RIEMANN-STIELTJES INTEGRAL INEQUALITIES OF TRAPEZOID TYPE WITH APPLICATIONS

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Abstract. In this paper we provide some bounds for the error in approximating the Riemann-Stieltjes integral $\int_a^b f(t)\,g(t)\,du(t)$ by the trapezoidal rule

$$\frac{f(a)+f(b)}{2}\int_a^b g(t)\,du(t)$$

under various assumptions for the integrands f and g, and the integrator u for which the above integral exists. Applications for continuous functions of selfadjoint operators in Hilbert spaces are provided as well.

1. Introduction

The following theorem generalizing the classical trapezoid inequality to the Riemann-Stieltjes integral for integrators of bounded variation and Hölder-continuous integrands was obtained by the author in 2001, see [4]:

Theorem 1. *Let $f:[a,b]\to\mathbb{C}$ be a p-H-Hölder type function, that is, it satisfies the condition*

$$|f(x)-f(y)|\le H|x-y|^p \text{ for all } x,\ y\in[a,b], \tag{1.1}$$

where $H>0$ and $p\in(0,1]$ are given, and $u:[a,b]\to\mathbb{C}$ is a function of bounded variation on $[a,b]$. Then we have the inequality:

$$\left|\frac{f(a)+f(b)}{2}[u(b)-u(a)]-\int_a^b f(t)\,du(t)\right|\le\frac{1}{2^p}H(b-a)^p\bigvee_a^b(u). \tag{1.2}$$

The constant $C=1$ on the right hand side of (1.2) cannot be replaced by a smaller quantity.

The case when the integrator is Lipschitzian is as follows, [8]:

Theorem 2. *Let $f:[a,b]\to\mathbb{C}$ be a p-H-Hölder type mapping where $H>0$ and $p\in(0,1]$ are given, and $u:[a,b]\to\mathbb{C}$ is a Lipschitzian function on $[a,b]$, this means that*

$$|u(x)-u(y)|\le L|x-y| \text{ for all } x,\ y\in[a,b], \tag{1.3}$$

where $L>0$ is given. Then we have the inequality:

$$\left|\frac{f(a)+f(b)}{2}[u(b)-u(a)]-\int_a^b f(t)\,du(t)\right|\le\frac{1}{p+1}HL(b-a)^{p+1}. \tag{1.4}$$

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In the case when u is monotonic nondecreasing, we have the following result as well, [8]:

Theorem 3. *Let $f:[a,b]\to\mathbb{C}$ be a p-H-Hölder type mapping where $H>0$ and $p\in(0,1]$ are given, and $u:[a,b]\to\mathbb{R}$ a monotonic nondecreasing function on $[a,b]$. Then we have the inequality:*

$$\begin{aligned}(1.5)\quad &\left|\frac{f(a)+f(b)}{2}[u(b)-u(a)]-\int_a^b f(t)\,du(t)\right|\\ &\leq\frac{1}{2}H\left\{(b-a)^p[u(b)-u(a)]-p\int_a^b\left[\frac{(b-t)^{1-p}-(t-a)^{1-p}}{(b-t)^{1-p}(t-a)^{1-p}}\right]u(t)\,dt\right\}\\ &\leq\frac{1}{2^p}H(b-a)^p[u(b)-u(a)].\end{aligned}$$

The inequalities in (1.5) are sharp.

For other similar results, see [2]-[8].

In this paper we provide some bounds for the error in approximating the Riemann-Stieltjes integral $\int_a^b f(t)\,g(t)\,du(t)$ by the trapezoidal rule

$$\frac{f(a)+f(b)}{2}\int_a^b g(t)\,du(t)$$

under various assumptions for the integrands f and g, and the integrator u for which the above integral exists. Applications for continuous functions of selfadjoint operators in Hilbert spaces are provided as well.

2. Some Preliminary Facts

Assume that $u,\ f:[a,b]\to\mathbb{C}$. If the Riemann-Stieltjes integral $\int_a^b f(u)\,du(t)$ exists, we write for simplicity, like in [1, p. 142] that $f\in\mathcal{R}_{\mathbb{C}}(u,[a,b])$, or $\mathcal{R}_{\mathbb{C}}(u)$ when the interval is implicitly known. If the functions $u,\ f$ are real valued, then we write $f\in\mathcal{R}(u,[a,b])$, or $\mathcal{R}(u)$, respectively.

We start with the following simple fact:

Lemma 1. *Let $f,\ g,\ v:[a,b]\to\mathbb{C}$, $\lambda,\ \mu\in\mathbb{C}$ and $x\in[a,b]$. If $fg,\ g\in\mathcal{R}_{\mathbb{C}}(v,[a,x])\cap\mathcal{R}_{\mathbb{C}}(v,[x,b])$, then $fg,\ g\in\mathcal{R}_{\mathbb{C}}(v,[a,b])$ and*

$$\begin{aligned}(2.1)\quad \int_a^b f(t)\,g(t)\,dv(t)&=\lambda\int_a^x g(t)\,dv(t)+\mu\int_x^b g(t)\,dv(t)\\ &\quad+\int_a^x[f(t)-\lambda]\,g(t)\,dv(t)+\int_x^b[f(t)-\mu]\,g(t)\,dv(t)\\ &=\mu\int_a^b g(t)\,dv(t)+(\lambda-\mu)\int_a^x g(t)\,dv(t)\\ &\quad+\int_a^x[f(t)-\lambda]\,g(t)\,dv(t)+\int_x^b[f(t)-\mu]\,g(t)\,dv(t).\end{aligned}$$

In particular, for $\mu = \lambda$, we have

$$\int_a^b f(t)\,g(t)\,dv(t) = \lambda \int_a^b g(t)\,dv(t) + \int_a^x [f(t)-\lambda]\,g(t)\,dv(t) + \int_x^b [f(t)-\lambda]\,g(t)\,dv(t) = \lambda \int_a^b g(t)\,dv(t) + \int_a^b [f(t)-\lambda]\,g(t)\,dv(t). \tag{2.2}$$

Proof. The integrability follows by Theorem 7. 4 from [1] which says that if a function is Riemann-Stieltjes integrable on the intervals $[a,x]$, $[x,b]$ with $x \in [a,b]$, then it is integrable on the whole interval $[a,b]$.

Using the properties of the Riemann-Stieltjes integral, we have

$$\begin{aligned}
&\int_a^x [f(t)-\lambda]\,g(t)\,dv(t) + \int_x^b [f(t)-\mu]\,g(t)\,dv(t) \\
&= \int_a^x f(t)\,g(t)\,dv(t) - \lambda \int_a^x g(t)\,dv(t) + \int_x^b f(t)\,g(t)\,dv(t) - \mu \int_x^b g(t)\,dv(t) \\
&= \int_a^b f(t)\,g(t)\,dv(t) - \lambda \int_a^x g(t)\,dv(t) - \mu \int_x^b g(t)\,dv(t),
\end{aligned}$$

which is equivalent to the first equality in (2.1).

The rest is obvious. □

Corollary 1. *Assume that f, $v : [a,b] \to \mathbb{C}$ and $x \in [a,b]$ are such that $f \in \mathcal{R}_{\mathbb{C}}(v,[a,x]) \cap \mathcal{R}_{\mathbb{C}}(v,[x,b])$. Then for any λ, $\mu \in \mathbb{C}$ we have the equality*

$$\int_a^b f(t)\,dv(t) = \lambda[v(x)-v(a)] + \mu[v(b)-v(x)] + \int_a^x [f(t)-\lambda]\,dv(t) + \int_x^b [f(t)-\mu]\,dv(t). \tag{2.3}$$

In particular, for $\mu = \lambda$, we have

$$\int_a^b f(t)\,dv(t) = \lambda[v(b)-v(a)] + \int_a^x [f(t)-\lambda]\,dv(t) + \int_x^b [f(t)-\lambda]\,dv(t) = \lambda[v(b)-v(a)] + \int_a^b [f(t)-\lambda]\,dv(t). \tag{2.4}$$

The proof follows by Lemma 1 for $g(t) = 1$, $t \in [a,b]$.

Remark 1. *We observe that, see* [1, Theorem 7.27], *if f, $g \in \mathcal{C}_{\mathbb{C}}[a,b]$, namely, are continuous on $[a,b]$ and $v \in \mathcal{BV}_{\mathbb{C}}[a,b]$, namely of bounded variation on $[a,b]$, then for any $x \in [a,b]$ the Riemann-Stieltjes integrals in Lemma 1 exist and the equalities (2.1) and (2.2) hold.*

If we use the equality (2.2) for $\lambda = \frac{f(a)+f(b)}{2}$, then we have

$$\int_a^b f(t)\,g(t)\,du(t) = \frac{f(a)+f(b)}{2}\int_a^b g(t)\,du(t) + \int_a^b \left[f(t) - \frac{f(a)+f(b)}{2}\right] g(t)\,du(t). \tag{2.5}$$

In particular, for $g(t) = 1$, $t \in [a,b]$, we have

$$\int_a^b f(t)\,du(t) = [u(b) - u(a)]\frac{f(a)+f(b)}{2} + \int_a^b \left[f(t) - \frac{f(a)+f(b)}{2}\right] du(t), \tag{2.6}$$

respectively.

3. Inequalities for Integrands of Bounded Variation

We have:

Theorem 4. *Assume that $f, g \in \mathcal{C}_{\mathbb{C}}[a,b]$ and $u \in \mathcal{BV}_{\mathbb{C}}[a,b]$. If $f \in \mathcal{BV}_{\mathbb{C}}[a,b]$, then*

$$\left|\int_a^b f(t)\,g(t)\,du(t) - \frac{f(a)+f(b)}{2}\int_a^b g(t)\,du(t)\right| \le \frac{1}{2}\bigvee_a^b(f)\int_a^b |g(t)|\,d\left(\bigvee_a^t(u)\right) \le \frac{1}{2}\max_{t\in[a,b]}|g(t)|\bigvee_a^b(f)\bigvee_a^b(u). \tag{3.1}$$

Proof. Since f is of bounded variation on $[a,b]$, hence

$$\left|f(t) - \frac{f(a)+f(b)}{2}\right| = \left|\frac{f(t)-f(a)+f(t)-f(b)}{2}\right| \le \frac{1}{2}\left(|f(t)-f(a)| + |f(b)-f(t)|\right) \le \frac{1}{2}\bigvee_a^b(f) \tag{3.2}$$

for any $t \in [a,b]$.

It is well known that if $p \in \mathcal{R}(u,[a,b])$ where $u \in \mathcal{BV}_{\mathbb{C}}[a,b]$ then we have [1, p. 177]

$$\left|\int_a^b p(t)\,du(t)\right| \le \int_a^b |p(t)|\,d\left(\bigvee_a^t(u)\right) \le \sup_{t\in[a,b]}|p(t)|\bigvee_a^b(u). \tag{3.3}$$

Using the equality (2.5), (3.2) and (3.3) we get

$$\left|\int_a^b f(t)\,g(t)\,du(t) - \frac{f(a)+f(b)}{2}\int_a^b g(t)\,du(t)\right| \le \int_a^b \left|f(t) - \frac{f(a)+f(b)}{2}\right| |g(t)|\,d\left(\bigvee_a^t(u)\right) \le \frac{1}{2}\bigvee_a^b(f)\int_a^b |g(t)|\,d\left(\bigvee_a^t(u)\right) \le \frac{1}{2}\max_{t\in[a,b]}|g(t)|\bigvee_a^b(f)\bigvee_a^b(u), \tag{3.4}$$

which proves (3.1). □

Remark 2. *If $g(t) = 1$, $t \in [a,b]$, then by (3.1) we get*

$$\left|\int_a^b f(t)\,du(t) - \frac{f(a)+f(b)}{2}[u(b)-u(a)]\right| \leq \frac{1}{2}\bigvee_a^b(f)\bigvee_a^b(u). \tag{3.5}$$

This result was obtained in [8] *in which the constant $\frac{1}{2}$ was also shown to be best.*

Corollary 2. *Assume that $f \in \mathcal{C}_{\mathbb{C}}[a,b] \cap \mathcal{BV}_{\mathbb{C}}[a,b]$ and $u \in \mathcal{BV}_{\mathbb{C}}[a,b]$. If g is such that $|g|$ is convex on $[a,b]$, then*

$$\begin{aligned}
&\left|\int_a^b f(t)\,g(t)\,du(t) - \frac{f(a)+f(b)}{2}\int_a^b g(t)\,du(t)\right| \\
&\leq \frac{1}{2(b-a)}\left[|g(a)|\int_a^b\left(\bigvee_a^t(u)\right)dt + |g(b)|\int_a^b\left(\bigvee_t^b(u)\right)dt\right]\bigvee_a^b(f) \\
&\qquad\leq \frac{|g(a)|+|g(b)|}{2}\bigvee_a^b(f)\bigvee_a^b(u).
\end{aligned} \tag{3.6}$$

Proof. Since $|g|$ is convex on $[a,b]$, then

$$|g(t)| = \left|g\left(\frac{(b-t)a+(t-a)b}{b-a}\right)\right| \leq \frac{(b-t)|g(a)|+(t-a)|g(b)|}{b-a}$$

for $t \in [a,b]$.

Since $\bigvee_a^{\cdot}(u)$ is monotonic nondecreasing, then

$$\begin{aligned}
&\int_a^b |g(t)|\,d\left(\bigvee_a^t(u)\right) \\
&\leq \int_a^b\left[\frac{(b-t)|g(a)|+(t-a)|g(b)|}{b-a}\right]d\left(\bigvee_a^t(u)\right) \\
&= \frac{|g(a)|}{b-a}\int_a^b(b-t)\,d\left(\bigvee_a^t(u)\right) + \frac{|g(b)|}{b-a}\int_a^b(t-a)\,d\left(\bigvee_a^t(u)\right).
\end{aligned} \tag{3.7}$$

Using the integration by parts formula, we have

$$\int_a^b(b-t)\,d\left(\bigvee_a^t(u)\right) = (b-t)\bigvee_a^t(u)\Big|_a^b + \int_a^b\left(\bigvee_a^t(u)\right)dt = \int_a^b\left(\bigvee_a^t(u)\right)dt$$

and

$$\begin{aligned}\int_a^b (t-a)\, d\left(\bigvee_a^t (u)\right) &= (t-a)\bigvee_a^t (u)\Big|_a^b - \int_a^b \left(\bigvee_a^t (u)\right) dt\\ &= (b-a)\bigvee_a^b (u) - \int_a^b \left(\bigvee_a^t (u)\right) dt\\ &= \int_a^b \left(\bigvee_a^b (u) - \bigvee_a^t (u)\right) dt = \int_a^b \left(\bigvee_t^b (u)\right) dt.\end{aligned}$$

By making use of (3.7) we get the first inequality in (3.6).

Also, observe that

$$\int_a^b \left(\bigvee_a^t (u)\right) dt \le (b-a)\bigvee_a^b (u) \text{ and } \int_a^b \left(\bigvee_t^b (u)\right) dt \le (b-a)\bigvee_a^b (u),$$

which proves the last part of (3.6). ☐

4. Inequalities for Lipschitzian Integrands

The following result also holds:

Theorem 5. *Assume that f satisfies the end-point Lipschitzian conditions*

$$|f(t) - f(a)| \le L_a (t-a)^\alpha \text{ and } |f(b) - f(t)| \le L_b (b-t)^\beta \tag{4.1}$$

for any $t \in (a,b)$ where the constants $L_a,\ L_b > 0$ and $\alpha,\ \beta > 0$ are given. If $g \in \mathcal{C}_{\mathbb{C}}[a,b]$ and $u \in \mathcal{BV}_{\mathbb{C}}[a,b]$, then

$$\begin{aligned}&\left|\int_a^b f(t)\, g(t)\, du(t) - \frac{f(a)+f(b)}{2}\int_a^b g(t)\, du(t)\right| \\ &\le \frac{1}{2}\left[L_a \int_a^b (t-a)^\alpha |g(t)|\, d\left(\bigvee_a^t (u)\right) + L_b \int_a^b (b-t)^\beta |g(t)|\, d\left(\bigvee_a^t (u)\right)\right]\\ &\le \frac{1}{2}\max_{t\in[a,b]} |g(t)| \left[\alpha L_a \int_a^b (t-a)^{\alpha-1}\left(\bigvee_t^b (u)\right) dt + \beta L_b \int_a^b (b-t)^{\beta-1}\left(\bigvee_a^t (u)\right) dt\right]\\ &\le \frac{1}{2}\max_{t\in[a,b]} |g(t)| \left[L_a (b-a)^\alpha + L_b (b-a)^\beta\right] \bigvee_a^b (u).\end{aligned} \tag{4.2}$$

Proof. Since f satisfies the condition (4.1) on $[a,b]$, hence

$$\begin{aligned}\left|f(t) - \frac{f(a)+f(b)}{2}\right| &= \left|\frac{f(t)-f(a)+f(t)-f(b)}{2}\right|\\ &\le \frac{1}{2}\left(|f(t)-f(a)| + |f(b)-f(t)|\right)\\ &\le \frac{1}{2}\left[L_a (t-a)^\alpha + L_\beta (b-t)^\beta\right]\end{aligned} \tag{4.3}$$

for any $t \in (a,b)$.

Using the first part of inequality (3.4) and the inequality (4.3), then we have

$$\begin{aligned}
&\left|\int_a^b f(t)\,g(t)\,du(t) - \frac{f(a)+f(b)}{2}\int_a^b g(t)\,du(t)\right| \\
&\leq \int_a^b \left|f(t) - \frac{f(a)+f(b)}{2}\right| |g(t)|\, d\left(\bigvee_a^t(u)\right) \\
&\leq \frac{1}{2}\int_a^b \left[L_a (t-a)^{\alpha} + L_b (b-t)^{\beta}\right] |g(t)|\, d\left(\bigvee_a^t(u)\right) \\
&= \frac{1}{2}\left[L_a \int_a^b (t-a)^{\alpha} |g(t)|\, d\left(\bigvee_a^t(u)\right) + L_b \int_a^b (b-t)^{\beta} |g(t)|\, d\left(\bigvee_a^t(u)\right)\right] =: B(g,u),
\end{aligned}$$

which proves the first inequality in (4.2).

We also have

$$B(g,u) \leq \frac{1}{2}\max_{t\in[a,b]} |g(t)| \left[L_a \int_a^b (t-a)^{\alpha}\, d\left(\bigvee_a^t(u)\right) + L_b \int_a^b (b-t)^{\beta}\, d\left(\bigvee_a^t(u)\right)\right]. \tag{4.4}$$

Using the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\begin{aligned}
&\int_a^b (t-a)^{\alpha}\, d\left(\bigvee_a^t(u)\right) \\
&= (t-a)^{\alpha} \bigvee_a^t(u)\Big|_a^b - \alpha\int_a^b (t-a)^{\alpha-1}\left(\bigvee_a^t(u)\right) dt \\
&= (b-a)^{\alpha} \bigvee_a^b(u) - \alpha\int_a^b (t-a)^{\alpha-1}\left(\bigvee_a^t(u)\right) dt \\
&= \alpha \bigvee_a^b(u) \int_a^b (t-a)^{\alpha-1}\, dt - \alpha\int_a^b (t-a)^{\alpha-1}\left(\bigvee_a^t(u)\right) dt \\
&= \alpha\int_a^b (t-a)^{\alpha-1}\left(\bigvee_a^b(u) - \bigvee_a^t(u)\right) dt = \alpha\int_a^b (t-a)^{\alpha-1}\left(\bigvee_t^b(u)\right) dt
\end{aligned}$$

and

$$\begin{aligned}
\int_a^b (b-t)^{\beta}\, d\left(\bigvee_a^t(u)\right) &= (b-t)^{\beta} \bigvee_a^t(u)\Big|_a^b + \beta\int_a^b (b-t)^{\beta-1}\left(\bigvee_a^t(u)\right) dt \\
&= \beta\int_a^b (b-t)^{\beta-1}\left(\bigvee_a^t(u)\right) dt
\end{aligned}$$

and by (4.4) we obtain the second part of (4.2).

Using the fact that the function $\bigvee_a^{\cdot}(u)$ is nondecreasing and $\bigvee_{\cdot}^{b}(u)$ is nonincreasing, then

$$\begin{aligned}
&\alpha L_a \int_a^b (t-a)^{\alpha-1}\left(\bigvee_t^b (u)\right) dt + \beta L_b \int_a^b (b-t)^{\beta-1}\left(\bigvee_a^t (u)\right) dt \\
&\leq \alpha L_a \bigvee_a^b (u) \int_a^b (t-a)^{\alpha-1}\, dt + \beta L_b \bigvee_a^b (u) \int_a^b (b-t)^{\beta-1}\, dt \\
&= \left[L_a (b-a)^{\alpha} + L_b (b-a)^{\beta}\right] \bigvee_a^b (u),
\end{aligned}$$

which proves the last part of (4.2). □

Remark 3. *If $g(t)=1$, $t\in[a,b]$, then by (4.2) we get*

$$\begin{aligned}
(4.5)\quad &\left|\int_a^b f(t)\,du(t) - \frac{f(a)+f(b)}{2}\left[u(b)-u(a)\right]\right| \\
&\leq \frac{1}{2}\left[\alpha L_a \int_a^b (t-a)^{\alpha-1}\left(\bigvee_t^b (u)\right) dt + \beta L_b \int_a^b (b-t)^{\beta-1}\left(\bigvee_a^t (u)\right) dt\right] \\
&\leq \frac{1}{2}\left[L_a (b-a)^{\alpha} + L_b (b-a)^{\beta}\right] \bigvee_a^b (u),
\end{aligned}$$

where f satisfies the condition (4.1) and u is of bounded variation on $[a,b]$.

If we assume that f is Lipschitzian with the constant $L>0$, then by taking $\alpha=\beta=1$ and $L_a=L_b=L$ in the first inequality in (4.2), we get

$$\begin{aligned}
(4.6)\quad &\left|\int_a^b f(t)\,g(t)\,du(t) - \frac{f(a)+f(b)}{2}\int_a^b g(t)\,du(t)\right| \\
&\leq \frac{1}{2} L (b-a) \int_a^b |g(t)|\, d\left(\bigvee_a^t (u)\right)
\end{aligned}$$

Corollary 3. *Assume that f satisfies the end-point Lipschitzian conditions*

$$(4.7)\qquad |f(t)-f(a)| \leq L_a (t-a)^{\alpha} \text{ and } |f(b)-f(t)| \leq L_b (b-t)^{\alpha}$$

for any $t \in (a,b)$ *where the constants* L_a, $L_b > 0$ *and* $\alpha > 0$ *are given. If* $g \in \mathcal{C}_{\mathbb{C}}[a,b]$ *and* $u \in \mathcal{BV}_{\mathbb{C}}[a,b]$, *then*

$$\left|\int_a^b f(t)\,g(t)\,du(t) - \frac{f(a)+f(b)}{2}\int_a^b g(t)\,du(t)\right| \tag{4.8}$$
$$\leq \frac{1}{2}\left[L_a\int_a^b (t-a)^{\alpha}|g(t)|\,d\left(\bigvee_a^t(u)\right) + L_b\int_a^b (b-t)^{\alpha}|g(t)|\,d\left(\bigvee_a^t(u)\right)\right]$$
$$\leq \frac{1}{2}\max\{L_a,L_b\}\int_a^b [(t-a)^{\alpha}+(b-t)^{\alpha}]\,|g(t)|\,d\left(\bigvee_a^t(u)\right)$$
$$\leq \frac{1}{2}\max\{L_a,L_b\}\max_{t\in[a,b]}|g(t)|\int_a^b [(t-a)^{\alpha}+(b-t)^{\alpha}]\,d\left(\bigvee_a^t(u)\right).$$

We also have

Corollary 4. *Assume that* f *satisfies the end-point Lipschitzian conditions (4.1) and* $u \in \mathcal{BV}_{\mathbb{C}}[a,b]$,. *If* g *is such that* $|g|$ *is convex on* $[a,b]$, *then*

$$\left|\int_a^b f(t)\,g(t)\,du(t) - \frac{f(a)+f(b)}{2}\int_a^b g(t)\,du(t)\right| \leq I(g,u) \tag{4.9}$$

where

$$I(g,u) := \frac{1}{2}\frac{L_a}{b-a}\left[|g(a)|\int_a^b [(\alpha+1)t-a-\alpha b](t-a)^{\alpha-1}\bigvee_a^t(u)\,dt\right.$$
$$\left. + |g(b)|(\alpha+1)\int_a^b (t-a)^{\alpha}\bigvee_t^b(u)\,dt\right]$$
$$+\frac{1}{2}\frac{L_b}{b-a}\left[|g(a)|(\beta+1)\int_a^b (b-t)^{\beta}\bigvee_a^t(u)\,dt\right.$$
$$\left. + |g(b)|\int_a^b [(\beta+1)t-\beta a-b](b-t)^{\beta-1}\bigvee_a^t(u)\,dt\right].$$

Proof. By the convexity of $|g|$ we have

$$\int_a^b (t-a)^{\alpha}|g(t)|\,d\left(\bigvee_a^t(u)\right) \tag{4.10}$$
$$\leq \int_a^b (t-a)^{\alpha}\left[\frac{(b-t)|g(a)|+(t-a)|g(b)|}{b-a}\right]d\left(\bigvee_a^t(u)\right)$$
$$= \frac{1}{b-a}\int_a^b \left[(t-a)^{\alpha}(b-t)|g(a)| + (t-a)^{\alpha+1}|g(b)|\right]d\left(\bigvee_a^t(u)\right)$$
$$= \frac{1}{b-a}\left[|g(a)|\int_a^b (t-a)^{\alpha}(b-t)\,d\left(\bigvee_a^t(u)\right) + |g(b)|\int_a^b (t-a)^{\alpha+1}\,d\left(\bigvee_a^t(u)\right)\right]$$

and

$$\begin{aligned}
(4.11)\quad & \int_a^b (b-t)^\beta |g(t)|\, d\left(\bigvee_a^t (u)\right) \\
& \leq \int_a^b (b-t)^\beta \left[\frac{(b-t)|g(a)|+(t-a)|g(b)|}{b-a}\right] d\left(\bigvee_a^t (u)\right) \\
& = \frac{1}{b-a}\int_a^b \left[(b-t)^{\beta+1}|g(a)|+(t-a)(b-t)^\beta |g(b)|\right] d\left(\bigvee_a^t (u)\right) \\
& = \frac{1}{b-a}\left[|g(a)|\int_a^b (b-t)^{\beta+1}\, d\left(\bigvee_a^t (u)\right)+|g(b)|\int_a^b (t-a)(b-t)^\beta\, d\left(\bigvee_a^t (u)\right)\right].
\end{aligned}$$

Using the integration by parts formula for Riemann-Stieltjes integral, we have

$$\begin{aligned}
\int_a^b (t-a)^\alpha (b-t)\, d\left(\bigvee_a^t (u)\right) &= -\int_a^b \left[\alpha (t-a)^{\alpha-1}(b-t)-(t-a)^\alpha\right]\left(\bigvee_a^t (u)\right) dt \\
&= \int_a^b \left[(t-a)^\alpha-\alpha (t-a)^{\alpha-1}(b-t)\right]\left(\bigvee_a^t (u)\right) dt \\
&= \int_a^b \left[(\alpha+1)t-a-\alpha b\right](t-a)^{\alpha-1}\left(\bigvee_a^t (u)\right) dt,
\end{aligned}$$

$$\begin{aligned}
\int_a^b (t-a)^{\alpha+1}\, d\left(\bigvee_a^t (u)\right) &= \int_a^b (t-a)^{\alpha+1}\, d\left(\bigvee_a^b (u)-\bigvee_t^b (u)\right) \\
&= -\int_a^b (t-a)^{\alpha+1}\, d\left(\bigvee_t^b (u)\right) \\
&= (\alpha+1)\int_a^b (t-a)^\alpha \left(\bigvee_t^b (u)\right) dt,
\end{aligned}$$

$$\int_a^b (b-t)^{\beta+1}\, d\left(\bigvee_a^t (u)\right) = (\beta+1)\int_a^b (b-t)^\beta \left(\bigvee_a^t (u)\right) dt$$

and

$$\begin{aligned}
\int_a^b (t-a)(b-t)^\beta\, d\left(\bigvee_a^t (u)\right) &= -\int_a^b \frac{d}{dt}\left[(t-a)(b-t)^\beta\right]\bigvee_a^t (u)\, dt \\
&= \int_a^b \left[\beta (t-a)-(b-t)\right](b-t)^{\beta-1}\bigvee_a^t (u)\, dt \\
&= \int_a^b \left[(\beta+1)t-\beta a-b\right](b-t)^{\beta-1}\bigvee_a^t (u)\, dt
\end{aligned}$$

Therefore, by (4.10) and (4.11) we have

$$
\begin{aligned}
&\frac{1}{2}\left[L_a\int_a^b (t-a)^{\alpha}|g(t)|\,d\left(\bigvee_a^t(u)\right)+L_b\int_a^b (b-t)^{\beta}|g(t)|\,d\left(\bigvee_a^t(u)\right)\right]\\
&\le \frac{1}{2}\frac{L_a}{b-a}\left[|g(a)|\int_a^b (t-a)^{\alpha}(b-t)\,d\left(\bigvee_a^t(u)\right)+|g(b)|\int_a^b (t-a)^{\alpha+1}d\left(\bigvee_a^t(u)\right)\right]\\
&+\frac{1}{2}\frac{L_b}{b-a}\left[|g(a)|\int_a^b (b-t)^{\beta+1}d\left(\bigvee_a^t(u)\right)+|g(b)|\int_a^b (t-a)(b-t)^{\beta}d\left(\bigvee_a^t(u)\right)\right]\\
&\qquad\le \frac{1}{2}\frac{L_a}{b-a}\left[|g(a)|\int_a^b [(\alpha+1)t-a-\alpha b](t-a)^{\alpha-1}\left(\bigvee_a^t(u)\right)dt\right.\\
&\qquad\qquad\left.+|g(b)|(\alpha+1)\int_a^b (t-a)^{\alpha}\left(\bigvee_t^b(u)\right)dt\right]\\
&\qquad\quad+\frac{1}{2}\frac{L_b}{b-a}\left[|g(a)|(\beta+1)\int_a^b (b-t)^{\beta}\left(\bigvee_a^t(u)\right)dt\right.\\
&\qquad\qquad\left.+|g(b)|\int_a^b [(\beta+1)t-\beta a-b](b-t)^{\beta-1}\left(\bigvee_a^t(u)\right)dt\right],
\end{aligned}
$$

which proves the required inequality (4.9). □

Remark 4. *For $\alpha=\beta=1$ and $L_a=L_b=L$, we have*

$$
I(g,u):=\frac{L}{b-a}\left[|g(b)|\int_a^b (t-a)\bigvee_t^b(u)\,dt+|g(a)|\int_a^b (b-t)\bigvee_a^t(u)\,dt\right].
$$

Therefore, if f is Lipschitzian with the constant $L>0$, u of bounded variation and g is such that $|g|$ is convex on $[a,b]$, then we have the simple inequality of interest

$$
\begin{aligned}
&\left|\int_a^b f(t)\,g(t)\,du(t)-\frac{f(a)+f(b)}{2}\int_a^b g(t)\,du(t)\right| \qquad (4.12)\\
&\le \frac{L}{b-a}\left[|g(b)|\int_a^b (t-a)\bigvee_t^b(u)\,dt+|g(a)|\int_a^b (b-t)\bigvee_a^t(u)\,dt\right]\\
&\qquad\le \frac{|g(a)|+|g(b)|}{2}L(b-a)\bigvee_a^b(u).
\end{aligned}
$$

5. Applications for Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Let $A\in\mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda\in\mathbb{R}$ as follows

$$
\varphi_\lambda(s):=\begin{cases}1, & \text{for } -\infty<s\le\lambda,\\ \\ 0, & \text{for } \lambda<s<+\infty.\end{cases}
$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$E_{\lambda} := \varphi_{\lambda}(A) \tag{5.1}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [9, p. 256]:

Theorem 6 (Spectral Representation Theorem)**.** *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a = \min\{\lambda \,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $b = \max\{\lambda \,|\lambda \in Sp(A)\} =: \max Sp(A)$. Then there exists a family of projections $\{E_{\lambda}\}_{\lambda \in \mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_{\lambda} \leq E_{\lambda'}$ *for* $\lambda \leq \lambda'$;
b) $E_{a-0} = 0, E_b = I$ *and* $E_{\lambda+0} = E_{\lambda}$ *for all* $\lambda \in \mathbb{R}$;
c) *We have the representation*

$$A = \int_{a-0}^{b} \lambda dE_{\lambda}.$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ there exists a unique operator $\varphi(A) \in \mathcal{B}(H)$ such that for every $\varepsilon > 0$ there exists a $\delta > 0$ satisfying the inequality

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi\left(\lambda_k'\right)\left[E_{\lambda_k} - E_{\lambda_{k-1}}\right] \right\| \leq \varepsilon$$

whenever

$$\begin{cases} \lambda_0 < a = \lambda_1 < \dots < \lambda_{n-1} < \lambda_n = b, \\ \\ \lambda_k - \lambda_{k-1} \leq \delta \text{ for } 1 \leq k \leq n, \\ \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \leq k \leq n \end{cases}$$

this means that

$$\varphi(A) = \int_{a-0}^{b} \varphi(\lambda)\, dE_{\lambda}, \tag{5.2}$$

where the integral is of Riemann-Stieltjes type.

Corollary 5. *With the assumptions of Theorem 6 for A, E_{λ} and φ we have the representations*

$$\varphi(A)\, x = \int_{a-0}^{b} \varphi(\lambda)\, dE_{\lambda} x \quad \text{for all } x \in H$$

and

$$\langle \varphi(A)\, x, y\rangle = \int_{a-0}^{b} \varphi(\lambda)\, d\langle E_{\lambda} x, y\rangle \quad \text{for all } x,\ y \in H. \tag{5.3}$$

In particular,

$$\langle \varphi(A)\, x, x\rangle = \int_{a-0}^{b} \varphi(\lambda)\, d\langle E_{\lambda} x, x\rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(A)x\|^2 = \int_{a-0}^{b} |\varphi(\lambda)|^2 \, d\|E_\lambda x\|^2 \quad \text{for all } x \in H.$$

We need the following result that provides an upper bound for the total variation of the function $\mathbb{R} \ni \lambda \mapsto \langle E_\lambda x, y\rangle \in \mathbb{C}$ on an interval $[\alpha, \beta]$, see [7].

Lemma 2. *Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A. Then for any $x,\ y \in H$ and $\alpha < \beta$ we have the inequality*

$$\left[\bigvee_{\alpha}^{\beta}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right]^2 \le \langle (E_\beta - E_\alpha)x, x\rangle \langle (E_\beta - E_\alpha)y, y\rangle, \tag{5.4}$$

where $\bigvee_{\alpha}^{\beta}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$ denotes the total variation of the function $\left\langle E_{(\cdot)}x, y\right\rangle$ on $[\alpha, \beta]$.

Remark 5. *For $\alpha = a - \varepsilon$ with $\varepsilon > 0$ and $\beta = b$ we get from (5.4) the inequality*

$$\bigvee_{a-\varepsilon}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \le \langle (I - E_{a-\varepsilon})x, x\rangle^{1/2} \langle (I - E_{a-\varepsilon})y, y\rangle^{1/2} \tag{5.5}$$

for any $x,\ y \in H$.

This implies, for any $x,\ y \in H$, that

$$\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \le \|x\| \, \|y\|, \tag{5.6}$$

where $\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$ denotes the limit $\lim_{\varepsilon\to 0+}\left[\bigvee_{a-\varepsilon}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right]$.

We can state the following result for functions of selfadjoint operators:

Theorem 7. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a = \min\{\lambda \,|\, \lambda \in Sp(A)\} =: \min Sp(A)$ and $b = \max\{\lambda \,|\, \lambda \in Sp(A)\} =: \max Sp(A)$. Also, assume that $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A and $f : I \to \mathbb{C}$ is continuous on I, $[a, b] \subset \mathring{I}$ (the interior of I) with f of locally bounded variation on I.*

(i) *If $g : [a, b] \to \mathbb{C}$ is continuous on $[a, b]$, then*

$$\begin{aligned} &\left|\langle f(A)\,g(A)x, y\rangle - \frac{f(a) + f(b)}{2}\langle g(A)x, y\rangle\right| \\ &\qquad \le \frac{1}{2}\bigvee_{a}^{b}(f)\int_{a}^{b}|g(t)|\, d\left(\bigvee_{a-0}^{t}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right) \\ &\qquad \le \frac{1}{2}\max_{t\in[a,b]}|g(t)|\bigvee_{a}^{b}(f)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \le \frac{1}{2}\max_{t\in[a,b]}|g(t)|\bigvee_{a}^{b}(f)\,\|x\|\,\|y\| \end{aligned} \tag{5.7}$$

for all $x, y \in H$.

(ii) *If* $|g|$ *is convex on* $[a,b]$, *then*

$$\left|\langle f(A)\,g(A)\,x,y\rangle-\frac{f(a)+f(b)}{2}\langle g(A)\,x,y\rangle\right| \tag{5.8}$$
$$\leq\frac{1}{2(b-a)}\bigvee_{a}^{b}(f)$$
$$\times\left[|g(a)|\int_{a}^{b}\left(\bigvee_{a-0}^{t}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right)dt+|g(b)|\int_{a}^{b}\left(\bigvee_{t}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right)dt\right]$$
$$\leq\frac{|g(a)|+|g(b)|}{2}\bigvee_{a}^{b}(f)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\leq\frac{|g(a)|+|g(b)|}{2}\bigvee_{a}^{b}(f)\,\|x\|\,\|y\|$$

for all $x,y\in H$.

Proof. (i) If we use the inequality (3.1), we have for small $\varepsilon>0$ and for any x, $y\in H$ that

$$\left|\int_{a-\varepsilon}^{b}f(t)\,g(t)\,d\langle E_{t}x,y\rangle-\frac{f(a-\varepsilon)+f(b)}{2}\int_{a-\varepsilon}^{b}g(t)\,d\langle E_{t}x,y\rangle\right|$$
$$\leq\frac{1}{2}\bigvee_{a-\varepsilon}^{b}(f)\int_{a-\varepsilon}^{b}|g(t)|\,d\left(\bigvee_{a-\varepsilon}^{t}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right)\leq\frac{1}{2}\max_{t\in[a-\varepsilon,b]}|g(t)|\bigvee_{a-\varepsilon}^{b}(f)\bigvee_{a-\varepsilon}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right).$$

Taking the limit over $\varepsilon\to 0+$ and using the continuity of f, g and the Spectral Representation Theorem, we deduce the desired result (5.7).

(ii) Goes in a similar way by utilising the inequalities (3.6). □

Remark 6. *The above inequalities (5.7) and (5.8) can produce several particular examples of interest.*

For instance, if we take $g(t)=t-\frac{a+b}{2}$, *then by (5.7) we get*

$$\left|\left\langle f(A)\left(A-\frac{a+b}{2}1_{H}\right)x,y\right\rangle-\frac{f(a)+f(b)}{2}\left\langle\left(A-\frac{a+b}{2}1_{H}\right)x,y\right\rangle\right| \tag{5.9}$$
$$\leq\frac{1}{4}(b-a)\bigvee_{a}^{b}(f)\,\|x\|\,\|y\|$$

for $x,y\in H$.

If in this inequality we assume that $[a,b]\subset(0,\infty)$ *and take* $f(t)=\ln t$, *then we get*

$$\left|\left\langle\left(A-\frac{a+b}{2}1_{H}\right)\ln Ax,y\right\rangle-\frac{\ln a+\ln b}{2}\left\langle\left(A-\frac{a+b}{2}1_{H}\right)x,y\right\rangle\right| \tag{5.10}$$
$$\leq\frac{1}{4}(b-a)(\ln b-\ln a)\,\|x\|\,\|y\|$$

for $x,y\in H$.

Also, if $f(t) = t^r$ with $r > 0$ and $[a,b] \subset (0,\infty)$, then by (5.11) we get

$$\left|\left\langle \left(A - \frac{a+b}{2}1_H\right) A^r x, y\right\rangle - \frac{a^r+b^r}{2}\left\langle \left(A - \frac{a+b}{2}1_H\right) x, y\right\rangle\right| \le \frac{1}{4}(b-a)(b^r-a^r)\|x\|\|y\| \tag{5.11}$$

for $x, y \in H$.

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