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CERTAIN PROPERTIES OF THE NIELSEN'S β-FUNCTION

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ABSTRACT. In this paper, we present some properties of the Nielsen's β-function. The obtained results are analogous to some known works involving the gamma and digamma functions.

1. INTRODUCTION

In 1974, Gautschi [3] presented an interesting inequality involving the classical Euler's Gamma function, $\Gamma(x)$. He proved that, for $x > 0$, the harmonic mean of $\Gamma(x)$ and $\Gamma(1/x)$ is always greater than or equal to 1. That is,

$$1 \leq \frac{2\Gamma(x)\Gamma(1/x)}{\Gamma(x)+\Gamma(1/x)}, \quad x > 0, \tag{1}$$

with equality if $x = 1$. As a direct consequence of (1), the inequalities

$$2 \leq \Gamma(x) + \Gamma(1/x), \quad x > 0, \tag{2}$$

and

$$1 \leq \Gamma(x)\Gamma(1/x), \quad x > 0, \tag{3}$$

are obtained. Then recently, Alzer and Jameson [1] established a striking companion of (1) which involves the digamma function, $\psi(x)$. They proved that the inequality

$$-\gamma \leq \frac{2\psi(x)\psi(1/x)}{\psi(x)+\psi(1/x)}, \quad x > 0, \tag{4}$$

holds, with equality if $x = 1$, where $\gamma = 0.57721, ...$ is the Euler-Mascheroni constant. In addition, they proved that

$$P(x) = \psi(x) + \psi(1/x), \tag{5}$$

is strictly concave on $(0, \infty)$ and that

$$\psi(x) + \psi(1/x) < -2\gamma, \quad x > 0, x \neq 1. \tag{6}$$

$$\psi(1+y)\psi(1-y) < \gamma^2, \quad y \in (0, 1). \tag{7}$$

$$\psi(x)\psi(1/x) < \gamma^2, \quad x > 0, x \neq 1. \tag{8}$$

Also, in [11], it was established among other things that the function

$$h_1 = \psi\left(x + \frac{1}{2}\right) - \psi(x) - \frac{1}{2x}, \tag{9}$$

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is strictly decreasing and convex on $(0, \infty)$. Motivated by the result (9), Mortici [6] proved that the generalized function

$$f_a = \psi(x + a) - \psi(x) - \frac{a}{x}, \quad a \in (0, 1), \tag{10}$$

is strictly completely monotonic on $(0, \infty)$.
Inspired by the above results, the purpose of this paper is to establish analogous results for the Nielsen's β-function.

2. Preliminary Definitions

The Nielsen's β-function may be defined by any of the following equivalent forms (see [2], [4], [7], [10]).

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t}\, dt, \quad x > 0, \tag{11}$$

$$= \int_0^\infty \frac{e^{-xt}}{1+e^{-t}}\, dt, \quad x > 0, \tag{12}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+x}, \quad x > 0, \tag{13}$$

$$= \frac{1}{2}\left\{\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right\}, \quad x > 0, \tag{14}$$

where $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$ is the digamma or psi function and $\Gamma(x)$ is the Euler's Gamma function. It is known to satisfy the properties:

$$\beta(x+1) = \frac{1}{x} - \beta(x), \tag{15}$$

$$\beta(x) + \beta(1-x) = \frac{\pi}{\sin \pi x}. \tag{16}$$

Some particular values of the function are $\beta(1) = \ln 2$, $\beta\left(\frac{1}{2}\right) = \frac{\pi}{2}$, $\beta\left(\frac{3}{2}\right) = 2 - \frac{\pi}{2}$ and $\beta(2) = 1 - \ln 2$.
By differentiating n-times of (11), (12), (13), (14) and (15), one obtains

$$\beta^{(n)}(x) = \int_0^1 \frac{(\ln t)^n t^{x-1}}{1+t}\, dt, \quad x > 0 \tag{17}$$

$$= (-1)^n \int_0^\infty \frac{t^n e^{-xt}}{1+e^{-t}}\, dt, \quad x > 0 \tag{18}$$

$$= (-1)^n n! \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+x)^{n+1}}, \quad x > 0 \tag{19}$$

$$= \frac{1}{2^{n+1}}\left\{\psi^{(n)}\left(\frac{x+1}{2}\right) - \psi^{(n)}\left(\frac{x}{2}\right)\right\}, \quad x > 0 \tag{20}$$

$$\beta^{(n)}(x+1) = \frac{(-1)^n n!}{x^{n+1}} - \beta^{(n)}(x), \quad x > 0 \tag{21}$$

where $n \in \mathbb{N}_0$ and $\beta^{(0)}(x) = \beta(x)$.

For additional information on this special function, one may refer to [7], [8], [9] and the related references therein.

3. Main Results

Lemma 3.1. *The function* $x\beta(x)$ *is decreasing and convex on* $(0,\infty)$. *Consequently, the inequalities*

$$\beta(x) + x\beta'(x) < 0, \quad x > 0, \tag{22}$$

and

$$2\beta'(x) + x\beta''(x) > 0, \quad x > 0, \tag{23}$$

are satisfied.

Proof. In Theorem 3 of [9], the function $x\left|\beta^{(m)}(x)\right|$, $x>0$, $m \in \mathbb{N}_0$ was proved to be completely monotonic. Thus, $x\beta(x)$, the case where $m=0$, is completely monotonic. Since every completely monotonic function is decreasing and convex [5], we conclude that $x\beta(x)$ is decreasing and convex. This give rise to inequalities (22) and (23).

Theorem 3.2. *The function*

$$Q(x) = \beta(x) + \beta(1/x) \tag{24}$$

is strictly convex on $(0,\infty)$.

Proof. By direct differentiation, and by applying (23), we obtain

$$\begin{aligned} Q'(x) &= \beta'(x) - \frac{1}{x^2}\beta'(1/x), \\ Q''(x) &= \beta''(x) + \frac{2}{x^3}\beta'(1/x) + \frac{1}{x^4}\beta''(1/x) \\ &= \beta''(x) + \frac{1}{x^3}\left[2\beta'(1/x) + \frac{1}{x}\beta''(1/x)\right] > 0, \end{aligned}$$

which completes the proof.

Theorem 3.3. *The inequality*

$$\beta(x) + \beta(1/x) \geq 2\ln 2 \tag{25}$$

holds for $x > 0$.

Proof. Since $Q''(x) > 0$, then $(Q'(x))' > 0$ which implies that $Q'(x)$ is increasing. Then $Q'(x) \leq Q'(1) = 0$ for $x \in (0,1]$ and $Q'(x) \geq Q'(1) = 0$ for $x \in [1,\infty)$. These imply that $Q(x)$ is decreasing on $(0,1]$ and increasing on $[1,\infty)$. Therefore, in either case, we have $Q(x) \geq Q(1) = 2\ln 2$ which gives the desired result.

Theorem 3.4. *The inequality*

$$\beta(1+s)\beta(1-s) \geq (\ln 2)^2 \tag{26}$$

holds for $s \in [0,1)$.

Proof. Since $\beta(x)$ is logarithmically convex (see [7]), we have

$$\beta\left(\frac{x+y}{2}\right) \leq \sqrt{\beta(x)\beta(y)}, \tag{27}$$

for $x > 0$ and $y > 0$. Now, by letting $x = 1 + s$ and $y = 1 - s$ in (27), we obtain the desired result (26).

Theorem 3.5. *The inequality*

$$\beta(x)\beta(1/x) \geq (\ln 2)^2 \tag{28}$$

holds for $x > 0$.

Proof. If $x \geq 1$, then $0 < 1/x \leq 1$. Also, if $0 < x \leq 1$, then $1/x \geq 1$. Hence it suffices to prove (28) for $x \geq 1$. For $x \geq 1$ and $s \in [0, 1)$, let $x = 1 + s$ and $1/x = 1 - s$. Then by (26), we obtain

$$\beta(x)\beta(1/x) = \beta(1 + s)\beta(1 - s) \geq (\ln 2)^2,$$

which concludes the proof.

Theorem 3.6. *For $x, y \in (1, \infty)$, the harmonic mean of $\beta(x)$ and $\beta(y)$ is less than 1 . In other words, the inequality*

$$\frac{2\beta(x)\beta(y)}{\beta(x) + \beta(y)} < 1 \tag{29}$$

holds for $x, y \in (1, \infty)$.

Proof. Note that for $t \in (1, \infty)$, we have $\beta(t) < \beta(1) = \ln 2$, since $\beta(x)$ is decreasing. Let $x, y \in (1, \infty)$. Then by the AM-GM inequality, we have

$$\sqrt{\beta(x)\beta(y)} \leq \frac{\beta(x) + \beta(y)}{2}.$$

This implies that

$$2\beta(x)\beta(y) \leq [\beta(x)]^2 + [\beta(y)]^2 < \beta(x) + \beta(y),$$

which gives the desired result. Note that $\beta(v) \in (0, 1)$ for all $v \in (1, \infty)$. Hence, $[\beta(v)]^2 < \beta(v)$ for all $v \in (1, \infty)$.

In view of the harmonic mean inequalities (1) and (4), we give the following conjecture.

Conjecture 3.7. For $x \in (0, \infty)$, the inequality

$$\frac{2\beta(x)\beta(1/x)}{\beta(x) + \beta(1/x)} \leq \ln 2, \tag{30}$$

is satisfied.

Theorem 3.8. *The double inequality*

$$\frac{1}{x} - \ln 2 < \beta(x) < \frac{1}{x} \tag{31}$$

holds for $x > 0$.

Proof. As a a direct consequence of (15), we obtain

$$\beta(x) < \frac{1}{x}. \tag{32}$$

for $x > 0$. Also, by (15), we obtain the limit

$$\lim_{x\to 0^+}\left\{\frac{1}{x} - \beta(x)\right\} = \ln 2. \tag{33}$$

Now, let $\theta(x) = \frac{1}{x} - \beta(x)$ for $x > 0$. Then by (21), we obtain

$$\theta'(x) = -\frac{1}{x^2} - \beta'(x) < 0,$$

which shows that $\theta(x)$ is decreasing. Hence for $x > 0$, we obtain

$$\frac{1}{x} - \beta(x) = \theta(x) < \lim_{x\to 0^+}\theta(x) = \ln 2 \tag{34}$$

Then, by combining (32) and (34), we obtain the result (31).

Theorem 3.9. *The limit*

$$\lim_{z\to 0^+}\frac{1}{z}\left\{\frac{1}{\beta(1-z)} - \frac{1}{\beta(1+z)}\right\} = -\frac{\pi^2}{6(\ln 2)^2} \tag{35}$$

is valid for $z \in (0,1)$.

Proof. It can be shown from relation (14) that $\beta'(1) = -\frac{\pi^2}{12}$. Then by L'Hopital's rule, we obtain

$$\begin{aligned}\lim_{z\to 0^+}\frac{1}{z}\left\{\frac{1}{\beta(1-z)} - \frac{1}{\beta(1+z)}\right\} &= \lim_{z\to 0^+}\left\{\frac{\beta'(1-z)}{[\beta(1-z)]^2} + \frac{\beta'(1+z)}{[\beta(1+z)]^2}\right\} \\ &= -\frac{\pi^2}{6(\ln 2)^2}.\end{aligned}$$

Theorem 3.10. *For* $a > 0$ *and* $x > 0$*, let* f_a *be defined as*

$$f_a(x) = \beta(x+a) - \beta(x) - \frac{a}{x}. \tag{36}$$

Then $-f_a$ *is strictly completely monotonic.*

Proof. Recall that a function $f : (0,\infty) \to \mathbb{R}$ is said to be completely monotonic on $(0,\infty)$ if f has derivatives of all order and $(-1)^n f^{(n)}(x) \geq 0$ for all $x \in (0,\infty)$ and $n \in \mathbb{N}$. Let

$$h_a(x) = -f_a(x) = \frac{a}{x} + \beta(x) - \beta(x+a).$$

Then by repeated differentiation and by using (18), we obtain

$$\begin{aligned} h_a^{(n)}(x) &= (-1)^n a \frac{n!}{x^{n+1}} + \beta^{(n)}(x) - \beta^{(n)}(x+a) \\ &= (-1)^n a \int_0^\infty t^n e^{-xt}\,dt + (-1)^n \int_0^\infty \frac{t^n e^{-xt}}{1+e^{-t}}\,dt \\ &\quad - (-1)^n \int_0^\infty \frac{t^n e^{-(x+a)t}}{1+e^{-t}}\,dt, \\ (-1)^n h_a^{(n)}(x) &= a \int_0^\infty t^n e^{-xt}\,dt + \int_0^\infty \frac{t^n e^{-xt}}{1+e^{-t}}\,dt - \int_0^\infty \frac{t^n e^{-(x+a)t}}{1+e^{-t}}\,dt \\ &= \int_0^\infty \left[a + \frac{1-e^{-at}}{1+e^{-t}} \right] t^n e^{-xt}\,dt > 0, \end{aligned}$$

which completes the proof.

Corollary 3.11. *The inequality*

$$0 < \beta(x) - \beta(x+a) + \frac{a}{x} \leq \ln 2 + a - \frac{1}{a} + \beta(a) \tag{37}$$

holds for $a > 0$ and $x \in [1, \infty)$.

Proof. Since $h_a(x)$ is completely monotonic on $(0, \infty)$, then it is decreasing on $(0, \infty)$. Then for $x \in [1, \infty)$, we have

$$\begin{aligned} 0 = \lim_{x\to\infty} h_a(x) < h_a(x) \leq h_a(1) &= a + \beta(1) - \beta(1+a) \\ &= \ln 2 + a - \frac{1}{a} + \beta(a) \end{aligned}$$

yielding the desired result.

Remark 3.12. In particular, if $a = \frac{1}{2}$, we obtain the sharp inequality

$$0 < \beta(x) - \beta\left(x + \frac{1}{2}\right) + \frac{1}{2x} \leq \ln 2 + \frac{\pi - 3}{2} \tag{38}$$

for $x \in [1, \infty)$. If $x \in (0, 1]$, then the right-hand sides of (37) and (38) are reversed.

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