag{3.5}$$
$$\leq \left(\int_a^b (t-a)|h(t)|^p\,dV(t)\right)^{1/p}\left(\int_a^b [V(b)-V(t)]|g'(t)|^q\,dt\right)^{1/q}.$$

If $g(b)=0$, then

$$\int_a^b |h(t)\,g(t)|\,dV(t) \leq \left(\int_a^b (b-t)\,|f'(t)|^p\,dV(t)\right)^{1/p}\left(\int_a^b V(t)\,|g'(t)|^q\,dt\right)^{1/q}. \tag{3.6}$$

By utilising (3.4)-(3.6) we get the desired results (3.2) and (3.3). □

The case $h\equiv 1$ is of interest since in this case

$$\int_a^b (t-a)\,dV(t) = (b-a)\,V(b) - \int_a^b V(t)\,dt = \int_a^b (V(b)-V(t))\,dt = \int_a^b \overline{V}(t)\,dt$$

and

$$\int_a^b (b-t)\,dV(t) = \int_a^b V(t)\,dt.$$

We then can state:

Corollary 6. *Assume that g is absolutely continuous on $[a,b]$ and v is of bounded variation on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b g(t)\,dv(t)$ exists and for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have*

(i) *If $g(a)=0$, then*

$$\left|\int_a^b g(t)\,dv(t)\right| \leq \left(\int_a^b \overline{V}(t)\,dt\right)^{1/p}\left(\int_a^b \overline{V}(t)\,|g'(t)|^q\,dt\right)^{1/q}. \tag{3.7}$$

(ii) *If $g(b)=0$, then*

$$\left|\int_a^b g(t)\,dv(t)\right| \leq \left(\int_a^b V(t)\,dt\right)^{1/p}\left(\int_a^b V(t)\,|g'(t)|^q\,dt\right)^{1/q}. \tag{3.8}$$

We also have:

Proposition 2. *Assume that $h:[a,b]\to\mathbb{C}$ is continuous, g is absolutely continuous on $[a,b]$ with $g(a)=g(b)=0$ and v is of bounded variation on $[a,b]$, then for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ we have*

$$\left|\int_a^b h(t)\,g(t)\,dv(t)\right| \leq \frac{1}{2}(b-a)^{1/p}\left[\bigvee_a^b(v)\right]^{1/q}\left(\int_a^b |h(t)|^p\,dV(t)\right)^{1/p}\left(\int_a^b |g'(t)|^q\,dt\right)^{1/q} \tag{3.9}$$

and

$$\left|\int_a^b h(t)\,g(t)\,dv(t)\right| \leq \left(\frac{1}{2}(b-a)\int_a^b |h(t)|^p\,dV(t) - \int_a^b \left|t-\frac{a+b}{2}\right| |h(t)|^p\,dV(t)\right)^{1/p} \times \left(\int_a^b \left|V(t)-V\left(\frac{a+b}{2}\right)\right| |g'(t)|^q\,dt\right)^{1/q}. \tag{3.10}$$

The proof follows by Theorem 5 and Lemma 1.

Corollary 7. *Assume that g is absolutely continuous on $[a,b]$ with $g(a)=g(b)=0$ and v is of bounded variation on $[a,b]$, then for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ we have*

$$\left|\int_a^b g(t)\,dv(t)\right| \leq \frac{1}{2}(b-a)^{1/p}\left(\int_a^b |g'(t)|^q\,dt\right)^{1/q} \bigvee_a^b (v) \tag{3.11}$$

and

$$\left|\int_a^b g(t)\,dv(t)\right| \leq \left(\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right) V(t)\,dt\right)^{1/p} \left(\int_a^b \left|V(t)-V\left(\frac{a+b}{2}\right)\right| |g'(t)|^q\,dt\right)^{1/q}. \tag{3.12}$$

Proof. Is similar to the one from Corollary 5. □

4. Some Inequalities for the Čebyšev Functional

Consider now the *Čebyšev functional*

$$C_u(f,g) := \frac{1}{u(b)-u(a)}\int_a^b f(t)\,g(t)\,du(t) - \frac{1}{u(b)-u(a)}\int_a^b f(t)\,du(t)\,\frac{1}{u(b)-u(a)}\int_a^b g(t)\,du(t) \tag{4.1}$$

where f, $g : [a,b] \to \mathbb{C}$ are continuous and u is monotonic nondecreasing with $u(a) \neq u(b)$.

In [3], Cerone and Dragomir obtained some reverses of Grüss inequality for positive measures. If we write these inequalities for the Riemann-Stieltjes integral of monotonic integrators, we have

$$
\begin{aligned}
(4.2)\quad & \left|C_u(f,g)\right| \\
&\leq \frac{1}{2}(M-m)\frac{1}{u(b)-u(a)}\int_a^b \left| g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| du(t) \\
&\leq \frac{1}{2}(M-m)\left[\frac{1}{u(b)-u(a)}\int_a^b \left| g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|^p du(t)\right]^{\frac{1}{p}} \\
&\leq \frac{1}{2}(M-m)\operatorname*{essup}_{t\in[a,b]}\left| g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|
\end{aligned}
$$

for $p>1$, provided $-\infty<m\leq f(t)\leq M<\infty$ for a.e. $t\in[a,b]$ and the corresponding integrals are finite. The constant $\frac{1}{2}$ is sharp in all the inequalities in (4.2) in the sense that it cannot be replaced by a smaller constant.

In addition, if $-\infty<n\leq g(t)\leq N<\infty$ for a.e. $t\in[a,b]$, then the following refinement of the Grüss inequality for Riemann-Stieltjes integral is obtained:

$$
\begin{aligned}
(4.3)\quad & \left|C_u(f,g)\right| \\
&\leq \frac{1}{2}(M-m)\frac{1}{u(b)-u(a)}\int_a^b \left| g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| du(t) \\
&\leq \frac{1}{2}(M-m) \\
&\quad\times\left[\frac{1}{u(b)-u(a)}\int_a^b g^2(t)\,du(t)-\left(\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right)^2\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}(M-m)(N-n).
\end{aligned}
$$

Here, the constants $\frac{1}{2}$ and $\frac{1}{4}$ are also sharp in the sense mentioned above.

We have the following inequality for the Čebyšev functional C_u.

Theorem 5. *Assume that $g:[a,b]\to\mathbb{C}$ is absolutely continuous on $[a,b]$, $f:[a,b]\to\mathbb{C}$ is continuous on $[a,b]$ and u is monotonic nondecreasing on $[a,b]$, where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$
\begin{aligned}
(4.4)\quad & \left|C_u(f,g)\right| \\
&\leq \left(\frac{1}{u(b)-u(a)}\int_a^b (t-a)\left| f(t)-\frac{1}{u(b)-u(a)}\int_a^b f(s)\,du(s)\right|^p du(t)\right)^{1/p} \\
&\quad\times\left(\frac{1}{u(b)-u(a)}\int_a^b \left|g'(t)\right|^q\left[u(b)-u(t)\right]dt\right)^{1/q}
\end{aligned}
$$

and

$$|C_u(f,g)| \le \left(\frac{1}{u(b)-u(a)}\int_a^b (b-t)\left|f(t)-\frac{1}{u(b)-u(a)}\int_a^b f(s)\,du(s)\right|^p du(t)\right)^{1/p} \times \left(\frac{1}{u(b)-u(a)}\int_a^b |g'(t)|^q [u(t)-u(a)]\,dt\right)^{1/q}. \tag{4.5}$$

Proof. We use the following *Sonin type identity* for the Riemann-Stieltjes integral

$$C_u(f,g) = \frac{1}{u(b)-u(a)}\int_a^b \left(f(t)-\frac{1}{u(b)-u(a)}\int_a^b f(s)\,du(s)\right)[g(t)-\gamma]\,du(t), \tag{4.6}$$

for $\gamma \in \mathbb{C}$, which can be proved directly on calculating the integral from the right hand side.

Using the inequality (2.1) for $\gamma = g(a)$, we have

$$\begin{aligned} |C_u(f,g)| &\le \frac{1}{u(b)-u(a)}\int_a^b \left|f(t)-\frac{1}{u(b)-u(a)}\int_a^b f(s)\,du(s)\right| |g(t)-g(a)|\,du(t) \\ &\le \frac{1}{u(b)-u(a)}\left(\int_a^b (t-a)\left|f(t)-\frac{1}{u(b)-u(a)}\int_a^b f(s)\,du(s)\right|^p du(t)\right)^{1/p} \\ &\qquad \times \left(\int_a^b |g'(t)|^q [u(b)-u(t)]\,dt\right)^{1/q} \end{aligned}$$

that proves the inequality in (4.4).

Using the inequality (2.2) for $\gamma = g(b)$, we have

$$\begin{aligned} |C_u(f,g)| &\le \frac{1}{u(b)-u(a)}\int_a^b \left|f(t)-\frac{1}{u(b)-u(a)}\int_a^b f(s)\,du(s)\right| |g(t)-g(b)|\,du(t) \\ &\le \frac{1}{u(b)-u(a)}\left(\int_a^b (b-t)\left|f(t)-\frac{1}{u(b)-u(a)}\int_a^b f(s)\,du(s)\right|^p du(t)\right)^{1/p} \\ &\qquad \times \left(\int_a^b [u(t)-u(a)]\,|g'(t)|^q\,dt\right)^{1/q} \end{aligned}$$

that proves the inequality (4.5). □

Corollary 8. *Assume that $g : [a,b] \to \mathbb{C}$ is absolutely continuous on $[a,b]$, $f : [a,b] \to \mathbb{C}$ is continuous on $[a,b]$ and u is monotonic nondecreasing on $[a,b]$, where*

$p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Then

$$
\begin{aligned}
(4.7)\quad |C_u(f,g)| &\le \frac{1}{2}(b-a)^{1/p}\left(\int_a^b |g'(t)|^q\,dt\right)^{1/q}\\
&\times\left(\frac{1}{u(b)-u(a)}\int_a^b \left|f(t)-\frac{1}{u(b)-u(a)}\int_a^b f(s)\,du(s)\right|^p du(t)\right)^{1/p}.
\end{aligned}
$$

Proof. If we add the inequalities (4.4) and (4.5), then we get

$$
\begin{aligned}
&2|C_u(f,g)|\\
&\le\left(\frac{1}{u(b)-u(a)}\int_a^b (t-a)\left|f(t)-\frac{1}{u(b)-u(a)}\int_a^b f(s)\,du(s)\right|^p du(t)\right)^{1/p}\\
&\times\left(\frac{1}{u(b)-u(a)}\int_a^b |g'(t)|^q[u(b)-u(t)]\,dt\right)^{1/q}\\
&+\left(\frac{1}{u(b)-u(a)}\int_a^b (b-t)\left|f(t)-\frac{1}{u(b)-u(a)}\int_a^b f(s)\,du(s)\right|^p du(t)\right)^{1/p}\\
&\times\left(\frac{1}{u(b)-u(a)}\int_a^b |g'(t)|^q[u(t)-u(a)]\,dt\right)^{1/q}\\
&=: E.
\end{aligned}
$$

Using the inequality (2.15), we have

$$
\begin{aligned}
E &\le\left(\frac{1}{u(b)-u(a)}\int_a^b (t-a)\left|f(t)-\frac{1}{u(b)-u(a)}\int_a^b f(s)\,du(s)\right|^p du(t)\right.\\
&\left.+\frac{1}{u(b)-u(a)}\int_a^b (b-t)\left|f(t)-\frac{1}{u(b)-u(a)}\int_a^b f(s)\,du(s)\right|^p du(t)\right)^{1/p}\\
&\times\left(\frac{1}{u(b)-u(a)}\int_a^b |g'(t)|^q[u(b)-u(t)]\,dt\right.\\
&\left.+\frac{1}{u(b)-u(a)}\int_a^b |g'(t)|^q[u(t)-u(a)]\,dt\right)^{1/q}\\
&=(b-a)^{1/p}\left(\frac{1}{u(b)-u(a)}\int_a^b \left|f(t)-\frac{1}{u(b)-u(a)}\int_a^b f(s)\,du(s)\right|^p du(t)\right)^{1/p}\\
&\times\left(\int_a^b |g'(t)|^q\,dt\right)^{1/q},
\end{aligned}
$$

which proves (4.7). □

Remark 3. *If $p=q=2$, then by (4.7) we get*

$$|C_u(f,g)| \le \frac{1}{2}(b-a)^{1/2}\left(\int_a^b |g'(t)|^2 dt\right)^{1/2} \times \left(\frac{1}{u(b)-u(a)}\int_a^b |f(t)|^2 du(t) - \left|\frac{1}{u(b)-u(a)}\int_a^b f(s)\,du(s)\right|^2\right)^{1/2}. \tag{4.8}$$

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