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ON THE HERMITE-HADAMARD TYPE INEQUALITIES FOR n-TIMES DIFFERENTIABLE r-CONVEX FUNCTIONS

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ABSTRACT. In this paper, some new integral inequalities for n-times differentiable r-convex functions are obtained.

1. INTRODUCTION

The following inequality is well known in literature as Hermite-Hadamard inequality for convex functions:

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. Then the following double inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

For Hermite-Hadamard inequality, see [2]-[4], [6], [7] and [9]-[11] where further references are listed.

In [7], the power mean $M_r(x, y; \lambda)$ of order r of positive numbers x, y is defined as the following:

$$M_r(x,y;\lambda) = \begin{cases} (\lambda x^r + (1-\lambda)y^r), & if\ r \neq 0 \\ \left(x^{\lambda}y^{1-\lambda}\right), & if\ r = 0 \end{cases}. \tag{1.2}$$

In [3], Gill et. all used the definition of $M_r(x, y; \lambda)$ to introduce the concept of r-convex functions.

Definition 1. *A positive function f is $r-$convex on $[a, b]$ if for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$*

(1.3)

$$f(\lambda x + (1-\lambda)y) \leq M_r(f(x), f(y); \lambda) = \begin{cases} (\lambda f^r(x) + (1-\lambda)f^r(y))^{\frac{1}{r}}, & r \neq 0 \\ f^{\lambda}(x)f^{1-\lambda}(y), & r = 0 \end{cases}.$$

In the definition of r-convex functions if we choose $r = 1$ and $r = 0$, we have ordinary convex functions and log-convex functions respectively.

It is obvious that if f is r-convex in $[a, b]$ where $r > 0$, then f^r is convex on $[a, b]$.

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The generalized logarithmic means of order r of positive numbers x, y defined by

$$L_r(x,y)=\begin{cases}\frac{r}{r+1}\frac{x-y}{\ln x-\ln y}, & r\neq 0,-1; x\neq y\\ \frac{x-y}{\ln x-\ln y}, & r=0, x\neq y\\ xy\frac{x-y}{\ln x-\ln y}, & r=-1, x\neq y\\ x & x=y\end{cases}\tag{1.4}$$

,see [3].

Gill et. all proved the following theorem for r-convex functions:

Theorem 1. *Suppose f is a positive r-convex function on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(x)dx\leq L_r(f(a),f(b)).\tag{1.5}$$

If f is a positive r-concave function, then the inequality is reversed, see [3].

For several results concerning of r-convexity, see [3] and [5]-[11] where further references are listed.

The main aim of this paper is to obtain some new integral inequalities for r-convex functions by using Lemma 1.

Lemma 1. *Suppose $f: I\subseteq\mathbb{R}\to\mathbb{R}$ is a function such that $f^{(n)}$ exists on I° for $n\in\mathbb{N}$, $n\geq 1$. If $f^{(n)}$ is integrable on $[a,b]$, for $a,b\in I$ with $a<b$, the equality holds*

$$\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right]}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx=\frac{(-1)(b-a)^n}{n!}\int_0^1 K_n(t)f^{(n)}\left(ta+(1-t)b\right)dt,\tag{1.6}$$

where

$$K_n(t):=\begin{cases}t^n, & t\in\left[0,\frac{1}{2}\right]\\ (t-1)^n & t\in\left(\frac{1}{2},1\right],\end{cases}$$

see [4].

2. New Results for r-Convexity

Theorem 2. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ is a function such that $f^{(n)}$ exists on I° for $n\in\mathbb{N}$, $n\geq 1$ and $f^{(n)}\in L[a,b]$ where $a,b\in I$ with $a<b$. If $\left|f^{(n)}\right|$ is r-convex function on $[a,b]$ for $r>1$, then the inequality holds:*

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right]}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ &\frac{(b-a)^n}{n!}\left\{\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right\}\\ &\times\left(\frac{r}{2^{(nr+r+1)/r}(nr+r+1)}+\beta_{1/2}\left(n+1,\frac{1}{r}+1\right)\right),\end{aligned}$$

where

$$\beta_z(a,b)=\int_0^z u^{a-1}(1-u)^{b-1}du$$

is the incomplete Beta function which is a generalization of the complete Beta function.

Proof. From Lemma 1 and using properties of modulus, we can write

$$\begin{aligned} A &= \left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right]}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\le \frac{(b-a)^n}{n!}\left[\int_0^{\frac12}t^n\left|f^{(n)}(ta+(1-t)b)\right|dt+\int_{\frac12}^1(1-t)^n\left|f^{(n)}(ta+(1-t)b)\right|dt\right]. \end{aligned}$$

If we use r-convexity of $\left|f^{(n)}\right|$, then we have

$$\begin{aligned} A &\le \frac{(b-a)^n}{n!}\left[\int_0^{\frac12}t^n\left(t\left|f^{(n)}(a)\right|^r+(1-t)\left|f^{(n)}(b)\right|^r\right)^{1/r}dt\right. \\ &\quad \left.+\int_{\frac12}^1(1-t)^n\left(t\left|f^{(n)}(a)\right|^r+(1-t)\left|f^{(n)}(b)\right|^r\right)^{1/r}dt\right]. \end{aligned}$$

If we use
(2.1)

$$\sum_{i=1}^n(a_i+b_i)^k\le\sum_{i=1}^n a_i^k+\sum_{i=1}^n b_i^k \text{ for } 0<k<1; a_1,a_2,...,a_n\ge 0 \text{ and } b_1,b_2,...,b_n\ge 0,$$

we obtain

$$\begin{aligned} A &\le \frac{(b-a)^n}{n!}\left[\int_0^{\frac12}\left(t^{(nr+1)/r}\left|f^{(n)}(a)\right|+t^n(1-t)^{1/r}\left|f^{(n)}(b)\right|\right)dt\right. \\ &\quad \left.+\int_{\frac12}^1\left((1-t)^n t^{1/r}\left|f^{(n)}(a)\right|+(1-t)^{(nr+1)/r}\left|f^{(n)}(b)\right|\right)dt\right]. \end{aligned}$$

If we calculate the above integrals, then the proof is completed. □

Theorem 3. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is a function such that $f^{(n)}$ exists on I° for $n\in\mathbb{N}$, $n\ge 1$ and $f^{(n)}\in L[a,b]$ where $a,b\in I$ with $a<b$. If $\left|f^{(n)}\right|^q$ is r-convex function on $[a,b]$ for $r>1$ and $q>1$; $\frac1p+\frac1q=1$, then the inequality holds:*

$$\begin{aligned} &\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right]}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\le \frac{(b-a)^n}{n!}\left(\frac{1}{(np+1)2^{np+1}}\right)^{1/p}\left(\frac{r}{(r+1)2^{(r+1)/r}}\right)^{1/q} \\ &\quad\times\left\{\left[\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q(2^{(r+1)/r}-1)\right]^{1/q}\right. \\ &\quad\left.+\left[\left|f^{(n)}(a)\right|^q(2^{(r+1)/r}-1)+\left|f^{(n)}(b)\right|^q\right]^{1/q}\right\}. \end{aligned}$$

Proof. From Lemma 1 and using the properties of modulus and well known Hölder inequality, we can write

$$\begin{aligned} A &\leq \frac{(b-a)^n}{n!}\left[\left(\int_0^{\frac{1}{2}} t^{np}dt\right)^{1/p}\left(\int_0^{\frac{1}{2}}\left|f^{(n)}\left(ta+(1-t)b\right)\right|^q dt\right)^{1/q}\right. \\ &\left.+\left(\int_{\frac{1}{2}}^1 (1-t)^{np}dt\right)^{1/p}\left(\int_{\frac{1}{2}}^1\left|f^{(n)}\left(ta+(1-t)b\right)\right|^q dt\right)^{1/q}\right]. \end{aligned}$$

If we use r-convexity of $\left|f^{(n)}\right|^q$, then we have

$$\begin{aligned} A &\leq \frac{(b-a)^n}{n!}\left[\left(\int_0^{\frac{1}{2}} t^{np}dt\right)^{1/p}\left(\int_0^{\frac{1}{2}}\left[t\left|f^{(n)}(a)\right|^{qr}+(1-t)\left|f^{(n)}(b)\right|^{qr}\right]^{1/r} dt\right)^{1/q}\right. \\ &\left.+\left(\int_{\frac{1}{2}}^1 (1-t)^{np}dt\right)^{1/p}\left(\int_{\frac{1}{2}}^1\left[t\left|f^{(n)}(a)\right|^{qr}+(1-t)\left|f^{(n)}(b)\right|^{qr}\right]^{1/r} dt\right)^{1/q}\right]. \end{aligned}$$

Applying (2.1) in the above inequality we can write

$$\begin{aligned} A &\leq \frac{(b-a)^n}{n!}\left[\left(\int_0^{\frac{1}{2}} t^{np}dt\right)^{1/p}\left(\int_0^{\frac{1}{2}}\left[t^{1/r}\left|f^{(n)}(a)\right|^{q}+(1-t)^{1/r}\left|f^{(n)}(b)\right|^{q}\right] dt\right)^{1/q}\right. \\ &\left.+\left(\int_{\frac{1}{2}}^1 (1-t)^{np}dt\right)^{1/p}\left(\int_{\frac{1}{2}}^1\left[t^{1/r}\left|f^{(n)}(a)\right|^{q}+(1-t)^{1/r}\left|f^{(n)}(b)\right|^{q}\right] dt\right)^{1/q}\right]. \end{aligned}$$

If we calculate above integrals, then the proof is completed. □

Theorem 4. *Under the assumptions of Theorem 3 we have the following inequality*

$$\begin{aligned} &\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right]}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq \frac{(b-a)^n}{n!}\left(\frac{1}{2}\right)^{1/p} \\ &\times\left\{\left(\left|f^{(n)}(a)\right|^q\frac{r}{r+qnr+1}\left(\frac{1}{2}\right)^{(r+qnr+1)/r}\right.\right. \\ &\left.+\left|f^{(n)}(b)\right|^q\beta_{1/2}\left(qn+1,\frac{1}{r}+1\right)\right)^{1/q} \\ &+\left(\left|f^{(n)}(a)\right|^q\beta_{1/2}\left(qn+1,\frac{1}{r}+1\right)\right. \\ &\left.\left.+\left|f^{(n)}(b)\right|^q\frac{r}{r+qnr+1}\left(\frac{1}{2}\right)^{(r+qnr+1)/r}\right)^{1/q}\right\}. \end{aligned}$$

Proof. From Lemma 1 and using the properties of modulus and well known Hölder inequality, we can write

$$\begin{aligned} A &\leq \frac{(b-a)^n}{n!}\left[\left(\int_0^{\frac{1}{2}} dt\right)^{1/p}\left(\int_0^{\frac{1}{2}} t^{nq}\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{1/q}\right. \\ &\left. +\left(\int_{\frac{1}{2}}^1 dt\right)^{1/p}\left(\int_{\frac{1}{2}}^1 (1-t)^{nq}\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{1/q}\right]. \end{aligned}$$

If we use $\left|f^{(n)}\right|^q$ of r-convexity and applying (2.1) in the above inequality we can write

$$\begin{aligned} A &\leq \frac{(b-a)^n}{n!}\left[\left(\int_0^{\frac{1}{2}} dt\right)^{1/p}\left(\left|f^{(n)}(a)\right|^q\int_0^{\frac{1}{2}} t^{(nqr+1)/r}dt+\left|f^{(n)}(b)\right|^q\int_0^{\frac{1}{2}} t^{nq}(1-t)^{1/r}dt\right)^{1/q}\right. \\ &\left. +\left(\int_{\frac{1}{2}}^1 dt\right)^{1/p}\left(\left|f^{(n)}(a)\right|^q\int_{\frac{1}{2}}^1 (1-t)^{nq}t^{1/r}dt+\left|f^{(n)}(b)\right|^q\int_{\frac{1}{2}}^1 (1-t)^{(nqr+1)/r}dt\right)^{1/q}\right]. \end{aligned}$$

If we calculate above integrals, then the proof is completed. □

Theorem 5. *Under the assumptions of Theorem 3 we have the following inequality*

$$\begin{aligned} &\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right]}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq \frac{(b-a)^n}{n!}\left(\frac{1}{2^{n+1}(n+1)}\right)^{1/p} \\ &\times\left\{\left(\left|f^{(n)}(a)\right|^q\frac{r}{(r+nr+1)2^{(r+nr+1)/r}}\right.\right. \\ &\left.+\left|f^{(n)}(b)\right|^q\beta_{1/2}\left(n+1,\frac{1}{r}+1\right)\right)^{1/q} \\ &+\left(\left|f^{(n)}(a)\right|^q\beta_{1/2}\left(n+1,\frac{1}{r}+1\right)\right. \\ &\left.\left.+\left|f^{(n)}(b)\right|^q\frac{r}{(r+nr+1)2^{(r+nr+1)/r}}\right)^{1/q}\right\}. \end{aligned}$$

Proof. From Lemma 1 and using the properties of modulus and well known Hölder inequality, we can write

$$A \leq \frac{(b-a)^n}{n!}\left[\left(\int_0^{\frac{1}{2}} t^n dt\right)^{1/p}\left(\int_0^{\frac{1}{2}} t^n \left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{1/q} + \left(\int_{\frac{1}{2}}^1 (1-t)^n dt\right)^{1/p}\left(\int_{\frac{1}{2}}^1 (1-t)^n \left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{1/q}\right].$$

If we use $\left|f^{(n)}\right|^q$ of r-convexity and applying (2.1) in the above inequality we can write

$$A \leq \frac{(b-a)^n}{n!}\left[\left(\int_0^{\frac{1}{2}} t^n dt\right)^{1/p}\left(\int_0^{\frac{1}{2}} t^n \left(t^{1/r}\left|f^{(n)}(a)\right|^q + (1-t)^{1/r}\left|f^{(n)}(b)\right|^q\right) dt\right)^{1/q} + \left(\int_{\frac{1}{2}}^1 (1-t)^n dt\right)^{1/p}\left(\int_{\frac{1}{2}}^1 (1-t)^n \left(t^{1/r}\left|f^{(n)}(a)\right|^q + (1-t)^{1/r}\left|f^{(n)}(b)\right|^q\right) dt\right)^{1/q}\right].$$

If we calculate above integrals, then the proof is completed. □

Theorem 6. *Under the assumptions of Theorem 3 we have the following inequality*

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right]}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^n}{n!}\left(\frac{1}{2^{(n-1)p+2}\left[(n-1)p+2\right]}\right)^{1/p}$$
$$\times\left\{\left(\left|f^{(n)}(a)\right|^q \frac{r}{2^{(2r+1)/r}(2r+1)} + \left|f^{(n)}(b)\right|^q \frac{r}{(r+1)(2r+1)}\left[\frac{2^{(2r+1)/r}r-3r-1}{2^{(2r+1)/r}}\right]\right)^{1/q} + \left(\left|f^{(n)}(a)\right|^q \frac{r}{(r+1)(2r+1)}\left[\frac{2^{(2r+1)/r}r-3r-1}{2^{(2r+1)/r}}\right] + \left|f^{(n)}(b)\right|^q \frac{r}{2^{(2r+1)/r}(2r+1)}\right)^{1/q}\right\}.$$

Proof. From Lemma 1 and using the properties of modulus and well known Hölder inequality, we can write

$$A \leq \frac{(b-a)^n}{n!}\left[\left(\int_0^{\frac{1}{2}} t^{(n-1)p} t dt\right)^{1/p}\left(\int_0^{\frac{1}{2}} t\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{1/q} + \left(\int_{\frac{1}{2}}^1 (1-t)^{(n-1)p}(1-t)dt\right)^{1/p}\left(\int_{\frac{1}{2}}^1 (1-t)\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{1/q}\right].$$

If we use $\left|f^{(n)}\right|^q$ of r-convexity and applying (2.1) in the above inequality we can write

$$A \leq \frac{(b-a)^n}{n!}\left[\left(\int_0^{\frac{1}{2}} t^{(n-1)p} t dt\right)^{1/p}\left(\int_0^{\frac{1}{2}} t\left(t^{1/r}\left|f^{(n)}(a)\right|^q+(1-t)^{1/r}\left|f^{(n)}(b)\right|^q\right) dt\right)^{1/q}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1 (1-t)^{(n-1)p}(1-t) dt\right)^{1/p}\left(\int_{\frac{1}{2}}^1 (1-t)\left(t^{1/r}\left|f^{(n)}(a)\right|^q+(1-t)^{1/r}\left|f^{(n)}(b)\right|^q\right) dt\right)^{1/q}\right].$$

If we calculate above integrals, then the proof is completed. □

Theorem 7. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a function such that $f^{(n)}$ exists on I° for $n \in \mathbb{N}$, $n \geq 1$ such that $f^{(n)} \in L[a,b]$ where $a, b \in I$ with $a < b$. If $\left|f^{(n)}\right|^q$ is r-convex function on $[a,b]$ for $q > 1$; $\frac{1}{p}+\frac{1}{q}=1$, then the inequality holds:*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k+1\right]}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a} \int_a^b f(x) dx\right|$$
$$\leq \frac{(b-a)^n}{2(n)!}\left(\frac{1}{2^n(n+1)}\right)^{1/p}$$
$$\left\{\left[L_r\left(\left|f^{(n)}(b)\right|^q,\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right)\right]^{1/q}\right.$$
$$\left.+\left[L_r\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right)\right]^{1/q}\right\}.$$

Proof. From Lemma 1 and using the properties of modulus and well known Hölder inequality, we can write

$$A \leq \frac{(b-a)^n}{n!}\left[\left(\int_0^{\frac{1}{2}} t^{np} dt\right)^{1/p}\left(\int_0^{\frac{1}{2}}\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{1/q}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1 (1-t)^{np} dt\right)^{1/p}\left(\int_{\frac{1}{2}}^1\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{1/q}\right].$$

If we use r-convextity of $\left|f^{(n)}\right|^q$; we can write the following inequalities

$$\int_0^{1/2}\left|f^{(n)}(ta+(1-t)b)\right|^q dt \leq \frac{1}{2} L_r\left(\left|f^{(n)}(b)\right|^q,\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right)$$

and

$$\int_{1/2}^1\left|f^{(n)}(ta+(1-t)b)\right|^q dt \leq \frac{1}{2} L_r\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right)$$

via (1.5). The proof is completed. □

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