Received 02/07/18

SIMPLE WEIGHTED INTEGRAL INEQUALITIES RELATED TO HARDY'S RESULT

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we provide some weighted inequalities that can be obtained in a simple way from the Hardy and Knopp inequalities. Some applications for natural weights are also provided.

1. Introduction

The following results is well-known in the literature as the Hardy's integral inequality, see [1, p. 240] or [3, p. 19]:

Theorem 1. *Let $f:[0,\infty)\to[0,\infty)$ and $p>1$ such that $\int_0^\infty f^p(x)\,dx$ is convergent. Then*

$$\int_0^\infty\left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx\le\left(\frac{p}{p-1}\right)^p\int_0^\infty f^p(x)\,dx. \tag{1.1}$$

The following theorem (cf. [3, Theorem 2]) generalizes the classical Hardy's integral inequality by introducing power weights x^α. It also provides a lower bound in the case of decreasing functions.

Theorem 2. *Let $f:[0,\infty)\to[0,\infty)$ and $p\ge1$, $\alpha<p-1$ such that $\int_0^\infty f^p(x)\,x^\alpha dx$ is convergent. Then*

$$\int_0^\infty\left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p x^\alpha dx\le\left(\frac{p}{p-\alpha-1}\right)^p\int_0^\infty f^p(x)\,x^\alpha dx. \tag{1.2}$$

Moreover, if f is a positive decreasing function, then also

$$\frac{p}{p-\alpha-1}\int_0^\infty f^p(x)\,x^\alpha dx\le\int_0^\infty\left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p x^\alpha dx. \tag{1.3}$$

The following limiting case when $p\to\infty$ was obtained by K. Knopp in [2]

Theorem 3. *Let $f:[0,\infty)\to(0,\infty)$ and such that $\int_0^\infty f(x)\,dx$ is convergent. Then*

$$\int_0^\infty\exp\left(\frac{1}{x}\int_0^x\ln f(t)\,dt\right)dx\le e\int_0^\infty f(x)\,dx. \tag{1.4}$$

In this paper we provide some weighted inequalities that can be obtained in a simple way from the inequalities above. Some applications for natural weights are also provided.

1991 *Mathematics Subject Classification.* 26D15; 26D10.

Key words and phrases. Hardy inequality, Knopp inequality.

RGMIA Res. Rep. Coll. 21 (2018), Art. 72, 6 pp.

2. The Results

We have:

Theorem 4. *Let $h:[0,\infty)\to[0,\infty)$ be a continuous strictly increasing function that is differentiable on $(0,\infty)$ and such $\lim_{t\to\infty}h(t)=\infty$. If $g:[0,\infty)\to[0,\infty)$ and $p>1$ are such that $\int_0^\infty g^p(x)\,h'(x)\,dx$ is convergent, then*

$$\int_0^\infty\left(\frac{1}{h(x)}\int_0^x g(t)\,h'(t)\,dt\right)^p h'(x)\,dx\le\left(\frac{p}{p-1}\right)^p\int_0^\infty g^p(x)\,h'(x)\,dx. \tag{2.1}$$

Proof. If we make the change of variable $x=h(y)$, $y\in[0,\infty)$, then $dx=h'(y)\,dy$ and by (1.1) we get

$$\int_0^\infty\left(\frac{1}{h(y)}\int_0^{h(y)} f(t)\,dt\right)^p h'(y)\,dy \le\left(\frac{p}{p-1}\right)^p\int_0^\infty[(f\circ h)(y)]^p\,h'(y)\,dy. \tag{2.2}$$

If we write the inequality (2.2) for $f:=g\circ h^{-1}:[0,\infty)\to[0,\infty)$, then we get

$$\int_0^\infty\left(\frac{1}{h(y)}\int_0^{h(y)}\left(g\circ h^{-1}\right)(t)\,dt\right)^p h'(y)\,dy \le\left(\frac{p}{p-1}\right)^p\int_0^\infty[g(y)]^p\,h'(y)\,dy. \tag{2.3}$$

Now, consider the change of variable $u=h^{-1}(t)$, $t\in[0,h(y)]$, then $t=h(u)$, $dt=h'(u)\,du$ and

$$\int_0^{h(y)}\left(g\circ h^{-1}\right)(t)\,dt=\int_0^y g(u)\,h'(u)\,du$$

and by (2.3) we get (2.1). □

We have the following weighted version of Hardy's inequality (1.1):

Corollary 1. *Let $w:[0,\infty)\to(0,\infty)$ be continuous and such that $\int_0^\infty w(s)\,ds=\infty$. If $g:[0,\infty)\to[0,\infty)$ and $p>1$ are such that $\int_0^\infty g^p(x)\,w(x)\,dx$ is convergent, then*

$$\int_0^\infty\left(\frac{1}{\int_0^x w(t)\,dt}\int_0^x g(t)\,w(t)\,dt\right)^p w(x)\,dx \le\left(\frac{p}{p-1}\right)^p\int_0^\infty g^p(x)\,w(x)\,dx. \tag{2.4}$$

The proof follows by Theorem 4 by taking $h(x):=\int_0^x w(t)\,dt$, $x\ge 0$.

The following result generalizing Theorem 2 also holds:

Theorem 5. *Let $h:[0,\infty)\to[0,\infty)$ be a continuous strictly increasing function that is differentiable on $(0,\infty)$ and such $\lim_{t\to\infty}h(t)=\infty$. If $g:[0,\infty)\to[0,\infty)$*

and $p \geq 1$, $\alpha < p-1$ *are such that* $\int_0^\infty g^p(x) h^\alpha(x) h'(x)\,dx$ *is convergent, then*

$$\int_0^\infty \left(\frac{1}{h(x)}\int_0^x g(t) h'(t)\,dt\right)^p h^\alpha(x) h'(x)\,dx \leq \left(\frac{p}{p-\alpha-1}\right)^p \int_0^\infty g^p(x) h^\alpha(x) h'(x)\,dx. \tag{2.5}$$

Moreover, if g *is a positive decreasing function, then also*

$$\frac{p}{p-\alpha-1}\int_0^\infty g^p(x) h^\alpha(x) h'(x)\,dx \leq \int_0^\infty \left(\frac{1}{h(x)}\int_0^x g(t) h'(t)\,dt\right)^p h^\alpha(x) h'(x)\,dx. \tag{2.6}$$

Proof. If we make the change of variable $x = h(y)$, $y \in [0,\infty)$, then $dx = h'(y)\,dy$ and by (1.2) we get

$$\int_0^\infty \left(\frac{1}{h(y)}\int_0^{h(y)} f(t)\,dt\right)^p h^\alpha(y) h'(y)\,dy \leq \left(\frac{p}{p-\alpha-1}\right)^p \int_0^\infty [f \circ h(y)]^p h^\alpha(y) h'(y)\,dy. \tag{2.7}$$

If we write the inequality (2.7) for $f := g \circ h^{-1} : [0,\infty) \to [0,\infty)$, then we get

$$\int_0^\infty \left(\frac{1}{h(y)}\int_0^{h(y)} \left(g \circ h^{-1}\right)(t)\,dt\right)^p h^\alpha(y) h'(y)\,dy \leq \left(\frac{p}{p-\alpha-1}\right)^p \int_0^\infty [g(y)]^p h^\alpha(y) h'(y)\,dy. \tag{2.8}$$

Now, consider the change of variable $u = h^{-1}(t)$, $t \in [0, h(y)]$, then $t = h(u)$, $dt = h'(u)\,du$ and

$$\int_0^{h(y)} \left(g \circ h^{-1}\right)(t)\,dt = \int_0^y g(u) h'(u)\,du$$

which together with (2.8) gives (2.5).

If g is a positive decreasing function, then $g \circ h^{-1}$ is also decreasing and by utilising inequality (1.3) we obtain (2.6). □

We have the following weighted inequalities as well:

Corollary 2. *Let* $w : [0,\infty) \to (0,\infty)$ *be continuous and such that* $\int_0^\infty w(s)\,ds = \infty$. *If* $g : [0,\infty) \to [0,\infty)$ *and* $p \geq 1$, $\alpha < p-1$ *are such that* $\int_0^\infty g^p(x) \left(\int_0^x w(s)\,ds\right)^\alpha w(x)\,dx$ *is convergent, then*

$$\int_0^\infty \left(\frac{1}{\int_0^x w(s)\,ds}\int_0^x g(t) w(t)\,dt\right)^p \left(\int_0^x w(s)\,ds\right)^\alpha w(x)\,dx \leq \left(\frac{p}{p-\alpha-1}\right)^p \int_0^\infty g^p(x) \left(\int_0^x w(s)\,ds\right)^\alpha w(x)\,dx. \tag{2.9}$$

Moreover, if g is a positive decreasing function, then also

$$\frac{p}{p-\alpha-1}\int_0^\infty g^p(x)\left(\int_0^x w(s)\,ds\right)^\alpha w(x)\,dx \leq \int_0^\infty\left(\frac{1}{\int_0^x w(s)\,ds}\int_0^x g(t)\,w(t)\,dt\right)^p\left(\int_0^x w(s)\,ds\right)^\alpha w(x)\,dx. \tag{2.10}$$

The proof follows by Theorem 5 by taking $h(x) := \int_0^x w(t)\,dt$, $x \geq 0$.

The following composite version of Knopp's inequality also holds.

Theorem 6. *Let $h : [0,\infty) \to [0,\infty)$ be a continuous strictly increasing function that is differentiable on $(0,\infty)$ and such $\lim_{t\to\infty} h(t) = \infty$. If $g : [0,\infty) \to (0,\infty)$ and $p > 1$ are such that $\int_0^\infty g(x)\,h'(x)\,dx$ is convergent, then*

$$\int_0^\infty \exp\left(\frac{1}{h(x)}\int_0^x h'(t)\ln g(t)\,dt\right)h'(x)\,dx \leq e\int_0^\infty g(x)\,h'(x)\,dx. \tag{2.11}$$

Proof. If we make the change of variable $x = h(y)$, $y \in [0,\infty)$, then $dx = h'(y)\,dy$ and by (1.4) we get

$$\int_0^\infty \exp\left(\frac{1}{h(y)}\int_0^{h(y)} \ln f(t)\,dt\right)h'(y)\,dy \leq e\int_0^\infty (f\circ h)(y)\,h'(y)\,dy. \tag{2.12}$$

If we write the inequality (2.12) for $f := g\circ h^{-1} : [0,\infty) \to (0,\infty)$, then we get

$$\int_0^\infty \exp\left(\frac{1}{h(y)}\int_0^{h(y)} \ln\left(g\circ h^{-1}\right)(t)\,dt\right)h'(y)\,dy \leq e\int_0^\infty g(y)\,h'(y)\,dy. \tag{2.13}$$

Now, consider the change of variable $u = h^{-1}(t)$, $t \in [0, h(y)]$, then $t = h(u)$, $dt = h'(u)\,du$ and by (2.13) we get (2.11). □

We have the following weighted version of Knopp's inequality:

Corollary 3. *Let $w : [0,\infty) \to (0,\infty)$ be continuous and such that $\int_0^\infty w(s)\,ds = \infty$. If $g : [0,\infty) \to (0,\infty)$ is such that $\int_0^\infty g(x)\,w(x)\,dx$ is convergent, then*

$$\int_0^\infty \exp\left(\frac{1}{\int_0^x w(s)\,ds}\int_0^x w(t)\ln g(t)\,dt\right)w(x)\,dx \leq e\int_0^\infty g(x)\,w(x)\,dx. \tag{2.14}$$

3. Some Examples

We can give some examples by utilising Theorem 4 as follows:

a). If we take $h : [0,\infty) \to [0,\infty)$, $h(t) = \ln(t+1)$, and $g : [0,\infty) \to [0,\infty)$ and $p > 1$ are such that $\int_0^\infty \frac{g^p(x)}{x+1}dx$ is convergent, then by (2.1) we have

$$\int_0^\infty\left(\frac{1}{\ln(x+1)}\int_0^x \frac{g(t)}{t+1}dt\right)^p \frac{1}{x+1}dx \leq \left(\frac{p}{p-1}\right)^p\int_0^\infty \frac{g^p(x)}{x+1}dx. \tag{3.1}$$

b). If we take $h : [0,\infty) \to [0,\infty)$, $h(t) = t^r$, $r > 0$, and $g : [0,\infty) \to [0,\infty)$ and $p > 1$ are such that $\int_0^\infty g^p(x)\,x^{r-1}dx$ is convergent, then by (2.1) we have

$$\int_0^\infty\left(\int_0^x g(t)\,t^{r-1}dt\right)^p x^{r-pr-1}dx \leq \left(\frac{p}{r(p-1)}\right)^p\int_0^\infty g^p(x)\,x^{r-1}dx. \tag{3.2}$$

c). If we take $h:[0,\infty)\to[0,\infty)$, $h(t)=\exp(\alpha t)-1$, $\alpha>0$ and $g:[0,\infty)\to[0,\infty)$ and $p>1$ are such that $\int_0^\infty g^p(x)\exp(\alpha x)\,dx$ is convergent, then by (2.1) we have

$$\int_0^\infty\left(\frac{1}{\exp(\alpha x)-1}\int_0^x g(t)\exp(\alpha t)\,dt\right)^p\exp(\alpha x)\,dx \leq\left(\frac{p}{\alpha(p-1)}\right)^p\int_0^\infty g^p(x)\exp(\alpha x)\,dx. \tag{3.3}$$

We have the following examples that follows by Theorem 5 as well:

d). If we take $h:[0,\infty)\to[0,\infty)$, $h(t)=t^r$, $r>0$, and $g:[0,\infty)\to[0,\infty)$ and $p\geq1$, $\alpha<p-1$ are such that $\int_0^\infty g^p(x)\,x^{\alpha r+r-1}dx$ is convergent, then by (2.5) we have

$$\int_0^\infty\left(\int_0^x g(t)\,t^{r-1}dt\right)^p x^{\alpha r+r-pr-1}dx \leq\left(\frac{p}{r(p-\alpha-1)}\right)^p\int_0^\infty g^p(x)\,x^{\alpha r+r-1}dx. \tag{3.4}$$

If g is a positive decreasing function, then also

$$\frac{p}{r^p(p-\alpha-1)}\int_0^\infty g^p(x)\,x^{\alpha r+r-1}dx \leq\int_0^\infty\left(\int_0^x g(t)\,t^{r-1}dt\right)^p x^{\alpha r+r-pr-1}dx. \tag{3.5}$$

e). If we take $h:[0,\infty)\to[0,\infty)$, $h(t)=\exp(\beta t)-1$, $\beta>0$ and $g:[0,\infty)\to[0,\infty)$ and $p\geq1$, $\alpha<p-1$ are such that $\int_0^\infty g^p(x)\left[\exp(\beta t)-1\right]^\alpha\exp(\beta t)\,dx$ is convergent, then

$$\int_0^\infty\left(\frac{1}{\exp(\beta t)-1}\int_0^x g(t)\exp(\beta t)\,dt\right)^p\left[\exp(\beta t)-1\right]^\alpha\exp(\beta t)\,dx \leq\left(\frac{p}{\beta(p-\alpha-1)}\right)^p\int_0^\infty g^p(x)\left[\exp(\beta t)-1\right]^\alpha\exp(\beta t)\,dx. \tag{3.6}$$

If g is a positive decreasing function, then also

$$\frac{p}{\beta^p(p-\alpha-1)}\int_0^\infty g^p(x)\left[\exp(\beta t)-1\right]^\alpha\exp(\beta t)\,dx \leq\int_0^\infty\left(\frac{1}{\exp(\beta t)-1}\int_0^x g(t)\exp(\beta t)\,dt\right)^p\left[\exp(\beta t)-1\right]^\alpha\exp(\beta t)\,dx. \tag{3.7}$$

From the Knopp's type inequality (2.11) we get the following particular inequalities as well:

f). If we take $h:[0,\infty)\to[0,\infty)$, $h(t)=t^r$, $r>0$, and $g:[0,\infty)\to(0,\infty)$ is such that $\int_0^\infty g(x)\,x^{r-1}dx$ is convergent, then

$$\int_0^\infty\exp\left(\frac{r}{x^r}\int_0^x t^{r-1}\ln g(t)\,dt\right)x^{r-1}dx\leq e\int_0^\infty g(x)\,x^{r-1}dx. \tag{3.8}$$

g). If we take $h:[0,\infty)\to[0,\infty)$, $h(t)=\exp(\beta t)-1$, $\beta>0$ and $g:[0,\infty)\to(0,\infty)$ with $\int_0^\infty g(x)\exp(\beta x)\,dx$ is convergent, then

$$\int_0^\infty \exp\left(\frac{\beta}{\exp(\beta x)-1}\int_0^x \exp(\beta t)\ln g(t)\,dt\right)\exp(\beta x)\,dx \leq e\int_0^\infty g(x)\exp(\beta x)\,dx. \tag{3.9}$$

References

[1] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge at the University Press, 1934.

[2] K. Knopp, Über Reihen mit positiven Gliedern, *J. Lond. Math. Soc.* **3** (1928), 205-211.

[3] A. Kufner, L. Maligranda and L.-E. Person, *The Hardy Inequality: About its History and Some Related Results,* Pilsen, 2007.

[4] E. Landau, A note on a terorem concerning series of positive terms, *J. Lond. Math. Soc.* **1** (1926), 38-39.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa