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SOME WEIGHTED INTEGRAL INEQUALITIES RELATED TO STEFFENSEN'S RESULT

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Abstract. In this paper we provide several weighted inequalities that can be obtained in a simple way from certain Steffensen type inequalities. Some applications for natural weights are also provided.

1. Introduction

In 1918, J. F. Steffensen [7] obtained the following inequality:

Theorem 1. *Suppose that g is integrable functions on $[a,b]$, f is nonincreasing on $[a,b]$ and $0 \leq g(t) \leq 1$ for all $t \in [a,b]$. Then*

$$\int_{b-\lambda}^{b} f(t)\,dt \leq \int_{a}^{b} f(t)\,g(t)\,dt \leq \int_{a}^{a+\lambda} f(t)\,dt, \tag{1.1}$$

where

$$\lambda = \int_{a}^{b} g(t)\,dt.$$

By using the substitution g/A for g in (1.1) with $A > 0$, one can get Hayashi's inequality [3]:

Theorem 2. *Suppose that g is integrable functions on $[a,b]$, f is nonincreasing on $[a,b]$ and $0 \leq g(t) \leq A$ for all $t \in [a,b]$, where $A > 0$. Then*

$$A\int_{b-\lambda}^{b} f(t)\,dt \leq \int_{a}^{b} f(t)\,g(t)\,dt \leq A\int_{a}^{a+\lambda} f(t)\,dt, \tag{1.2}$$

where

$$\lambda = \frac{1}{A}\int_{a}^{b} g(t)\,dt.$$

In 1982, J. Pečarić [4] (see also [6, p. 48]) established the following correction of an earlier inequality published by Bellman in [1]:

Theorem 3. *Let $f : [0,1] \to \mathbb{R}$ be a nonnegative and nonincreasing function and let $g : [0,1] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$ on $[0,1]$. If $p \geq 1$ and*

$$\lambda = \left(\int_{0}^{1} g(t)\,dt\right)^{p}, \tag{1.3}$$

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then

$$\left(\int_0^1 f(t) g(t) dt\right)^p \leq \int_0^\lambda f^p(t) dt. \tag{1.4}$$

Now, if $h : [a, b] \to \mathbb{R}$ is nonnegative and nonincreasing and $w : [a, b] \to \mathbb{R}$ is integrable and such that $0 \leq w \leq 1$ on $[a, b]$, then by putting $f(t) = h[(1-t)a + tb]$, $g(t) = w[(1-t)a + tb]$ and observing that, by the change of variable $x = (1-t)a + tb$, $t \in [0, 1]$, we have

$$\int_0^1 g(t) dt = \int_0^1 w[(1-t)a + tb] dt = \frac{1}{b-a}\int_a^b w(x) dx,$$

$$\int_0^1 f(t) g(t) dt = \frac{1}{b-a}\int_a^b h(x) w(x) dx$$

and

$$\int_0^\lambda f^p(t) dt = \frac{1}{b-a}\int_a^{(1-\lambda)a+\lambda b} f^p(x) dx = \frac{1}{b-a}\int_a^{a+(b-a)\lambda} f^p(x) dx.$$

Therefore, by Theorem 3 we have the following version of Pečarić's inequality for functions defined on $[a, b]$, see also [5]:

Corollary 1. *If $h : [a, b] \to \mathbb{R}$ is nonnegative and nonincreasing on $[a, b]$ and $w : [a, b] \to \mathbb{R}$ is integrable and such that $0 \leq w \leq 1$ on $[a, b]$, then for $p \geq 1$ and*

$$\nu := \frac{1}{(b-a)^{p-1}}\left(\int_a^b w(t) dt\right)^p \tag{1.5}$$

we have

$$\left(\int_a^b h(t) w(t) dt\right)^p \leq (b-a)^{p-1}\int_a^{a+\nu} h^p(t) dt. \tag{1.6}$$

In 1991, Cao [2] obtained another correction of Bellman's results as follows:

Theorem 4. *Let f be nonnegative and nonincreasing function on $[a, b]$ and $f \in L_p[a, b]$ for $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let g be measurable with $g \geq 0$ on $[a, b]$ and $\int_a^b g^q(t) dt \leq 1$. Then*

$$\left(\int_a^b f(t) g(t) dt\right)^p \leq \int_a^{a+\lambda} f^p(t) dt, \tag{1.7}$$

where

$$\lambda = \begin{cases} \left(\frac{\lim_{s\to a+} f(s)}{\lim_{u\to b-} f(u)}\right)^{p-1}\left(\int_a^b g(t) dt\right)^p & \text{if } \lim_{u\to b-} f(u) > 0, \\ b-a \text{ if } \lim_{u\to b-} f(u) = 0. & \end{cases}$$

In this paper we provide several weighted inequalities that can be obtained in a simple way from the Steffensen type inequalities stated above. Some applications for natural weights are also provided.

2. The Results

We have:

Theorem 5. *Let $h:[a,b]\to[h(a),h(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). Suppose that g is integrable function with $0\le g(t)\le A$ for all $t\in[a,b]$, where $A>0$ and f is nonincreasing on $[a,b]$. Then*

$$A\int_{h^{-1}[h(b)-\nu]}^{b} f(t)\,h'(t)\,dt \le \int_a^b f(t)\,g(t)\,h'(t)\,dt \le A\int_a^{h^{-1}[h(a)+\nu]} f(t)\,h'(t)\,dt, \tag{2.1}$$

where

$$\nu := \frac{1}{A}\int_a^b g(t)\,h'(t)\,dt.$$

Proof. The function $f\circ h^{-1}$ is nonincreasing on $[h(a),h(b)]$, $g\circ h^{-1}$ is integrable and $0\le\left(g\circ h^{-1}\right)(z)\le A$ for $z\in[h(a),h(b)]$. Define

$$\nu := \frac{1}{A}\int_{h(a)}^{h(b)}\left(g\circ h^{-1}\right)(z)\,dz.$$

Then by (1.2) we get

$$A\int_{h(b)-\nu}^{h(b)}\left(f\circ h^{-1}\right)(z)\,dz \le \int_{h(a)}^{h(b)}\left(f\circ h^{-1}\right)(z)\left(g\circ h^{-1}\right)(z)\,dz \le A\int_{h(a)}^{h(a)+\nu}\left(f\circ h^{-1}\right)(z)\,dz. \tag{2.2}$$

Consider the change of variable $t=h^{-1}(z)$, then $z=h(t)$, which gives $dz=h'(t)\,dt$.

Therefore

$$\nu = \frac{1}{A}\int_{h(a)}^{h(b)}\left(g\circ h^{-1}\right)(z)\,dz = \frac{1}{A}\int_a^b g(t)\,h'(t)\,dt,$$

$$\int_{h(b)-\nu}^{h(b)}\left(f\circ h^{-1}\right)(z)\,dz = \int_{h^{-1}[h(b)-\nu]}^{b} f(t)\,h'(t)\,dt,$$

$$\int_{h(a)}^{h(b)}\left(f\circ h^{-1}\right)(z)\left(g\circ h^{-1}\right)(z)\,dz = \int_a^b f(t)\,g(t)\,h'(t)\,dt$$

and

$$\int_{h(a)}^{h(a)+\nu}\left(f\circ h^{-1}\right)(z)\,dz = \int_a^{h^{-1}[h(a)+\nu]} f(t)\,h'(t)\,dt.$$

By utilising (2.2), we then get (2.1). □

If $w:[a,b]\to\mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W:[a,b]\to[0,\infty)$, $W(t):=\int_a^t w(s)\,ds$ is strictly increasing and differentiable on (a,b). We have $W'(t)=w(t)$ for any $t\in(a,b)$.

Corollary 2. *Assume that* $w:[a,b]\to\mathbb{R}$ *is continuous and positive on the interval* $[a,b]$. *Suppose that* g *is integrable function with* $0\le g(t)\le A$ *where* $A>0$ *for all* $t\in[a,b]$ *and* f *is nonincreasing on* $[a,b]$. *Then*

$$A\int_{b_i}^{b} f(t)\,w(t)\,dt\le\int_a^b f(t)\,g(t)\,w(t)\,dt\le A\int_a^{a_s} f(t)\,w(t)\,dt, \tag{2.3}$$

where

$$b_i:=W^{-1}\left[\frac{1}{A}\int_a^b (A-g(t))\,w(t)\,dt\right]$$

and

$$a_s:=W^{-1}\left[\frac{1}{A}\int_a^b g(t)\,w(t)\,dt\right].$$

We have the following composite version of Pečarić's inequality (1.6).

Theorem 6. *Let* $h:[a,b]\to[h(a),h(b)]$ *be a continuous strictly increasing function that is differentiable on* (a,b). *Suppose that* g *is integrable functions on* $[a,b]$ *with* $0\le g(t)\le 1$ *for all* $t\in[a,b]$ *and* f *is nonincreasing and nonnegative on* $[a,b]$. *Then for* $p\ge 1$ *we have*

$$\left(\int_a^b f(t)\,g(t)\,h'(t)\,dt\right)^p\le[h(b)-h(a)]^{p-1}\int_a^{h^{-1}(h(a)+\sigma)} f^p(t)\,h'(t)\,dt, \tag{2.4}$$

where

$$\sigma:=\frac{1}{[h(b)-h(a)]^{p-1}}\left(\int_a^b g(t)\,h'(t)\,dt\right)^p.$$

Proof. The function $f\circ h^{-1}$ is nonincreasing and nonnegative on $[h(a),h(b)]$, $g\circ h^{-1}$ is integrable and $0\le\left(g\circ h^{-1}\right)(z)\le 1$ for $z\in[h(a),h(b)]$. Define

$$\nu:=\frac{1}{[h(b)-h(a)]^{p-1}}\left(\int_{h(a)}^{h(b)} g\circ h^{-1}(z)\,dz\right)^p.$$

Then by (1.6) for $f\circ h^{-1}$ and $g\circ h^{-1}$ we have

$$\left(\int_{h(a)}^{h(b)} f\circ h^{-1}(z)\,g\circ h^{-1}(z)\,dz\right)^p \le[h(b)-h(a)]^{p-1}\int_{h(a)}^{h(a)+\nu}\left[f\circ h^{-1}(z)\right]^p dz. \tag{2.5}$$

Consider the change of variable $t=h^{-1}(z)$, then $z=h(t)$, which gives $dz=h'(t)\,dt$.

Therefore

$$\int_{h(a)}^{h(b)} g\circ h^{-1}(z)\,dz=\int_a^b g(t)\,h'(t)\,dt,$$

$$\int_{h(a)}^{h(b)} f\circ h^{-1}(z)\,g\circ h^{-1}(z)\,dz=\int_a^b f(t)\,g(t)\,h'(t)\,dt$$

and

$$\int_{h(a)}^{h(a)+\nu}\left[f\circ h^{-1}(z)\right]^p dz=\int_a^{h^{-1}(h(a)+\sigma)} f^p(t)\,h'(t)\,dt,$$

and by (2.5) we obtain the desired result (2.4). □

We have the following weighted version of Pečarić's inequality (1.6).

Corollary 3. *Assume that* $w : [a,b] \to \mathbb{R}$ *is continuous and positive on the interval* $[a,b]$. *Suppose that* g *is integrable function with* $0 \le g(t) \le 1$ *for all* $t \in [a,b]$ *and* f *is nonincreasing and nonnegative on* $[a,b]$. *Then for* $p \ge 1$ *we have*

$$\left(\int_a^b f(t)\,g(t)\,w(t)\,dt\right)^p \le \left(\int_a^b w(s)\,ds\right)^{p-1} \int_a^{W^{-1}(\eta)} f^p(t)\,w(t)\,dt, \tag{2.6}$$

where

$$\eta := \frac{1}{\left(\int_a^b w(s)\,ds\right)^{p-1}} \left(\int_a^b g(t)\,w(t)\,dt\right)^p$$

and $W : [a,b] \to [0,\infty)$, $W(t) := \int_a^t w(s)\,ds$.

We have:

Theorem 7. *Let* $h : [a,b] \to [h(a), h(b)]$ *be a continuous strictly increasing function that is differentiable on* (a,b). *Assume that* g *is measurable function on* $[a,b]$ *with* $0 \le g$ *and* f *is nonincreasing and nonnegative on* $[a,b]$. *For* $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *assume also that we have* $f^p h'$ *is integrable on* $[a,b]$ *and* $\int_a^b g^q(t)\,h'(t)\,dt \le 1$, *then*

$$\left(\int_a^b f(t)\,g(t)\,h'(t)\,dt\right)^p \le \int_a^{h^{-1}(h(a)+\sigma)} f^p(t)\,h'(t)\,dt, \tag{2.7}$$

where

$$\sigma = \begin{cases} \left(\frac{\lim_{s\to a+} f(s)}{\lim_{u\to b-} f(u)}\right)^{p-1} \left(\int_a^b g(t)\,h'(t)\,dt\right)^p & \text{if } \lim_{u\to b-} f(u) > 0, \\ \\ h(b) - h(a) \ \text{ if } \lim_{u\to b-} f(u) = 0. \end{cases}$$

The proof follows in a similar way as above by utilising Theorem 4. The details are omitted.

Corollary 4. *Let* $w : [a,b] \to \mathbb{R}$ *be continuous and positive on the interval* $[a,b]$. *Assume that* g *is measurable function on* $[a,b]$ *with* $0 \le g$ *and* f *is nonincreasing and nonnegative on* $[a,b]$. *For* $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *assume also that we have* $f^p w$ *is integrable on* $[a,b]$ *and* $\int_a^b g^q(t)\,w(t)\,dt \le 1$, *then*

$$\left(\int_a^b f(t)\,g(t)\,w(t)\,dt\right)^p \le \int_a^{W^{-1}(\nu)} f^p(t)\,w(t)\,dt, \tag{2.8}$$

where

$$\nu = \begin{cases} \left(\frac{\lim_{s\to a+} f(s)}{\lim_{u\to b-} f(u)}\right)^{p-1} \left(\int_a^b g(t)\,w(t)\,dt\right)^p & \text{if } \lim_{u\to b-} f(u) > 0, \\ \\ \int_a^b w(s)\,ds \ \text{ if } \lim_{u\to b-} f(u) = 0 \end{cases}$$

and $W : [a,b] \to [0,\infty)$, $W(t) := \int_a^t w(s)\,ds$.

3. Some Examples

By making use of Theorem 5, we can give the following examples for particular weights of interest.

a). If we take $h : [a,b] \subset (0,\infty) \to \mathbb{R}$, $h(t) = \ln t$, and assume that g is integrable functions on $[a,b]$ with $0 \le g(t) \le A$ for all $t \in [a,b]$, where $A > 0$, f is nonincreasing on $[a,b]$, then by (2.1) we get

$$A \int_{b\exp(-\sigma)}^{b} \frac{f(t)}{t} dt \le \int_a^b \frac{f(t)g(t)}{t} dt \le A \int_a^{a\exp(\sigma)} \frac{f(t)}{t} dt, \tag{3.1}$$

where

$$\sigma := \frac{1}{A} \int_a^b \frac{g(t)}{t} dt.$$

b). If we take $h : [a,b] \to \mathbb{R}$, $h(t) = \exp t$, and assume that g is integrable functions on $[a,b]$, f is nonincreasing on $[a,b]$ and $0 \le g(t) \le A$ for all $t \in [a,b]$, where $A > 0$, then by (2.1) we get

$$\begin{aligned} A \int_{\ln[\exp(b)-\eta]}^{b} f(t) \exp t dt &\le \int_a^b f(t) g(t) \exp dt \\ &\le A \int_a^{\ln[\exp(a)+\eta]} f(t) \exp t dt, \end{aligned} \tag{3.2}$$

where

$$\eta := \frac{1}{A} \int_a^b g(t) \exp t dt.$$

c). If we take $h : [a,b] \subset (0,\infty) \to \mathbb{R}$, $h(t) = t^r$, $r > 0$ and assume that g is integrable functions on $[a,b]$, f is nonincreasing on $[a,b]$ and $0 \le g(t) \le A$ for all $t \in [a,b]$, where $A > 0$, then by (2.1) we get

$$\begin{aligned} A \int_{(b^r-\vartheta)^{1/r}}^{b} f(t) t^{r-1} dt &\le \int_a^b f(t) g(t) t^{r-1} dt \\ &\le A \int_a^{(a^r+\vartheta)^{1/r}} f(t) t^{r-1} dt, \end{aligned} \tag{3.3}$$

where

$$\vartheta := \frac{r}{A} \int_a^b g(t) t^{r-1} dt.$$

By utilising Theorem 6 we also have the following particular inequalities of interest.

d). If we take $h : [a,b] \subset (0,\infty) \to \mathbb{R}$, $h(t) = \ln t$, and suppose that g is integrable functions on $[a,b]$ with $0 \le g(t) \le 1$ for all $t \in [a,b]$ and f is nonincreasing and nonnegative on $[a,b]$. Then for $p \ge 1$ we have

$$\left(\int_a^b \frac{f(t)g(t)}{t} dt \right)^p \le \left[\ln\left(\frac{b}{a}\right) \right]^{p-1} \int_a^{a\exp(\beta)} \frac{f^p(t)}{t} dt, \tag{3.4}$$

where

$$\beta := \frac{1}{\left[\ln\left(\frac{b}{a}\right)\right]^{p-1}} \left(\int_a^b \frac{g(t)}{t} dt \right)^p.$$

e). If we take $h:[a,b]\rightarrow\mathbb{R}$, $h(t)=\exp t$, and suppose that g is integrable functions on $[a,b]$ with $0\leq g(t)\leq 1$ for all $t\in[a,b]$ and f is nonincreasing and nonnegative on $[a,b]$. Then for $p\geq 1$ we have

$$\left(\int_a^b f(t)\,g(t)\exp t dt\right)^p\leq(\exp b-\exp a)^{p-1}\int_a^{\ln(\exp a+\vartheta)}f^p(t)\exp(t)\,dt, \tag{3.5}$$

where

$$\vartheta:=\frac{1}{(\exp b-\exp a)^{p-1}}\left(\int_a^b g(t)\exp(t)\,dt\right)^p.$$

f). If we take $h:[a,b]\subset(0,\infty)\rightarrow\mathbb{R}$, $h(t)=t^r$, $r>0$ and suppose that g is integrable functions on $[a,b]$ with $0\leq g(t)\leq 1$ for all $t\in[a,b]$ and f is nonincreasing and nonnegative on $[a,b]$. Then for $p\geq 1$ we have

$$\left(\int_a^b f(t)\,g(t)\,t^{r-1}dt\right)^p\leq\left(\frac{b^r-a^r}{r}\right)^{p-1}\int_a^{(a^r+\delta)^{1/r}}f^p(t)\,t^{r-1}dt, \tag{3.6}$$

where

$$\delta:=\frac{r}{(b^r-a^r)^{p-1}}\left(\int_a^b g(t)\,t^{r-1}dt\right)^p.$$

By making use of Theorem 7 we can also state the following particular inequalities:

g). Take $h:[a,b]\subset(0,\infty)\rightarrow\mathbb{R}$, $h(t)=\ln t$, and assume that g is measurable function on $[a,b]$ with $0\leq g$ and f is nonincreasing and nonnegative on $[a,b]$. For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $\ell(t)=t$, $t\in[a,b]$, assume also that we have $\frac{f^p}{\ell}$ is integrable on $[a,b]$ and $\int_a^b\frac{g^q(t)}{t}dt\leq 1$, then

$$\left(\int_a^b\frac{f(t)\,g(t)}{t}dt\right)^p\leq\int_a^{a\gamma}\frac{f^p(t)}{t}dt, \tag{3.7}$$

where

$$\gamma=\begin{cases}\left(\frac{\lim_{s\rightarrow a+}f(s)}{\lim_{u\rightarrow b-}f(u)}\right)^{p-1}\left(\int_a^b\frac{g(t)}{t}dt\right)^p & \text{if } \lim_{u\rightarrow b-}f(u)>0,\\ \ln\left(\frac{b}{a}\right) & \text{if } \lim_{u\rightarrow b-}f(u)=0.\end{cases}$$

h). Take $h:[a,b]\rightarrow\mathbb{R}$, $h(t)=\exp t$, assume that g is measurable function on $[a,b]$ with $0\leq g$ and f is nonincreasing and nonnegative on $[a,b]$. For $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ assume also that we have $f^p\exp$ is integrable on $[a,b]$ and $\int_a^b g^q(t)\exp t dt\leq 1$, then

$$\left(\int_a^b f(t)\,g(t)\exp t dt\right)^p\leq\int_a^{\ln(\exp(a)+\sigma)}f^p(t)\exp t dt, \tag{3.8}$$

where

$$\sigma=\begin{cases}\left(\frac{\lim_{s\rightarrow a+}f(s)}{\lim_{u\rightarrow b-}f(u)}\right)^{p-1}\left(\int_a^b g(t)\exp t dt\right)^p & \text{if } \lim_{u\rightarrow b-}f(u)>0,\\ \exp b-\exp a & \text{if } \lim_{u\rightarrow b-}f(u)=0.\end{cases}$$

k). Take $h:[a,b]\subset(0,\infty)\rightarrow\mathbb{R}$, $h(t)=t^r$, $r>0$, assume that g is measurable function on $[a,b]$ with $0\leq g$ and f is nonincreasing and nonnegative on $[a,b]$. For

$p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ assume also that we have $f^{p}\ell^{r-1}$ is integrable on $[a,b]$ and $\int_{a}^{b}g^{q}\left(t\right)t^{r-1}dt\leq 1/r$, then

$$\left(\int_{a}^{b}f\left(t\right)g\left(t\right)t^{r-1}dt\right)^{p}\leq r^{1-p}\int_{a}^{\left(a^{r}+\eta\right)^{1/r}}f^{p}\left(t\right)t^{r-1}dt, \tag{3.9}$$

where

$$\eta=\left\{\begin{array}{l} r^{p}\left(\frac{\lim_{s\rightarrow a+}f(s)}{\lim_{u\rightarrow b-}f(u)}\right)^{p-1}\left(\int_{a}^{b}g\left(t\right)r^{p-1}dt\right)^{p} \text{ if } \lim_{u\rightarrow b-}f\left(u\right)>0, \\ \\ b^{r}-a^{r} \text{ if } \lim_{u\rightarrow b-}f\left(u\right)=0. \end{array}\right.$$

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