SOME WEIGHTED OSTROWSKI TYPE INEQUALITIES FOR RIEMANN-STIELTJES INTEGRAL

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Abstract. In this paper we provide some Ostrowski type inequalities to approximate the Riemann-Stieltjes integral of a product of two functions $\int_a^b f(t)\,g(t)\,dv(t)$. Applications for continuous functions of selfadjoint operators and functions of unitary operators on Hilbert spaces are also given.

1. Introduction

One can approximate the *Stieltjes integral* $\int_a^b f(t)\,du(t)$ with the following simpler quantities:

$$\frac{1}{b-a}[u(b)-u(a)]\cdot\int_a^b f(t)\,dt \qquad ([25],\ [26]) \tag{1.1}$$

$$f(x)[u(b)-u(a)] \qquad ([15],\ [16]) \tag{1.2}$$

or with

$$[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a) \qquad ([24]), \tag{1.3}$$

where $x\in[a,b]$.

In order to provide *a priory* sharp bounds for the *approximation error*, consider the functionals:

$$D(f,u;a,b):=\int_a^b f(t)\,du(t)-\frac{1}{b-a}[u(b)-u(a)]\cdot\int_a^b f(t)\,dt,$$

$$\Theta(f,u;a,b,x):=\int_a^b f(t)\,du(t)-f(x)[u(b)-u(a)]$$

and

$$T(f,u;a,b,x):=\int_a^b f(t)\,du(t)-[u(b)-u(x)]f(b)-[u(x)-u(a)]f(a).$$

If the *integrand* f is *Riemann integrable* on $[a,b]$ and the *integrator* $u:[a,b]\to\mathbb{R}$ is $L-$*Lipschitzian*, i.e.,

$$|u(t)-u(s)|\le L|t-s| \qquad \text{for each } t,s\in[a,b], \tag{1.4}$$

then the Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and, as pointed out in [25],

$$|D(f,u;a,b)|\le L\int_a^b\left|f(t)-\int_a^b\frac{1}{b-a}f(s)\,ds\right|dt. \tag{1.5}$$

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The inequality (1.5) is sharp in the sense that the multiplicative constant $C = 1$ in front of L cannot be replaced by a smaller quantity. Moreover, if there exists the constants m, $M \in \mathbb{R}$ such that $m \leq f(t) \leq M$ for a.e. $t \in [a, b]$, then [25]

$$|D(f, u; a, b)| \leq \frac{1}{2} L (M - m)(b - a). \tag{1.6}$$

The constant $\frac{1}{2}$ is best possible in (1.6).

A different approach in the case of integrands of bounded variation were considered by the same authors in 2001, [26], where they showed that

$$|D(f, u; a, b)| \leq \max_{t \in [a,b]} \left| f(t) - \frac{1}{b - a} \int_a^b f(s)\, ds \right| \bigvee_a^b (u), \tag{1.7}$$

provided that f is continuous and u is of bounded variation. Here $\bigvee_a^b (u)$ denotes the total variation of u on $[a, b]$. The inequality (1.7) is sharp.

If we assume that f is K-Lipschitzian, then [26]

$$|D(f, u; a, b)| \leq \frac{1}{2} K (b - a) \bigvee_a^b (u), \tag{1.8}$$

with $\frac{1}{2}$ the best possible constant in (1.8).

For various bounds on the error functional $D(f, u; a, b)$ where f and u belong to different classes of function for which the Stieltjes integral exists, see [21], [20], [19], and [8] and the references therein.

For the functional $\theta(f, u; a, b, x)$ we have the bound [15]:

$$\begin{aligned}
&|\theta(f, u; a, b, x)| \\
&\leq H \left[(x - a)^r \bigvee_a^x (f) + (b - x)^r \bigvee_x^b (f) \right] \\
&\leq H \times \begin{cases}
\left[(x - a)^r + (b - x)^r \right] \left[\frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right]; \\[2ex]
\left[(x - a)^{qr} + (b - x)^{qr} \right]^{\frac{1}{q}} \left[\left(\bigvee_a^x (f) \right)^p + \left(\bigvee_x^b (f) \right)^p \right]^{\frac{1}{p}} \\
\qquad\qquad\qquad\qquad \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\[2ex]
\left[\frac{1}{2}(b - a) + \left| x - \frac{a+b}{2} \right| \right]^r \bigvee_a^b (f),
\end{cases}
\end{aligned} \tag{1.9}$$

provided f is of bounded variation and u is of *r-H-Hölder type*, i.e.,

$$|u(t) - u(s)| \leq H |t - s|^r \quad \text{for each } \ t, s \in [a, b], \tag{1.10}$$

with given $H > 0$ and $r \in (0, 1]$.

If f is of q-K-Hölder type and u is of bounded variation, then [16]

$$|\theta(f, u; a, b, x)| \leq K \left[\frac{1}{2}(b - a) + \left| x - \frac{a + b}{2} \right| \right]^q \bigvee_a^b (u), \tag{1.11}$$

for any $x \in [a, b]$.

If u is monotonic nondecreasing and f of q-K-Hölder type, then the following refinement of (1.11) also holds [8]:

$$\begin{aligned}(1.12)\qquad |\theta(f,u;a,b,x)| &\le K\Bigg[(b-x)^q u(b)-(x-a)^q u(a)\\ &\quad + q\left\{\int_a^x \frac{u(t)\,dt}{(x-t)^{1-q}}-\int_x^b \frac{u(t)\,dt}{(t-x)^{1-q}}\right\}\Bigg]\\ &\le K\left[(b-x)^q[u(b)-u(x)]+(x-a)^q[u(x)-u(a)]\right]\\ &\le K\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^q[u(b)-u(a)],\end{aligned}$$

for any $x\in[a,b]$.

If f is monotonic nondecreasing and u is of r-H-Hölder type, then [8]:

$$\begin{aligned}(1.13)\qquad &|\theta(f,u;a,b,x)|\\ &\le H\Bigg[[(x-a)^r-(b-x)^r]f(x)\\ &\quad + r\left\{\int_a^x \frac{f(t)\,dt}{(b-t)^{1-r}}-\int_x^b \frac{f(t)\,dt}{(t-r)^{1-r}}\right\}\Bigg]\\ &\le H\left\{(b-x)^r[f(b)-f(x)]+(x-a)^r[f(x)-f(a)]\right\}\\ &\le H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r[f(b)-f(a)],\end{aligned}$$

for any $x\in[a,b]$.

The error functional $T(f,u;a,b,x)$ satisfies similar bounds, see [24], [8], [3] and [2] and the details are omitted.

Motivated by the above results, in this paper we provide some simple ways to approximate the Riemann-Stieltjes integral of a product of two functions $\int_a^b f(t)\,g(t)\,dv(t)$ by the use of simpler quantities and under several assumptions for the functions involved. Applications for continuous functions of selfadjoint operators and continuous functions of unitary operators on Hilbert spaces are also given.

2. Ostrowski Type Inequalities for Riemann-Stieltjes Integral

Assume that $u,\ f:[a,b]\to\mathbb{C}$. If the Riemann-Stieltjes integral $\int_a^b f(u)\,du(t)$ exists, we write for simplicity, like in [1, p. 142] that $f\in\mathcal{R}_{\mathbb{C}}(u,[a,b])$, or $\mathcal{R}_{\mathbb{C}}(u)$ when the interval is implicitly known. If the functions $u,\ f$ are real valued, then we write $f\in\mathcal{R}(u,[a,b])$, or $\mathcal{R}(u)$.

We have the following simple however useful representation of the Riemann-Stieltjes integral:

Lemma 1. *Assume that $u,\ f:[a,b]\to\mathbb{C}$ and $x\in[a,b]$ are such that $f\in\mathcal{R}_{\mathbb{C}}(u,[a,x])\cap\mathcal{R}_{\mathbb{C}}(u,[x,b])$. Then for any $\lambda,\ \mu\in\mathbb{C}$ we have the equality*

$$\begin{aligned}(2.1)\qquad \int_a^b f(t)\,du(t) &= \lambda[u(x)-u(a)]+\mu[u(b)-u(x)]\\ &\quad+\int_a^x[f(t)-\lambda]\,du(t)+\int_x^b[f(t)-\mu]\,du(t).\end{aligned}$$

Proof. For any $x \in [a,b]$ and $\lambda, \mu \in \mathbb{C}$ the Riemann-Stieltjes integrals $\int_a^b f(t)\,du(t)$, $\int_a^x [f(t)-\lambda]\,du(t)$ and $\int_x^b [f(t)-\mu]\,du(t)$ exist and we have

$$\begin{aligned}
&\int_a^x [f(t)-\lambda]\,du(t) + \int_x^b [f(t)-\mu]\,du(t) \\
&= \int_a^x f(t)\,du(t) - \lambda \int_a^x du(t) + \int_x^b f(t)\,du(t) - \mu \int_x^b du(t) \\
&= \int_a^b f(t)\,du(t) - \lambda [u(x)-u(a)] - \mu [u(b)-u(x)]
\end{aligned}$$

giving the desired result (2.1). □

Remark 1. *Assume that $v \in \mathcal{BV}_{\mathbb{C}}[a,b]$, namely of bounded variation on $[a,b]$ and f, $g \in \mathcal{R}(v,[a,b])$. Define $F(x) := \int_a^x f(t)\,dv(t)$ and $G(x) := \int_a^x g(t)\,dv(t)$ for $x \in [a,b]$. According to [1, p. 158-159], we then have that $f \in \mathcal{R}_{\mathbb{C}}(G,[a,b])$, $g \in \mathcal{R}_{\mathbb{C}}(G,[a,b])$, $fg \in \mathcal{R}_{\mathbb{C}}(v,[a,b])$ and the following equalities hold*

$$\int_a^b f(t)\,g(t)\,dv(t) = \int_a^b f(t)\,dG(t) = \int_a^b g(t)\,dF(t).$$

Now, by writing the equality (2.1) for f and $u = G$ we get

$$\begin{aligned}
\int_a^b f(t)\,g(t)\,dv(t) &= \lambda \int_a^x g(t)\,dv(t) + \mu \int_x^b g(t)\,dv(t) \\
&\quad + \int_a^x [f(t)-\lambda]\,g(t)\,dv(t) + \int_x^b [f(t)-\mu]\,g(t)\,dv(t) \\
&= \mu \int_a^b g(t)\,dv(t) + (\lambda-\mu) \int_a^x g(t)\,dv(t) \\
&\quad + \int_a^x [f(t)-\lambda]\,g(t)\,dv(t) + \int_x^b [f(t)-\mu]\,g(t)\,dv(t).
\end{aligned} \tag{2.2}$$

The case of $\lambda = \mu$ is of interest and generates the following particular cases:

Corollary 1. *With the assumptions of Lemma 1 we have*

$$\int_a^b f(t)\,du(t) = [u(b)-u(a)]\,\lambda + \int_a^b [f(t)-\lambda]\,du(t). \tag{2.3}$$

In particular, we have for any $x \in [a,b]$ that

$$\begin{aligned}
\int_a^b f(t)\,du(t) &= [u(b)-u(a)]\,f(x) \\
&\quad - \int_a^x [f(x)-f(t)]\,du(t) + \int_x^b [f(t)-f(x)]\,du(t).
\end{aligned} \tag{2.4}$$

Remark 2. *With the assumption that all Riemann-Stieltjes integrals below exists, we have the three functions equalities:*

$$\int_a^b f(t)\,g(t)\,du(t) = \lambda \int_a^b g(t)\,du(t) + \int_a^b [f(t)-\lambda]\,g(t)\,du(t). \tag{2.5}$$

In particular, we have for any $x\in[a,b]$ that

$$\int_a^b f(t)\,g(t)\,du(t)=f(x)\int_a^b g(t)\,du(t)-\int_a^x [f(x)-f(t)]\,g(t)\,du(t)+\int_x^b [f(t)-f(x)]\,g(t)\,du(t)\,. \tag{2.6}$$

Theorem 1. *Assume that $f\in\mathcal{BV}_{\mathbb{C}}[a,b]\cap\mathcal{C}_{\mathbb{C}}[a,b]$ and $u\in\mathcal{BV}_{\mathbb{C}}[a,b]$. Then for any $x\in[a,b]$ we have*

$$\begin{aligned}
&\left|\int_a^b f(t)\,du(t)-[u(b)-u(a)]\,f(x)\right| \\
&\le \bigvee_a^x(f)\bigvee_a^x(u)+\bigvee_x^b(f)\bigvee_x^b(u)-\int_a^x \bigvee_a^t(f)\,d\left(\bigvee_a^t(u)\right)-\int_x^b \bigvee_t^b(f)\,d\left(\bigvee_x^t(u)\right) \\
&\le \bigvee_a^x(f)\bigvee_a^x(u)+\bigvee_x^b(f)\bigvee_x^b(u) \\
&\le \begin{cases} \left(\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right)\bigvee_a^b(u) \\ \left(\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right)\bigvee_a^b(f) \end{cases} \le \bigvee_a^b(f)\bigvee_a^b(u)\,.
\end{aligned} \tag{2.7}$$

Proof. It is well known that if $p\in\mathcal{R}(u,[a,b])$ where $u\in\mathcal{BV}_{\mathbb{C}}[a,b]$ then we have [1, p. 177]

$$\left|\int_a^b p(t)\,du(t)\right|\le\int_a^b |p(t)|\,d\left(\bigvee_a^t(u)\right)\le\sup_{t\in[a,b]}|p(t)|\bigvee_a^b(u)\,. \tag{2.8}$$

Using the identity (2.4) and the property (2.8) we have for any $x\in[a,b]$ that

$$\begin{aligned}
&\left|\int_a^b f(t)\,du(t)-[u(b)-u(a)]\,f(x)\right| \\
&\le\left|\int_a^x [f(x)-f(t)]\,du(t)\right|+\left|\int_x^b [f(t)-f(x)]\,du(t)\right| \\
&\le\int_a^x |f(x)-f(t)|\,d\left(\bigvee_a^t(u)\right)+\int_x^b |f(t)-f(x)|\,d\left(\bigvee_x^t(u)\right)=:B(x)\,.
\end{aligned} \tag{2.9}$$

Now, since $f\in\mathcal{BV}_{\mathbb{C}}[a,b]$, then

$$|f(x)-f(t)|\le\bigvee_t^x(f)\ \text{ for } a\le t\le x$$

and

$$|f(x)-f(t)|\le\bigvee_x^t(f)\ \text{ for } x\le t\le b.$$

Therefore

$$B(x) \le \int_a^x \bigvee_t^x (f)\, d\left(\bigvee_a^t (u)\right) + \int_x^b \bigvee_x^t (f)\, d\left(\bigvee_x^t (u)\right) =: C(x). \tag{2.10}$$

We also have

$$\begin{aligned}\int_a^x \bigvee_t^x (f)\, d\left(\bigvee_a^t (u)\right) &= \int_a^x \left(\bigvee_a^x (f) - \bigvee_a^t (f)\right) d\left(\bigvee_a^t (u)\right) \\ &= \bigvee_a^x (f) \bigvee_a^x (u) - \int_a^x \bigvee_a^t (f)\, d\left(\bigvee_a^t (u)\right) \le \bigvee_a^x (f) \bigvee_a^x (u),\end{aligned} \tag{2.11}$$

since the function $\bigvee_a^{\cdot} (u)$ is nondecreasing on $[a,x]$ and

$$\begin{aligned}\int_x^b \bigvee_x^t (f)\, d\left(\bigvee_x^t (u)\right) &= \int_x^b \left(\bigvee_x^b (f) - \bigvee_t^b (f)\right) d\left(\bigvee_x^t (u)\right) \\ &= \bigvee_x^b (f) \bigvee_x^b (u) - \int_x^b \bigvee_t^b (f)\, d\left(\bigvee_x^t (u)\right) \le \bigvee_x^b (f) \bigvee_x^b (u),\end{aligned} \tag{2.12}$$

since the function $\bigvee_x^{\cdot} (u)$ is nondecreasing on $[x,b]$.

By making use of (2.11) and (2.12) we get

$$\begin{aligned}C(x) &\le \bigvee_a^x (f) \bigvee_a^x (u) - \int_a^x \bigvee_a^t (f)\, d\left(\bigvee_a^t (u)\right) \\ &\quad + \bigvee_x^b (f) \bigvee_x^b (u) - \int_x^b \bigvee_t^b (f)\, d\left(\bigvee_x^t (u)\right) \le \bigvee_a^x (f) \bigvee_a^x (u) + \bigvee_x^b (f) \bigvee_x^b (u)\end{aligned}$$

and the first inequality (2.7) is thus proved.

Observe that, by the elementary fact

$$mp + nq \le \max\{p,q\}(m+n) = \left(\frac{1}{2}(p+q) + \frac{1}{2}|p-q|\right)(m+n)$$

we deduce the last part of (2.7). □

Remark 3. *Assume that there exists $m \in [a,b]$ such that*

$$\bigvee_a^m (f) = \bigvee_m^b (f), \tag{2.13}$$

then by (2.7) we get

$$\left|\int_a^b f(t)\, du(t) - [u(b) - u(a)]\, f(m)\right| \le \frac{1}{2} \bigvee_a^b (f) \bigvee_a^b (u).$$

Also, if $p \in [a,b]$ *is such that* $\bigvee_a^p (u) = \bigvee_p^b (u)$, *then by (2.7) we get*

$$\left| \int_a^b f(t)\, du(t) - [u(b) - u(a)] f(p) \right| \leq \frac{1}{2} \bigvee_a^b (f) \bigvee_a^b (u)$$

Corollary 2. *Assume that* $f,\ g \in \mathcal{BV}_{\mathbb{C}}[a,b] \cap \mathcal{C}_{\mathbb{C}}[a,b]$ *and* $v \in \mathcal{BV}_{\mathbb{C}}[a,b]$. *Define* $G(x) := \int_a^x g(t)\, dv(t)$, *then for any* $x \in [a,b]$ *we have*

$$\left| \int_a^b f(t)\, g(t)\, dv(t) - f(x) \int_a^b g(t)\, dv(t) \right| \tag{2.14}$$
$$\leq \bigvee_a^x (f) \bigvee_a^x (G) + \bigvee_x^b (f) \bigvee_x^b (G) - \int_a^x \bigvee_a^t (f)\, d\left(\bigvee_a^t (G) \right) - \int_x^b \bigvee_t^b (f)\, d\left(\bigvee_x^t (G) \right)$$
$$\leq \bigvee_a^x (f) \bigvee_a^x (G) + \bigvee_x^b (f) \bigvee_x^b (G)$$
$$\leq \begin{cases} \left(\frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right) \bigvee_a^b (G) \\ \left(\frac{1}{2} \bigvee_a^b (G) + \frac{1}{2} \left| \bigvee_a^x (G) - \bigvee_x^b (G) \right| \right) \bigvee_a^b (f) \end{cases} \leq \bigvee_a^b (f) \bigvee_a^b (G).$$

The proof follows by Theorem 1 by taking $u = G$.

Remark 4. *If there exists* $m \in [a,b]$ *such that (2.13) is true, then by (2.14) we get*

$$\left| \int_a^b f(t)\, g(t)\, dv(t) - f(m) \int_a^b g(t)\, dv(t) \right| \leq \frac{1}{2} \bigvee_a^b (f) \bigvee_a^b (G). \tag{2.15}$$

If $q \in [a,b]$ *is such that* $\bigvee_a^q (G) = \bigvee_q^b (G)$, *then by (2.14) we get*

$$\left| \int_a^b f(t)\, g(t)\, dv(t) - f(q) \int_a^b g(t)\, dv(t) \right| \leq \frac{1}{2} \bigvee_a^b (f) \bigvee_a^b (G). \tag{2.16}$$

Let $[c,d] \subset [a,b]$ and let $\Delta : c = x_0 < x_1 < ... < x_{n-1} < x_n = d$ be a division of $[c,d]$. Then

$$\begin{aligned} \bigvee_c^d (G) &= \sup_{\Delta} \sum_{i=0}^{n-1} |G(x_{i+1}) - G(x_i)| = \sup_{\Delta} \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} g(t)\, dv(t) \right| \\ &\leq \sup_{\Delta} \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} |g(t)|\, d\left(\bigvee_c^t (v) \right) = \int_c^d |g(t)|\, d\left(\bigvee_c^t (v) \right) \\ &\leq \max_{t \in [c,d]} |g(t)| \bigvee_c^d (v), \end{aligned}$$

which implies that

$$\bigvee_a^x (G) \leq \int_a^x |g(t)| \, d\left(\bigvee_a^t (v)\right) \leq \max_{t\in[a,x]} |g(t)| \bigvee_a^x (v)$$

and

$$\bigvee_x^b (G) \leq \int_x^b |g(t)| \, d\left(\bigvee_x^t (v)\right) \leq \max_{t\in[x,b]} |g(t)| \bigvee_x^b (v)$$

for $x \in (a,b)$.

Using Corollary 2 and the above bounds, we can state the following result as well:

Proposition 1. *Assume that* $f,\ v \in \mathcal{BV}_{\mathbb{C}}[a,b]$ *and* $f,\ g \in \mathcal{C}_{\mathbb{C}}[a,b]$, *then we have*

$$\left|\int_a^b f(t)\, g(t)\, dv(t) - f(x)\int_a^b g(t)\, dv(t)\right| \tag{2.17}$$
$$\leq \bigvee_a^x (f) \bigvee_a^x (G) + \bigvee_x^b (f) \bigvee_x^b (G)$$
$$\leq \bigvee_a^x (f) \int_a^x |g(t)| \, d\left(\bigvee_a^t (v)\right) + \bigvee_x^b (f) \int_x^b |g(t)| \, d\left(\bigvee_x^t (v)\right)$$
$$\leq \bigvee_a^x (f) \bigvee_a^x (v) \max_{t\in[a,x]} |g(t)| + \bigvee_x^b (f) \bigvee_x^b (v) \max_{t\in[x,b]} |g(t)|$$

for $x \in (a,b)$.

From (2.17) we can get various upper bounds for the quantity

$$B(f,g,v;x) := \bigvee_a^x (f) \bigvee_a^x (v) \max_{t\in[a,x]} |g(t)| + \bigvee_x^b (f) \bigvee_x^b (v) \max_{t\in[x,b]} |g(t)|,\ x \in (a,b)$$

out of which we can use in Operator Theory in Hilbert spaces the following one

$$B(f,g,v;x) \leq \left[\bigvee_a^x (f) \max_{t\in[a,x]} |g(t)| + \bigvee_x^b (f) \max_{t\in[x,b]} |g(t)|\right] \bigvee_a^b (v) \tag{2.18}$$

for $x \in (a,b)$.

3. Applications for Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Let $A \in \mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda \in \mathbb{R}$ as follows

$$\varphi_\lambda(s) := \begin{cases} 1, \text{ for } -\infty < s \leq \lambda, \\ \\ 0, \text{ for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$E_\lambda := \varphi_\lambda(A) \tag{3.1}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [27, p. 256]:

Theorem 2 (Spectral Representation Theorem)**.** *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a = \min\{\lambda\,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $b = \max\{\lambda\,|\lambda \in Sp(A)\} =: \max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_\lambda \le E_{\lambda'}$ *for* $\lambda \le \lambda'$;

b) $E_{a-0} = 0, E_b = I$ *and* $E_{\lambda+0} = E_\lambda$ *for all* $\lambda \in \mathbb{R}$*;*

c) *We have the representation*

$$A = \int_{a-0}^{b} \lambda dE_\lambda.$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ there exists a unique operator $\varphi(A) \in \mathcal{B}(H)$ such that for every $\varepsilon > 0$ there exists a $\delta > 0$ satisfying the inequality

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi(\lambda_k')\left[E_{\lambda_k} - E_{\lambda_{k-1}}\right] \right\| \le \varepsilon$$

whenever

$$\begin{cases} \lambda_0 < a = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = b, \\ \\ \lambda_k - \lambda_{k-1} \le \delta \ \textit{for}\ 1 \le k \le n, \\ \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \ \textit{for}\ 1 \le k \le n \end{cases}$$

this means that

$$\varphi(A) = \int_{a-0}^{b} \varphi(\lambda)\, dE_\lambda, \tag{3.2}$$

where the integral is of Riemann-Stieltjes type.

Corollary 3. *With the assumptions of Theorem 2 for A, E_λ and φ we have the representations*

$$\varphi(A)x = \int_{a-0}^{b} \varphi(\lambda)\, dE_\lambda x \quad \textit{for all}\ x \in H$$

and

$$\langle \varphi(A)x, y\rangle = \int_{a-0}^{b} \varphi(\lambda)\, d\langle E_\lambda x, y\rangle \quad \textit{for all}\ x,\ y \in H. \tag{3.3}$$

In particular,

$$\langle \varphi(A)x, x\rangle = \int_{a-0}^{b} \varphi(\lambda)\, d\langle E_\lambda x, x\rangle \quad \textit{for all}\ x \in H.$$

Moreover, we have the equality

$$\|\varphi(A)x\|^2 = \int_{a-0}^{b} |\varphi(\lambda)|^2\, d\|E_\lambda x\|^2 \quad \textit{for all}\ x \in H.$$

We need the following result that provides an upper bound for the total variation of the function $\mathbb{R} \ni \lambda \mapsto \langle E_\lambda x, y\rangle \in \mathbb{C}$ on an interval $[\alpha, \beta]$, see [23].

Lemma 2. *Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A. Then for any $x,\ y\in H$ and $\alpha<\beta$ we have the inequality*

$$\left[\bigvee_{\alpha}^{\beta}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right]^2 \leq \left\langle \left(E_\beta - E_\alpha\right)x,x\right\rangle \left\langle \left(E_\beta - E_\alpha\right)y,y\right\rangle, \tag{3.4}$$

where $\bigvee_{\alpha}^{\beta}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$ denotes the total variation of the function $\left\langle E_{(\cdot)}x,y\right\rangle$ on $[\alpha,\beta]$.

Remark 5. *For $\alpha=a-\varepsilon$ with $\varepsilon>0$ and $\beta=b$ we get from (3.4) the inequality*

$$\bigvee_{a-\varepsilon}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \leq \left\langle \left(I-E_{a-\varepsilon}\right)x,x\right\rangle^{1/2} \left\langle \left(I-E_{a-\varepsilon}\right)y,y\right\rangle^{1/2} \tag{3.5}$$

for any $x,\ y\in H$.

This implies, for any $x,\ y\in H$, that

$$\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \leq \|x\|\,\|y\|, \tag{3.6}$$

where $\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$ denotes the limit $\lim_{\varepsilon\to 0+}\left[\bigvee_{a-\varepsilon}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right]$.

We can state the following result for functions of selfadjoint operators:

Theorem 3. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a=\min\{\lambda\,|\lambda\in Sp(A)\}=:\min Sp(A)$ and $b=\max\{\lambda\,|\lambda\in Sp(A)\}=:\max Sp(A)$. Also, assume that $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A and $f,\ g:I\to\mathbb{C}$ are continuous on I, $[a,b]\subset\mathring{I}$ (the interior of I) with f of locally bounded variation on I. Then for all $u\in[a,b]$,*

$$\begin{aligned}
&\left|\left\langle f(A)\,g(A)\,x,y\right\rangle - f(u)\left\langle g(A)\,x,y\right\rangle\right| \\
&\qquad\leq \left[\bigvee_{a}^{u}(f)\max_{t\in[a,u]}|g(t)| + \bigvee_{u}^{b}(f)\max_{t\in[u,b]}|g(t)|\right]\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&\qquad\qquad\leq \left[\bigvee_{a}^{u}(f)\max_{t\in[a,u]}|g(t)| + \bigvee_{u}^{b}(f)\max_{t\in[u,b]}|g(t)|\right]\|x\|\,\|y\|
\end{aligned} \tag{3.7}$$

for any $x,\ y\in H$.

Proof. Using the inequalities (2.17) and (2.18) we have for all $u\in[a,b]$, for small $\varepsilon>0$ and for any $x,\ y\in H$ that

$$\begin{aligned}
&\left|\int_{a-\varepsilon}^{b} f(t)\,g(t)\,d\left\langle E_t x,y\right\rangle - f(u)\int_{a-\varepsilon}^{b} g(t)\,d\left\langle E_t x,y\right\rangle\right| \\
&\leq \left[\bigvee_{a-\varepsilon}^{u}(f)\max_{t\in[a-\varepsilon,u]}|g(t)| + \bigvee_{u}^{b}(f)\max_{t\in[u,b]}|g(t)|\right]\bigvee_{a-\varepsilon}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right).
\end{aligned}$$

Taking the limit over $\varepsilon\to 0+$ and using the continuity of f, g and the Spectral Representation Theorem, we deduce the desired result (3.7). □

Remark 6. *The above inequality (3.7) can produce several particular examples of interest. For example if* $[a,b] \subset (0,\infty)$ *and we take* $f(t) = \ln t$ *and* $g(t) = t^p$, $p > 0$, *then by (3.7) we get*

$$|\langle A^p \ln Ax, y\rangle - \ln(u)\langle A^p x, y\rangle| \leq \left[u^p \ln\left(\frac{u}{a}\right) + b^p \ln\left(\frac{b}{u}\right)\right] \|x\| \|y\| \tag{3.8}$$

for any $x, y \in H$ *and* $u \in [a,b]$.

If in (3.8) we take $u = \sqrt{ab} \in [a,b]$, *then we get*

$$\left|\langle A^p \ln Ax, y\rangle - \frac{\ln a + \ln b}{2}\langle A^p x, y\rangle\right| \leq \frac{1}{2}(\ln b - \ln a)\sqrt{b^p}\left[\sqrt{a^p} + \sqrt{b^p}\right]\|x\|\|y\| \tag{3.9}$$

for any $x, y \in H$.

If we take $f(t) = t^q$ *and* $g(t) = t^p$, $p, q > 0$, *then by (3.7) we get*

$$\left|\left\langle A^{p+q}x, y\right\rangle - u^q\langle A^p x, y\rangle\right| \leq [(u^q - a^q)u^p + (b^q - u^q)b^p]\|x\|\|y\| \tag{3.10}$$

for any x, $y \in H$ *and* $u \in [a,b]$.

If we take $u = \left(\frac{a^q+b^q}{2}\right)^{1/q} \in [a,b]$ *in (3.10), then we get*

$$\left|\left\langle A^{p+q}x, y\right\rangle - \frac{a^q + b^q}{2}\langle A^p x, y\rangle\right| \leq \frac{1}{2}(b^q - a^q)\left[\left(\frac{a^q+b^q}{2}\right)^{p/q} + b^p\right]\|x\|\|y\| \tag{3.11}$$

for any x, $y \in H$.

4. Applications for Unitary Operators

A unitary operator is a bounded linear operator $U : H \to H$ on a Hilbert space H satisfying

$$U^*U = UU^* = 1_H$$

where U^* is the adjoint of U, and $1_H : H \to H$ is the identity operator. This property is equivalent to the following:

(i) U preserves the inner product $\langle \cdot, \cdot \rangle$ of the Hilbert space, i.e., for all vectors x and y in the Hilbert space, $\langle Ux, Uy\rangle = \langle x, y\rangle$ and
(ii) U is surjective.

The following result is well known [27, p. 275 - p. 276]:

Theorem 4 (Spectral Representation Theorem)**.** *Let* U *be a unitary operator on the Hilbert space* H. *Then there exists a family of projections* $\{P_\lambda\}_{\lambda\in[0,2\pi]}$, *called the spectral family of* U, *with the following properties*

a) $P_\lambda \leq P_{\lambda'}$ *for* $\lambda \leq \lambda'$;
b) $P_0 = 0, P_{2\pi} = I$ *and* $P_{\lambda+0} = P_\lambda$ *for all* $\lambda \in [0, 2\pi)$;
c) *We have the representation*

$$U = \int_0^{2\pi} \exp(i\lambda)\, dP_\lambda.$$

More generally, for every continuous complex-valued function φ defined on the unit circle $\mathcal{C}(0,1)$ there exists a unique operator $\varphi(U) \in \mathcal{B}(H)$ such that for every $\varepsilon > 0$ there exists a $\delta > 0$ satisfying the inequality

$$\left\| \varphi(U) - \sum_{k=1}^{n} \varphi\left(\exp\left(i\lambda_k'\right)\right)\left[P_{\lambda_k} - P_{\lambda_{k-1}}\right] \right\| \leq \varepsilon$$

whenever

$$\begin{cases} 0 = \lambda_1 < \ldots < \lambda_{n-1} < \lambda_n = 2\pi, \\ \lambda_k - \lambda_{k-1} \leq \delta \text{ for } 1 \leq k \leq n, \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \leq k \leq n \end{cases}$$

this means that

$$\varphi(U) = \int_0^{2\pi} \varphi(\exp(i\lambda))\, dP_\lambda, \tag{4.1}$$

where the integral is of Riemann-Stieltjes type.

Corollary 4. *With the assumptions of Theorem 4 for U, P_λ and φ we have the representations*

$$\varphi(U)\, x = \int_0^{2\pi} \varphi(\exp(i\lambda))\, dP_\lambda x \quad \text{for all } x \in H$$

and

$$\langle \varphi(U)\, x, y \rangle = \int_0^{2\pi} \varphi(\exp(i\lambda))\, d\langle P_\lambda x, y \rangle \quad \text{for all } x, y \in H. \tag{4.2}$$

In particular,

$$\langle \varphi(U)\, x, x \rangle = \int_0^{2\pi} \varphi(\exp(i\lambda))\, d\langle P_\lambda x, x \rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(U)\, x\|^2 = \int_0^{2\pi} |\varphi(\exp(i\lambda))|^2\, d\|P_\lambda x\|^2 \quad \text{for all } x \in H.$$

On making use of an argument similar to the one in [23, Theorem 6], we have:

Lemma 3. *Let $\{P_\lambda\}_{\lambda \in [0,2\pi]}$ be the spectral family of the unitary operator U on the Hilbert space H. Then for any x, $y \in H$ and $0 \leq \alpha < \beta \leq 2\pi$ we have the inequality*

$$\bigvee_{\alpha}^{\beta} \left(\left\langle P_{(\cdot)} x, y \right\rangle\right) \leq \langle (P_\beta - P_\alpha)\, x, x \rangle^{1/2} \langle (P_\beta - P_\alpha)\, y, y \rangle^{1/2}, \tag{4.3}$$

where $\bigvee_{\alpha}^{\beta} \left(\left\langle P_{(\cdot)} x, y \right\rangle\right)$ denotes the total variation of the function $\left\langle P_{(\cdot)} x, y \right\rangle$ on $[\alpha, \beta]$.

In particular,

$$\bigvee_{0}^{2\pi} \left(\left\langle P_{(\cdot)} x, y \right\rangle\right) \leq \|x\|\, \|y\| \tag{4.4}$$

for any x, $y \in H$.

We have:

Theorem 5. *Let U be a unitary operator on the Hilbert space H and $\{P_\lambda\}_{\lambda\in[0,2\pi]}$ the spectral family of projections of U. Also, assume that f, $g:\mathcal{C}(0,1)\to\mathbb{C}$ are continuous on $\mathcal{C}(0,1)$ with $f(\exp(i\cdot))$ of bounded variation on $[0,2\pi]$. If $u\in[0,2\pi]$, then*

$$\begin{aligned}(4.5)\quad & |\langle f(U)\,g(U)\,x,y\rangle - f(\exp(iu))\langle g(U)\,x,y\rangle| \\ & \le \left[\bigvee_0^u (f(\exp(it)))\max_{t\in[0,u]}|g(\exp(it))| + \bigvee_u^{2\pi}(\exp(it))\max_{t\in[u,2\pi]}|g(\exp(it))|\right] \\ & \qquad\qquad \times \bigvee_0^{2\pi}(\langle P_{(\cdot)}x,y\rangle) \\ & \le \left[\bigvee_0^u (f(\exp(it)))\max_{t\in[0,u]}|g(\exp(it))| + \bigvee_u^{2\pi}(\exp(it))\max_{t\in[u,2\pi]}|g(\exp(it))|\right] \\ & \qquad\qquad \times \|x\|\,\|y\|\end{aligned}$$

for any x, $y\in H$.

The proof follows by (2.17) and (2.18) and the Spectral Representation Theorem for unitary operators in a similar way with the proof of Theorem 1 and we omit the details.

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