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SOME INTEGRAL INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. In this paper we show amongst other that, if $f:[a,b]\to\mathbb{R}$ is continuous convex with $f(a)=0$ and $f'_+(a)$ is finite, then

$$\frac{1}{2}\int_a^b \frac{f(t)}{t-a}dt \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \int_{\frac{a+b}{2}}^b \frac{f(t)\,dt}{t-a}.$$

Other related results are also provided. An example for logarithmic function is also given.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$h\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b h(x)dx \le \frac{h(a)+h(b)}{2}, \quad a,\ b\in\mathbb{R},\ a<b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [6]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [1]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [6]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the *Hermite-Hadamard inequality.* For a monograph devoted to this result see [2]. The recent survey paper [5] provides other related results.

Let $h:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ and assume that $h'_+(a)$ and $h'_-(b)$ are finite. We recall the following reverse inequality for the first Hermite-Hadamard result that has been established in [3]

$$0 \le \frac{1}{b-a}\int_a^b h(u)\,du - h\left(\frac{a+b}{2}\right) \le \frac{1}{8}(b-a)\left[h'_-(b)-h'_+(a)\right]. \tag{1.2}$$

The following inequality that provides a reverse of the second Hermite-Hadamard result has been obtained in [4]

$$0 \le \frac{h(a)+h(b)}{2} - \frac{1}{b-a}\int_a^b h(u)\,du \le \frac{1}{8}(b-a)\left[h'_-(b)-h'_+(a)\right]. \tag{1.3}$$

The constant $\frac{1}{8}$ is best possible in both (3.3) and (3.4).

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By making use of the above inequalities for convex functions, in this paper we establish certain inequalities involving the quantities

$$\frac{1}{2}\int_a^b \frac{h(t)}{t-a}dt,\ \frac{1}{b-a}\int_a^b h(t)\,dt \text{ and } \int_{\frac{a+b}{2}}^b \frac{h(t)\,dt}{t-a}$$

for continuous convex functions $h:[a,b]\to\mathbb{R}$ that satisfy the condition $h(a)=0$.

2. Some Preliminary Facts

We have:

Lemma 1. *Let $f:(a,b]\to\mathbb{R}$ be a measurable function and such that the improper integral $\int_a^b \frac{f(t)dt}{t-a}$ exists and the lateral limit $L:=\lim_{t\to a+} f(t)$ exists and is finite, then $L=0$. If the improper integral $\int_a^b \frac{f(t)dt}{t-a}$ exists, then $\lim_{t\to a+} f(t)$ can be neither ∞ nor $-\infty$.*

Proof. Assume that $L>0$. Then for any $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that for any $t\in(a,a+\delta(\varepsilon)]$ we have $|f(t)-L|<\varepsilon$ that is equivalent to

$$L-\varepsilon<f(t)<\varepsilon+L. \tag{2.1}$$

Take $0<\varepsilon<L$ and $0<\eta<\delta(\varepsilon)$. By the first inequality in (2.1) we get for $t\in[a+\eta,a+\delta(\varepsilon)]$ that

$$0<\frac{L-\varepsilon}{t-a}<\frac{f(t)}{t-a}.$$

By taking the integral on $[a+\eta,a+\delta(\varepsilon)]$ we get

$$0<(L-\varepsilon)\int_{a+\eta}^{a+\delta(\varepsilon)}\frac{1}{t-a}<\int_{a+\eta}^{a+\delta(\varepsilon)}\frac{f(t)}{t-a}dt,$$

which is equivalent to

$$0<(L-\varepsilon)[\ln\delta(\varepsilon)-\ln\eta]<\int_{a+\eta}^{a+\delta(\varepsilon)}\frac{f(t)}{t-a}dt. \tag{2.2}$$

By taking the limit over $\eta\to0+$ in (2.2) we get that

$$\infty\le\int_a^{a+\delta(\varepsilon)}\frac{f(t)}{t-a}dt,$$

which contradicts the fact that the improper integral $\int_a^b \frac{f(t)dt}{t-a}$ exists.

Also, assume that $L<0$. Take $0<\varepsilon<-L$ and $0<\eta<\delta(\varepsilon)$. Then by the second inequality we have

$$\int_{a+\eta}^{a+\delta(\varepsilon)}\frac{f(t)}{t-a}dt<(\varepsilon+L)\int_{a+\eta}^{a+\delta(\varepsilon)}\frac{1}{t-a}<0,$$

which is equivalent to

$$\int_{a+\eta}^{a+\delta(\varepsilon)}\frac{f(t)}{t-a}dt<(\varepsilon+L)[\ln\delta(\varepsilon)-\ln\eta]<0. \tag{2.3}$$

By taking the limit over $\eta\to0+$ in (2.3) we get that

$$\int_a^{a+\delta(\varepsilon)}\frac{f(t)}{t-a}dt\le-\infty$$

which contradicts the fact that the improper integral $\int_a^b \frac{f(t)dt}{t-a}$ exists.

Now, assume that $\lim_{t\to a+} f(t) = \infty$. This means that for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for any $t \in (a, a+\delta(\varepsilon)]$ we have $f(t) \geq \varepsilon$. Take $0 < \eta < \delta(\varepsilon)$. Then for $t \in [a+\eta, a+\delta(\varepsilon)]$ we have

$$\frac{f(t)}{t-a} \geq \frac{\varepsilon}{t-a}$$

and by taking the integral, we have

$$\int_{a+\eta}^{a+\delta(\varepsilon)} \frac{f(t)\,dt}{t-a} \geq \varepsilon \int_{a+\eta}^{a+\delta(\varepsilon)} \frac{dt}{t-a} = \varepsilon\left[\ln\delta(\varepsilon) - \ln\eta\right]. \tag{2.4}$$

By taking the limit over $\eta \to 0+$ in (2.4), we get that

$$\int_a^{a+\delta(\varepsilon)} \frac{f(t)\,dt}{t-a} \geq \infty$$

which contradicts the fact that the improper integral $\int_a^b \frac{f(t)dt}{t-a}$ exists.

The case $\lim_{t\to a+} f(t) = -\infty$ can be proved in the same way and the details are omitted. □

Lemma 2. *Let $f : (a,b] \to \mathbb{R}$ be an integrable function and such that the improper integral $\int_a^b \frac{f(t)dt}{t-a}$ exists and the lateral limit $L := \lim_{t\to a+} f(t)$ exists and is finite, then*

$$\begin{aligned}\int_a^b \frac{\int_a^t f(s)\,ds}{(t-a)^2}dt &= \int_a^b \frac{f(t)\,dt}{t-a} - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^b \left(\frac{b-t}{t-a}\right) f(t)\,dt.\end{aligned} \tag{2.5}$$

Proof. Let $\varepsilon > 0$ and such that $a+\varepsilon < b$. Using the integration by parts formula we have

$$\begin{aligned}\int_{a+\varepsilon}^b \frac{\int_a^t f(s)\,ds}{(t-a)^2}dt &= -\int_{a+\varepsilon}^b \left(\int_a^t f(s)\,ds\right) d\left(\frac{1}{t-a}\right) \\ &= -\left[\left(\int_a^t f(s)\,ds\right)\frac{1}{t-a}\bigg|_{a+\varepsilon}^b - \int_{a+\varepsilon}^b \frac{f(t)}{t-a}dt\right] \\ &= \int_{a+\varepsilon}^b \frac{f(t)}{t-a}dt - \frac{1}{b-a}\int_a^b f(s)\,ds + \frac{1}{\varepsilon}\int_a^{a+\varepsilon} f(s)\,ds.\end{aligned} \tag{2.6}$$

Using l'Hôpital's Theorem, we have by Lemma 1 that

$$\lim_{\varepsilon\to 0+}\left(\frac{1}{\varepsilon}\int_a^{a+\varepsilon} f(s)\,ds\right) = \lim_{\varepsilon\to 0+} f(a+\varepsilon) = \lim_{x\to a+} f(x) = 0.$$

By taking the limit over $\varepsilon \to 0+$ in (2.6), we get the first equality in (2.5).

We also have

$$\begin{aligned}&\int_a^b \frac{f(t)\,dt}{t-a} - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \int_a^b \left[\frac{1}{t-a} - \frac{1}{b-a}\right] f(t)\,dt = \frac{1}{b-a}\int_a^b \left(\frac{b-t}{t-a}\right) f(t)\,dt\end{aligned}$$

that proves the second part of (2.5). □

Remark 1. *We observe that if* $f:[a,b]\to\mathbb{R}$ *is continuous and* $\int_a^b \frac{f(t)dt}{t-a}$ *exists, then* $f(a)=0$. *If* $g:[a,b]\to\mathbb{R}$ *is continuous and if we put* $f(t)=g(t)-g(a)$, $t\in[a,b]$, *then* $f(a)=0$ *if we assume that* $\int_a^b \frac{g(t)-g(a)}{t-a}dt$ *exists, then by (2.6) we get*

$$\int_a^b \frac{\int_a^t g(s)\,ds-g(a)(t-a)}{(t-a)^2}dt \tag{2.7}$$

$$=\int_a^b \frac{g(t)-g(a)}{t-a}dt-\frac{1}{b-a}\int_a^b g(t)\,dt-g(a)=\frac{1}{b-a}\int_a^b (b-t)\left[\frac{g(t)-g(a)}{t-a}\right]dt.$$

3. Inequalities for Convex Functions

We have:

Theorem 1. *Let* $f:[a,b]\to\mathbb{R}$ *be continuous convex with* $f(a)=0$ *and* $f'_+(a)$ *is finite, then*

$$\frac{1}{2}f'_+(a)(b-a)\le\frac{1}{2}\int_a^b\frac{f(t)}{t-a}dt\le\frac{1}{b-a}\int_a^b f(t)\,dt\le\int_{\frac{a+b}{2}}^b\frac{f(t)\,dt}{t-a}. \tag{3.1}$$

The constants $\frac{1}{2}$ *in front of the left integral and* 1 *in front of the right integral are best possible.*

Proof. By the gradient inequality, we have

$$f'_+(a)(t-a)\le f(t)-f(a)=f(t),\ t\in(a,b],$$

which implies that $f'_+(a)\le\frac{f(t)}{t-a}$, $t\in(a,b]$, giving that

$$f'_+(a)(b-a)\le\int_a^b\frac{f(t)\,dt}{t-a}$$

that shows that the improper integral $\int_a^b\frac{f(t)dt}{t-a}$ is finite.

If we use Hermite-Hadamard inequality for f we have

$$f\left(\frac{a+t}{2}\right)\le\frac{1}{t-a}\int_a^t f(t)\,dt\le\frac{1}{2}f(t)$$

for all $t\in(a,b]$.

If we multiply with $\frac{1}{t-a}$ and integrate to get

$$\int_a^b\frac{1}{t-a}f\left(\frac{a+t}{2}\right)dt\le\int_a^b\frac{1}{(t-a)^2}\left(\int_a^t f(s)\,ds\right)dt\le\frac{1}{2}\int_a^b\frac{f(t)}{t-a}dt. \tag{3.2}$$

Using the first equality in (2.5), we get

$$\int_a^b\frac{1}{t-a}f\left(\frac{a+t}{2}\right)dt\le\int_a^b\frac{f(t)\,dt}{t-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\le\frac{1}{2}\int_a^b\frac{f(t)}{t-a}dt. \tag{3.3}$$

Using the change of variable $y=\frac{a+t}{2}$, then $dt=2dy$, $t-a=2(y-a)$ and

$$\int_a^b\frac{1}{t-a}f\left(\frac{a+t}{2}\right)dt=\int_a^{\frac{a+b}{2}}\frac{f(y)}{y-a}dy$$

and from (3.3) we get

$$\int_a^{\frac{a+b}{2}} \frac{f(t)}{t-a}dt \le \int_a^b \frac{f(t)\,dt}{t-a} - \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{1}{2}\int_a^b \frac{f(t)}{t-a}dt. \tag{3.4}$$

From the first inequality in (3.4) we get

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \int_a^b \frac{f(t)\,dt}{t-a} - \int_a^{\frac{a+b}{2}} \frac{f(t)}{t-a}dt = \int_{\frac{a+b}{2}}^b \frac{f(t)\,dt}{t-a}$$

that proves the second inequality in (3.1).

From the second inequality in (3.4) we get the first part of (3.1).

Now, assume that there exist C, $D > 0$ with

$$C\int_a^b \frac{f(t)}{t-a}dt \le \frac{1}{b-a}\int_a^b f(t)\,dt \le D\int_{\frac{a+b}{2}}^b \frac{f(t)\,dt}{t-a}. \tag{3.5}$$

Consider the convex function $f(t) = (t-a)^{\alpha+1}$ with $\alpha > 0$. Then by (3.5) we have

$$C\int_a^b (t-a)^{\alpha}\,dt \le \frac{1}{b-a}\int_a^b (t-a)^{\alpha+1}\,dt \le D\int_{\frac{a+b}{2}}^b (t-a)^{\alpha}\,dt,$$

namely

$$C\frac{1}{\alpha+1} \le \frac{1}{\alpha+2} \le D\frac{1-\left(\frac{1}{2}\right)^{\alpha+1}}{\alpha+1}.$$

By taking $\alpha \to 0+$ in this inequality we get

$$C \le \frac{1}{2} \le \frac{1}{2}D,$$

which shows that the constants $\frac{1}{2}$ in front of the left integral and 1 in front of the right integral are best possible. □

Corollary 1. *Let $g : [a,b] \to \mathbb{R}$ be continuous convex on $[a,b]$ and $g'_+(a)$ is finite, then*

$$\begin{aligned} \frac{1}{2}(b-a)\,g'_+(a) + g(a) &\le \frac{1}{2}\int_a^b \frac{g(t)-g(a)}{t-a}dt + g(a) \\ &\le \frac{1}{b-a}\int_a^b g(t)\,dt \le \int_{\frac{a+b}{2}}^b \frac{g(t)}{t-a}dt + \ln\left(\frac{e}{2}\right)g(a). \end{aligned} \tag{3.6}$$

Proof. If we write the inequality (3.1) for $f(t) = g(t) - g(a)$, then we get

$$\frac{1}{2}\int_a^b \frac{g(t)-g(a)}{t-a}dt \le \frac{1}{b-a}\int_a^b [g(t)-g(a)]\,dt \le \int_{\frac{a+b}{2}}^b \frac{g(t)-g(a)}{t-a}dt. \tag{3.7}$$

Since

$$\begin{aligned} \int_{\frac{a+b}{2}}^b \frac{g(t)-g(a)}{t-a}dt &= \int_{\frac{a+b}{2}}^b \frac{g(t)}{t-a}dt - g(a)\int_{\frac{a+b}{2}}^b \frac{dt}{t-a} \\ &= \int_{\frac{a+b}{2}}^b \frac{g(t)}{t-a}dt - g(a)\left[\ln(b-a) - \ln\left(\frac{b-a}{2}\right)\right] \\ &= \int_{\frac{a+b}{2}}^b \frac{g(t)}{t-a}dt - g(a)\ln 2, \end{aligned}$$

hence we obtain by (3.7) the desired inequality (3.6). □

Now, if we replace $g(t)$ by $h(a+b-t)$, then by (3.6) we have

$$\frac{1}{2}\int_a^b \frac{h(a+b-t)-h(b)}{t-a}dt + h(b) \leq \frac{1}{b-a}\int_a^b h(a+b-t)\,dt \leq \int_{\frac{a+b}{2}}^b \frac{h(a+b-t)}{t-a}dt + \ln\left(\frac{e}{2}\right)h(b). \tag{3.8}$$

By using the change of variable $u=a+b-t$, $t\in[a,b]$ we have $dt=-du$,

$$\int_a^b \frac{h(a+b-t)-h(b)}{t-a}dt = -\int_b^a \frac{h(u)-h(b)}{b-u}du = -\int_a^b \frac{h(b)-h(u)}{b-u}du,$$

$$\int_a^b h(a+b-t)\,dt = \int_a^b h(u)\,du$$

and

$$\int_{\frac{a+b}{2}}^b \frac{h(a+b-t)}{t-a}dt = -\int_{\frac{a+b}{2}}^a \frac{h(u)}{b-u}du = \int_a^{\frac{a+b}{2}} \frac{h(u)}{b-u}du.$$

Therefore, we can state the following result as well:

Corollary 2. *Let $g:[a,b]\to\mathbb{R}$ be continuous convex on $[a,b]$ with $g'_-(b)$ is finite, then*

$$g(b)-\frac{1}{2}(b-a)g'_-(b) \leq g(b)-\frac{1}{2}\int_a^b \frac{g(b)-g(t)}{b-t}dt \leq \frac{1}{b-a}\int_a^b g(t)\,dt \leq \int_a^{\frac{a+b}{2}} \frac{g(t)}{b-t}dt + \ln\left(\frac{e}{2}\right)g(b). \tag{3.9}$$

We also have:

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be continuous convex with $f(a)=0$ and $f'_+(a)$ is finite, then*

$$0 \leq \int_{\frac{a+b}{2}}^b \frac{f(t)\,dt}{t-a} - \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{1}{8}\left[f(b)-(b-a)f'_+(a)\right] \tag{3.10}$$

and

$$0 \leq \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{2}\int_a^b \frac{f(t)\,dt}{t-a} \leq \frac{1}{8}\left[f(b)-(b-a)f'_+(a)\right]. \tag{3.11}$$

Proof. From (1.2) we have

$$0 \leq \frac{1}{t-a}\int_a^t f(u)\,du - f\left(\frac{a+t}{2}\right) \leq \frac{1}{8}(t-a)\left[f'_-(t)-f'_+(a)\right]$$

for $t\in(a,b]$.

Divide by $t-a$ and integrate on $[a,b]$ to get

$$0 \leq \int_a^b \frac{1}{(t-a)^2}\left(\int_a^t f(u)\,du\right)dt - \int_a^b \frac{1}{t-a}f\left(\frac{a+t}{2}\right)dt \leq \frac{1}{8}\int_a^b \left[f'_-(t)-f'_+(a)\right]dt. \tag{3.12}$$

Since, as above

$$\int_a^b \frac{1}{(t-a)^2}\left(\int_a^t f(u)\,du\right)dt = \int_a^b \frac{f(t)\,dt}{t-a} - \frac{1}{b-a}\int_a^b f(t)\,dt,$$

$$\int_a^b \frac{1}{t-a} f\left(\frac{a+t}{2}\right)dt = \int_a^{\frac{a+b}{2}} \frac{f(t)}{t-a}dt$$

and

$$\int_a^b \left[f'_-(t) - f'_+(a)\right]dt = f(b) - (b-a)\,f'_+(a)$$

hence by (3.12) we get (3.10).

From (1.3) we get

$$0 \leq \frac{1}{2} f(t) - \frac{1}{t-a}\int_a^t f(u)\,du \leq \frac{1}{8}(t-a)\left[f'_-(t) - f'_+(a)\right].$$

for $t \in (a,b]$.

Divide by $t-a$ and integrate on $[a,b]$ to get

$$0 \leq \frac{1}{2}\int_a^b \frac{f(t)\,dt}{t-a} - \int_a^b \frac{1}{(t-a)^2}\left(\int_a^t f(u)\,du\right)dt \leq \frac{1}{8}\int_a^b \left[f'_-(t) - f'_+(a)\right]dt,$$

which gives (3.11). □

Corollary 3. *Let $g : [a,b] \to \mathbb{R}$ be continuous convex on $[a,b]$ and $g'_+(a)$ is finite, then*

$$\begin{aligned} 0 &\leq \frac{1}{b-a}\int_a^b g(t)\,dt - \frac{1}{2}\int_a^b \frac{g(t)-g(a)}{t-a}dt - g(a) \\ &\leq \frac{1}{8}\left[g(b) - g(a) - (b-a)\,g'_+(a)\right] \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} 0 &\leq \int_{\frac{a+b}{2}}^b \frac{g(t)}{t-a}dt + \ln\left(\frac{e}{2}\right) g(a) - \frac{1}{b-a}\int_a^b g(t)\,dt \\ &\leq \frac{1}{8}\left[g(b) - g(a) - (b-a)\,g'_+(a)\right]. \end{aligned} \tag{3.14}$$

4. An Example

Consider $[a,b] \subset (0,\infty)$ and take the convex function $g(t) = \frac{1}{t}$. Then

$$\begin{aligned} \int_a^b \frac{g(t)-g(a)}{t-a}dt &= \int_a^b \frac{\frac{1}{t}-\frac{1}{a}}{t-a}dt = \int_a^b \frac{a-t}{ta(t-a)}dt \\ &= -\frac{1}{a}\int_a^b \frac{dt}{t} = -\frac{1}{a}\ln\left(\frac{b}{a}\right), \end{aligned}$$

$$\frac{1}{b-a}\int_a^b g(t)\,dt = \frac{1}{b-a}\int_a^b \frac{1}{t}dt = \frac{\ln b - \ln a}{b-a}$$

and

$$\int_{\frac{a+b}{2}}^{b} \frac{g(t)}{t-a} dt = \int_{\frac{a+b}{2}}^{b} \frac{1}{t(t-a)} dt = \frac{1}{a} \int_{\frac{a+b}{2}}^{b} \left(\frac{1}{t-a} - \frac{1}{t} \right) dt$$
$$= \frac{1}{a} \left(\ln 2 - \ln b + \ln \left(\frac{a+b}{2} \right) \right) = \frac{1}{a} \ln \left(\frac{a+b}{b} \right).$$

Then by using (3.9) we get

$$\frac{1}{a} \left(1 - \frac{1}{2} \ln \left(\frac{b}{a} \right) \right) \leq \frac{\ln b - \ln a}{b-a} \leq \frac{1}{a} \ln \left(\frac{e(a+b)}{2b} \right), \tag{4.1}$$

for any $0 < a < b$.

By using (3.13) and (3.14) we also have

$$0 \leq \frac{\ln b - \ln a}{b-a} - \frac{1}{a} \left(1 - \frac{1}{2} \ln \left(\frac{b}{a} \right) \right) \leq \frac{1}{8a} (b-a)^2 \tag{4.2}$$

and

$$0 \leq \frac{1}{a} \ln \left(\frac{e(a+b)}{2b} \right) - \frac{\ln b - \ln a}{b-a} \leq \frac{1}{8a} (b-a)^2 \tag{4.3}$$

for any $0 < a < b$.

References

[1] E. F. Beckenbach, Convex functions, *Bull. Amer. Math. Soc.* **54** (1948), 439–460.

[2] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, 2000. [Online http://rgmia.org/monographs/hermite_hadamard.html].

[3] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.* **3** (2002), No. 2, Article 31. [Online https://www.emis.de/journals/JIPAM/article183.html?sid=183].

[4] S. S. Dragomir, An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.* **3** (2002), No. 3, Article 35. [Online https://www.emis.de/journals/JIPAM/article187.html?sid=187].

[5] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results. *Aust. J. Math. Anal. Appl.* **14** (2017), no. 1, Art. 1, 283 pp. [Online http://ajmaa.org/cgi-bin/paper.pl?string=v14n1/V14I1P1.tex].

[6] D. S. Mitrinović and I. B. Lacković, Hermite and convexity, *Aequationes Math.* **28** (1985), 229–232.

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