SOME INTEGRAL INEQUALITIES RELATED TO SCHWARZ AND HÖLDER INEQUALITIES

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Abstract. In this paper we establish some simple refinements and reverses of Cauchy-Bunyakovsky-Schwarz (CBS) and Hölder integral inequalities.

1. Introduction

In mathematics, the Cauchy-Bunyakovsky-Schwarz (CBS) inequality, is a useful inequality encountered in many different settings, such as linear algebra, analysis, probability theory, vector algebra and other areas. It is considered to be one of the most important inequalities in all of mathematics.

The inequality for sums was published by Augustin-Louis Cauchy in 1821, while the corresponding inequality for integrals was first proved by Viktor Bunyakovsky in 1859. The modern proof of the integral inequality in inner product spaces was given by Hermann Amandus Schwarz in 1888.

In what follows, we consider the CBS-inequality in $L_2[a,b]$, namely in the form

$$\left|\int_a^b f(t)\,g(t)\,dt\right| \leq \left(\int_a^b |f(t)|^2\,dt\right)^{1/2}\left(\int_a^b |g(t)|^2\,dt\right)^{1/2}, \tag{1.1}$$

where $f,\ g\in L_2[a,b]$, the Hilbert space of Lebesgue square integrable functions on $[a,b]$. For a survey on CBS-type inequalities see [1].

For $r\geq 1$ we define

$$L_r[a,b] := \left\{ f:[a,b]\to\mathbb{C} \text{ is Lebesgue measurable and } \int_a^b |f(t)|^r\,dt<\infty \right\}.$$

If $f\in L_p[a,b]$, $g\in L_q[a,b]$ where $\frac{1}{p}+\frac{1}{q}=1$, then the following Hölder's inequality is valid

$$\left|\int_a^b f(t)\,g(t)\,dt\right| \leq \left(\int_a^b |f(t)|^p\,dt\right)^{1/p}\left(\int_a^b |g(t)|^q\,dt\right)^{1/q}. \tag{1.2}$$

Hölder's inequality was first found by Leonard James Rogers in 1888, [5] and discovered independently by Otto Ludwig Hölder in 1889, [3]. The case $p=q=2$ produces the CBS-inequality (1.1).

In this paper we establish some simple refinements and reverses of these two fundamental inequalities.

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2. The Main Results

We need the following preliminary facts:

Lemma 1. *Let $f:(a,b]\to\mathbb{R}$ be a measurable function and such that the improper integral $\int_a^b \frac{f(t)dt}{t-a}$ exists and the lateral limit $L:=\lim_{t\to a+} f(t)$ exists and is finite, then $L=0$. If the improper integral $\int_a^b \frac{f(t)dt}{t-a}$ exists, then $\lim_{t\to a+} f(t)$ can be neither ∞ nor $-\infty$.*

Proof. Assume that $L>0$. Then for any $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that for any $t\in(a,a+\delta(\varepsilon)]$ we have $|f(t)-L|<\varepsilon$ that is equivalent to

$$L-\varepsilon<f(t)<\varepsilon+L. \tag{2.1}$$

Take $0<\varepsilon<L$ and $0<\eta<\delta(\varepsilon)$. By the first inequality in (2.1) we get for $t\in[a+\eta,a+\delta(\varepsilon)]$ that

$$0<\frac{L-\varepsilon}{t-a}<\frac{f(t)}{t-a}.$$

By taking the integral on $[a+\eta,a+\delta(\varepsilon)]$ we get

$$0<(L-\varepsilon)\int_{a+\eta}^{a+\delta(\varepsilon)}\frac{1}{t-a}<\int_{a+\eta}^{a+\delta(\varepsilon)}\frac{f(t)}{t-a}dt,$$

which is equivalent to

$$0<(L-\varepsilon)\left[\ln\delta(\varepsilon)-\ln\eta\right]<\int_{a+\eta}^{a+\delta(\varepsilon)}\frac{f(t)}{t-a}dt. \tag{2.2}$$

By taking the limit over $\eta\to0+$ in (2.2) we get that

$$\infty\le\int_a^{a+\delta(\varepsilon)}\frac{f(t)}{t-a}dt,$$

which contradicts the fact that the improper integral $\int_a^b \frac{f(t)dt}{t-a}$ exists.

Also, assume that $L<0$. Take $0<\varepsilon<-L$ and $0<\eta<\delta(\varepsilon)$. Then by the second inequality we have

$$\int_{a+\eta}^{a+\delta(\varepsilon)}\frac{f(t)}{t-a}dt<(\varepsilon+L)\int_{a+\eta}^{a+\delta(\varepsilon)}\frac{1}{t-a}<0,$$

which is equivalent to

$$\int_{a+\eta}^{a+\delta(\varepsilon)}\frac{f(t)}{t-a}dt<(\varepsilon+L)\left[\ln\delta(\varepsilon)-\ln\eta\right]<0. \tag{2.3}$$

By taking the limit over $\eta\to0+$ in (2.3) we get that

$$\int_a^{a+\delta(\varepsilon)}\frac{f(t)}{t-a}dt\le-\infty$$

which contradicts the fact that the improper integral $\int_a^b \frac{f(t)dt}{t-a}$ exists.

Now, assume that $\lim_{t\to a+} f(t)=\infty$. This means that for any $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ such that for any $t\in(a,a+\delta(\varepsilon)]$ we have $f(t)\ge\varepsilon$. Take $0<\eta<\delta(\varepsilon)$. Then for $t\in[a+\eta,a+\delta(\varepsilon)]$ we have

$$\frac{f(t)}{t-a}\ge\frac{\varepsilon}{t-a}$$

and by taking the integral, we have

$$\int_{a+\eta}^{a+\delta(\varepsilon)} \frac{f(t)\,dt}{t-a} \geq \varepsilon \int_{a+\eta}^{a+\delta(\varepsilon)} \frac{dt}{t-a} = \varepsilon\left[\ln\delta(\varepsilon) - \ln\eta\right]. \tag{2.4}$$

By taking the limit over $\eta \to 0+$ in (2.4), we get that

$$\int_{a}^{a+\delta(\varepsilon)} \frac{f(t)\,dt}{t-a} \geq \infty$$

which contradicts the fact that the improper integral $\int_a^b \frac{f(t)dt}{t-a}$ exists.

The case $\lim_{t\to a+} f(t) = -\infty$ can be proved in the same way and the details are omitted. □

Lemma 2. *Let $f:(a,b] \to \mathbb{R}$ be an integrable function and such that the improper integral $\int_a^b \frac{f(t)dt}{t-a}$ exists and the lateral limit $L := \lim_{t\to a+} f(t)$ exists and is finite, then*

$$\int_a^b \frac{\int_a^t f(s)\,ds}{(t-a)^2}dt = \int_a^b \frac{f(t)\,dt}{t-a} - \frac{1}{b-a}\int_a^b f(t)\,dt. \tag{2.5}$$

Proof. Let $\varepsilon > 0$ and such that $a+\varepsilon < b$. Using the integration by parts formula we have

$$\begin{aligned} \int_{a+\varepsilon}^b \frac{\int_a^t f(s)\,ds}{(t-a)^2}dt &= -\int_{a+\varepsilon}^b \left(\int_a^t f(s)\,ds\right) d\left(\frac{1}{t-a}\right) \\ &= -\left[\left(\int_a^t f(s)\,ds\right)\frac{1}{t-a}\bigg|_{a+\varepsilon}^b - \int_{a+\varepsilon}^b \frac{f(t)}{t-a}dt\right] \\ &= \int_{a+\varepsilon}^b \frac{f(t)}{t-a}dt - \frac{1}{b-a}\int_a^b f(s)\,ds + \frac{1}{\varepsilon}\int_a^{a+\varepsilon} f(s)\,ds. \end{aligned} \tag{2.6}$$

Using l'Hôpital's Theorem, we have by Lemma 1 that

$$\lim_{\varepsilon\to 0+}\left(\frac{1}{\varepsilon}\int_a^{a+\varepsilon} f(s)\,ds\right) = \lim_{\varepsilon\to 0+} f(a+\varepsilon) = \lim_{x\to a+} f(x) = 0.$$

By taking the limit over $\varepsilon \to 0+$ in (2.6), we get the first equality in (2.5). □

We have the following result related to Hölder's inequality:

Theorem 1. *Let $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L_p[a,b]$, $g \in L_q[a,b]$ and assume that the improper integrals $\int_a^b \frac{|f(t)|^p}{t-a}dt$, $\int_a^b \frac{|g(t)|^q}{t-a}dt$ exist and the lateral limits $\lim_{t\to a+} f(t)$, $\lim_{t\to a+} g(t)$ exists and are finite, then*

$$\begin{aligned} &\left|\int_a^b \frac{f(t)\,g(t)\,dt}{t-a} - \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt\right| \\ &\leq \left(\int_a^b \frac{|f(t)|^p\,dt}{t-a} - \frac{1}{b-a}\int_a^b |f(t)|^p\,dt\right)^{1/p}\left(\int_a^b \frac{|g(t)|^q\,dt}{t-a} - \frac{1}{b-a}\int_a^b |g(t)|^q\,dt\right)^{1/q}. \end{aligned} \tag{2.7}$$

Proof. By (2.5) we have

$$\int_a^b \frac{\int_a^t f(s)\,g(s)\,ds}{(t-a)^2}dt = \int_a^b \frac{f(t)\,g(t)\,dt}{t-a} - \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt, \tag{2.8}$$

$$0 \leq \int_a^b \frac{\int_a^t |f(s)|^p\, ds}{(t-a)^2} dt = \int_a^b \frac{|f(t)|^p\, dt}{t-a} - \frac{1}{b-a} \int_a^b |f(t)|^p\, dt, \tag{2.9}$$

and

$$0 \leq \int_a^b \frac{\int_a^t |g(s)|^q\, ds}{(t-a)^2} dt = \int_a^b \frac{|g(t)|^q\, dt}{t-a} - \frac{1}{b-a} \int_a^b |g(t)|^q\, dt. \tag{2.10}$$

Using the Hölder's integral inequality for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\left| \int_a^t f(s)\, g(s)\, ds \right| \leq \left(\int_a^t |f(s)|^p\, ds \right)^{1/p} \left(\int_a^t |g(s)|^q\, ds \right)^{1/q},$$

which implies that

$$\begin{aligned} &\left| \frac{1}{(t-a)^2} \int_a^t f(s)\, g(s)\, ds \right| \\ &\leq \left(\frac{1}{(t-a)^2} \int_a^t |f(s)|^p\, ds \right)^{1/p} \left(\frac{1}{(t-a)^2} \int_a^t |g(s)|^q\, ds \right)^{1/q}, \end{aligned}$$

for $t \in (a, b]$.

Taking the integral on $[a, b]$, we get

$$\begin{aligned} &\int_a^b \left| \frac{1}{(t-a)^2} \int_a^t f(s)\, g(s)\, ds \right| dt \\ &\leq \int_a^b \left(\frac{1}{(t-a)^2} \int_a^t |f(s)|^p\, ds \right)^{1/p} \left(\frac{1}{(t-a)^2} \int_a^t |g(s)|^q\, ds \right)^{1/q} dt. \end{aligned} \tag{2.11}$$

Using the modulus property, we have

$$\left| \int_a^b \frac{1}{(t-a)^2} \left(\int_a^t f(s)\, g(s)\, ds \right) dt \right| \leq \int_a^b \left| \frac{1}{(t-a)^2} \int_a^t f(s)\, g(s)\, ds \right| dt, \tag{2.12}$$

while by the the Hölder's integral inequality for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\begin{aligned} &\int_a^b \left(\frac{1}{(t-a)^2} \int_a^t |f(s)|^p\, ds \right)^{1/p} \left(\frac{1}{(t-a)^2} \int_a^t |g(s)|^q\, ds \right)^{1/q} dt \\ &\leq \left(\int_a^b \frac{1}{(t-a)^2} \left(\int_a^t |f(s)|^p\, ds \right) dt \right)^{1/p} \left(\int_a^b \frac{1}{(t-a)^2} \left(\int_a^t |g(s)|^q\, ds \right) dt \right)^{1/q}. \end{aligned} \tag{2.13}$$

By making use of (2.8)-(2.10) and (2.11)-(2.13), we get (2.7). □

We use the following inequality of Popoviciu, [4], see for instance [2, Inequality (26)]:

Lemma 3. *Let* $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, $a,\ a_i,\ b,\ b_i > 0$ *for* $i \in \{1, ..., n\}$ *and* $\sum_{i=1}^n a_i^p \leq a^p$, $\sum_{i=1}^n b_i^q \leq b^q$, *then*

$$\left(a^p - \sum_{i=1}^n a_i^p \right)^{1/p} \left(b^q - \sum_{i=1}^n b_i^q \right)^{1/q} \leq ab - \sum_{i=1}^n a_i b_i.$$

We can state and prove now the following reverse of Hölder's inequality (1.2).

Corollary 1. *Let* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1,\ f\in L_p[a,b],\ g\in L_q[a,b]$ *and assume that the improper integrals* $\int_a^b \frac{|f(t)|^p}{t-a}dt,\ \int_a^b \frac{|g(t)|^q}{t-a}dt$ *exist and the lateral limits* $\lim_{t\to a+} f(t),\ \lim_{t\to a+} g(t)$ *exists and are finite, then*

$$(2.14)\quad (b-a)\left[\left(\int_a^b \frac{|f(t)|^p}{t-a}dt\right)^{1/p}\left(\int_a^b \frac{|g(t)|^q}{t-a}dt\right)^{1/q}-\left|\int_a^b \frac{f(t)\,g(t)}{t-a}dt\right|\right]$$
$$\geq \left(\int_a^b |f(t)|^p\,dt\right)^{1/p}\left(\int_a^b |g(t)|^q\,dt\right)^{1/q}-\left|\int_a^b f(t)\,g(t)\,dt\right|\geq 0.$$

Proof. For $n=1$, Popoviciu's inequality becomes

$$(\alpha^p-\beta^p)^{1/p}(\gamma^q-\delta^q)^{1/q}\leq \alpha\gamma-\beta\delta \text{ for } 0\leq\beta\leq\alpha \text{ and } 0\leq\delta\leq\gamma,$$

where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Therefore

$$(2.15)\quad \left(\int_a^b \frac{|f(t)|^p\,dt}{t-a}-\frac{1}{b-a}\int_a^b |f(t)|^p\,dt\right)^{1/p}$$
$$\times\left(\int_a^b \frac{|g(t)|^q\,dt}{t-a}-\frac{1}{b-a}\int_a^b |g(t)|^q\,dt\right)^{1/q}$$
$$\leq \left(\int_a^b \frac{|f(t)|^p\,dt}{t-a}\right)^{1/p}\left(\int_a^b \frac{|g(t)|^q\,dt}{t-a}\right)^{1/q}$$
$$-\left(\frac{1}{b-a}\int_a^b |f(t)|^p\,dt\right)^{1/p}\left(\frac{1}{b-a}\int_a^b |g(t)|^q\,dt\right)^{1/q}.$$

Using the continuity property of the modulus,

$$|x|-|y|\leq ||x|-|y||\leq |x-y|,\ x,\ y\in\mathbb{R}$$

we have

$$(2.16)\quad \left|\int_a^b \frac{f(t)\,g(t)\,dt}{t-a}\right|-\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt\right|$$
$$\leq \left|\int_a^b \frac{f(t)\,g(t)\,dt}{t-a}-\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt\right|.$$

By using (2.7), (2.15) and (2.16) we get (2.14). □

The following refinement of (1.2) also holds

Corollary 2. *Let* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $f\in L_p[a,b]$, $g\in L_q[a,b]$, *then*

$$\begin{aligned}(2.17)\quad &\left(\int_a^b |f(t)|^p\,dt\right)^{1/p}\left(\int_a^b |g(t)|^q\,dt\right)^{1/q}-\left|\int_a^b f(t)\,g(t)\,dt\right|\\ &\geq \frac{1}{b-a}\left[\left(\int_a^b |f(t)|^p (t-a)\,dt\right)^{1/p}\left(\int_a^b |g(t)|^q (t-a)\,dt\right)^{1/q}\right.\\ &\qquad\left.-\left|\int_a^b f(t)\,g(t)\,(t-a)\,dt\right|\right]\geq 0.\end{aligned}$$

The proof follows by Corollary 1 on replacing $f(t)$ with $f(t)(t-a)^{1/p}$ and of $g(t)$ with $g(t)(t-a)^{1/q}$.

For $p=q=2$ we can state from Theorem 1 the following result related to the CBS-inequality:

Proposition 1. *Let* $f,\ g\in L_2[a,b]$ *and assume that the improper integrals* $\int_a^b \frac{|f(t)|^2}{t-a}dt$, $\int_a^b \frac{|g(t)|^2}{t-a}dt$ *exist and the lateral limits* $\lim_{t\to a+} f(t)$, $\lim_{t\to a+} g(t)$ *exists and are finite, then*

$$\begin{aligned}(2.18)\quad &\left|\int_a^b \frac{f(t)\,g(t)\,dt}{t-a}-\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt\right|^2\\ &\leq\left(\int_a^b \frac{|f(t)|^2\,dt}{t-a}-\frac{1}{b-a}\int_a^b |f(t)|^2\,dt\right)\left(\int_a^b \frac{|g(t)|^2\,dt}{t-a}-\frac{1}{b-a}\int_a^b |g(t)|^2\,dt\right).\end{aligned}$$

We have the following reverse of CBS-inequality (1.1):

Corollary 3. *Let* $f,\ g\in L_2[a,b]$ *and assume that the improper integrals* $\int_a^b \frac{|f(t)|^2}{t-a}dt$, $\int_a^b \frac{|g(t)|^2}{t-a}dt$ *exist and the lateral limits* $\lim_{t\to a+} f(t)$, $\lim_{t\to a+} g(t)$ *exists and are finite, then*

$$\begin{aligned}(2.19)\quad &(b-a)\left[\left(\int_a^b \frac{|f(t)|^2}{t-a}dt\right)^{1/2}\left(\int_a^b \frac{|g(t)|^2}{t-a}dt\right)^{1/2}-\left|\int_a^b \frac{f(t)\,g(t)}{t-a}dt\right|\right]\\ &\geq\left(\int_a^b |f(t)|^2\,dt\right)^{1/2}\left(\int_a^b |g(t)|^2\,dt\right)^{1/2}-\left|\int_a^b f(t)\,g(t)\,dt\right|\geq 0.\end{aligned}$$

We also have the following refinement of (1.1) result:

Corollary 4. *Let f, $g \in L_2[a,b]$, then*

$$\left(\int_a^b |f(t)|^2 \, dt\right)^{1/2} \left(\int_a^b |g(t)|^2 \, dt\right)^{1/2} - \left|\int_a^b f(t)\, g(t)\, dt\right| \tag{2.20}$$
$$\geq \frac{1}{b-a}\left[\left(\int_a^b |f(t)|^2 (t-a)\, dt\right)^{1/2} \left(\int_a^b |g(t)|^2 (t-a)\, dt\right)^{1/2} - \left|\int_a^b f(t)\, g(t)\,(t-a)\, dt\right|\right] \geq 0.$$

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