Patterns in Pythagorean Triples Using Single Variable Procedures

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Abstract

The Pythagoras theorem is very famous in the literature of mathematics. The aim of this work is to extend in symmetrical way the some Pythagorean triples. These symmetric extensions are in such a way that we reach to good patterns. In some cases, the final sum also give a good pattern. In some cases examples are pandigital palindromic-type patterns. The patterns are obtained based on six procedures for single variable functions. The two variable functions are dealth in next work [5].

Contents

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RGMIA Res. Rep. Coll. 21 (2018), Art. 81, 40 pp. Received 20/07/18

1 Introduction

By Pythagoras theorem it is understood that

$$a^2 + b^2 = c^2, \ \forall a,\ b,\ c \in N_+$$

For simplicity, let's write it as (a, b, c) . Let's consider the initial Pythagorean triple (3,4,5). This means that $3^2 + 4^2 = 5^2$.

The symmetric extensions of above triple we call **patterns in Pythagorean triples**. There two obvious ways of getting **patterns in Pythagorean triples**.

1.1 Simple Patterns

- **Multiplication by 10, 100, 1000, ... :**

See two examples

$$\begin{aligned} 3^2 + 4^2 &= 5^2 := 25 \\ 30^2 + 40^2 &= 50^2 := 2500 \\ 300^2 + 400^2 &= 500^2 := 250000 \end{aligned} \tag{1}$$

$$\begin{aligned} 9^2 + 40^2 &= 41^2 := 1681 \\ 90^2 + 400^2 &= 410^2 := 168100 \\ 900^2 + 4000^2 &= 4100^2 := 16810000 \end{aligned} \tag{2}$$

The difference in above two examples is that the first one is with single digit in each case, i.e., 3, 4 and 5, and in second example (2) not all the same digits are with same length, i.e., the first one is of length 1 and other two are with length 2. This process is generally true for all types of **Pythagorean triples**.

- **Repetition of Digits:**

$$\begin{aligned} 3^2 + 4^2 &= 5^2 := 25 \\ 33^2 + 44^2 &= 55^2 := 3025 \\ 333^2 + 444^2 &= 555^2 := 308035 \end{aligned} \tag{3}$$

$$\begin{aligned} 12^2 + 35^2 &= 37^2 := 1369 \\ 1212^2 + 3535^2 &= 3737^2 := 13965169 \\ 121212^2 + 353535^2 &= 373737^2 := 139679345169 \end{aligned} \tag{4}$$

$$\begin{aligned} 119^2 + 120^2 &= 169^2 &&:= 28561 \\ 119119^2 + 120120^2 &= 169169^2 &&:= 28618150561 \\ 119119119^2 + 120120120^2 &= 169169169^2 &&:= 28618207740150561 \end{aligned} \tag{5}$$

In the examples (3), (4) and (5), we observe that the repetition of digits give us patterned results when we work with same length triples. Here we have (3,4,5), (12, 35, 27) and (119, 120, 169). In the example (3, 4, 5) each digit is of length 1. In the example (12, 35, 27) each digit is of length 2, and the triple (119, 120, 169) each digit is of length 3. This theory doesn't work when the triples are with different lengths, for example, (5, 12, 13). This gives

$$\begin{aligned} 5^2 + 12^2 = 13^2 &\quad \Rightarrow 169 = 169 \\ 55^2 + 1212^2 \neq 1313^2 &\Rightarrow 1471969 \neq 1723969 \end{aligned} \tag{6}$$

In this case the pattern (6) is not extendable as in case examples (3), (4) and (5).

Remark 1. *Analysing the final sums, we observe that the examples (1) and (2) are good for* ***patterns with final sums****. The example (3) is not so good but acceptable. In case of examples (4) and (5), the patterns with final sums are not good. In this work, we shall write patterns with final sums only if they are good.*

The aim of this work is to give different procedures to find Pythagorean triples, and then give examples giving patterns based on functional representations. This work is limited to single variable cases. Two variable cases are done in another part.

2 Procedures

2.1 Procedure 1

Let's consider the following three functions:

$$\begin{aligned} f_1(n) &:= 2n(n+1) \\ g_1(n) &:= 2n+1 \\ h_1(n) &:= 2n^2+2n+1 \end{aligned} \tag{7}$$

We can easily check that

$$\begin{aligned} f_1(n)^2 + g_1(n)^2 &= 4n^2(n+1)^2 + (2n+1)^2 \\ &= 4n^4 + 8n^3 + 4n^2 + +4n^2 + 4n + 1 \\ &= 4n^4 + 8n^3 + 8n^2 + 4n + 1 \\ &= (2n^2+2n+1)^2 \\ &= h_1(n)^2 \end{aligned}$$

This proves that the triple (f_1, g_1, h_1) is a **Pythagorean triple**. Let's see some particular values. For $n = 1, 2, ..., 9, 10$ in (7), we get

$$\begin{aligned}
(f_1(1), g_1(1), h_1(1)) &= (4, 3, 5)\\
(f_1(2), g_1(2), h_1(2)) &= (12, 5, 13)\\
(f_1(3), g_1(3), h_1(3)) &= (24, 7, 25)\\
(f_1(4), g_1(4), h_1(4)) &= (40, 9, 41)\\
(f_1(5), g_1(5), h_1(5)) &= (60, 11, 61)\\
(f_1(6), g_1(6), h_1(6)) &= (85, 13, 85)\\
(f_1(7), g_1(7), h_1(7)) &= (112, 15, 113)\\
(f_1(8), g_1(8), h_1(8)) &= (114, 17, 145)\\
(f_1(9), g_1(9), h_1(9)) &= (180, 19, 181)\\
(f_1(10), g_1(10), h_1(10)) &= (220, 21, 221)
\end{aligned}$$

We observe that the **Pythagorean triples** are primitive and out of three two values are consecutive. This procedure is very well known in the literature [1]. Examples of patterns in Pythagorean triples based on this procedure are given in subsection 3.1.

2.2 Procedure 2

Let's consider the following three functions:

$$\begin{aligned}
f_2(m) &:= m(m+2)\\
g_2(m) &:= 2(m+1)\\
h_2(m) &:= m^2 + 2m + 2
\end{aligned} \tag{8}$$

We can easily check that

$$\begin{aligned}
f_2(m)^2 + g_2(m)^2 &= (m(m+2))^2 + (2(m+1))^2\\
&= m^4 + 4m^3 + 4m^2 + 4m^2 + 8m + 4\\
&= m^4 + 4m^3 + 8m^2 + 8m + 4\\
&= (m^2 + 2m + 2)^2\\
&= h_2(m)^2
\end{aligned}$$

This proves that the triple (f_2, g_2, h_2) is a **Pythagorean triple**. Let's see some particular values. For $m = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ in (8), we get

$$\begin{aligned}
(f_2(1), g_2(1), h_2(1)) &= (3, 4, 5)\\
(f_2(2), g_2(2), h_2(2)) &= (8, 6, 10)\\
(f_2(3), g_2(3), h_2(3)) &= (15, 8, 17)\\
(f_2(4), g_2(4), h_2(4)) &= (24, 10, 26)\\
(f_2(5), g_2(5), h_2(5)) &= (35, 12, 37)\\
(f_2(6), g_2(6), h_2(6)) &= (48, 14, 50)\\
(f_2(7), g_2(7), h_2(7)) &= (63, 16, 65)\\
(f_2(8), g_2(8), h_2(8)) &= (80, 18, 82)\\
(f_2(9), g_2(9), h_2(9)) &= (99, 20, 101)\\
(f_2(10), g_3(10), h_5(10)) &= (120, 22, 122)
\end{aligned}$$

We observe that not all **Pythagorean triples** are primitive. While, in case of Procedure 1 given in (7), all the triples are with primitive values. Examples of patterns in Pythagorean triples based on this procedure are given in subsection 3.2.

2.3 Procedure 3

Let's consider the following three functions:

$$\begin{aligned} f_3(m) &:= m^2 - 1 \\ g_3(m) &:= 2m \\ h_3(m) &:= m^2 + 1 \end{aligned} \tag{9}$$

Then we can easily check that

$$\begin{aligned} f_3(m)^2 + g_3(m)^2 &= \left(m^2 - 1\right)^2 + (2m)^2 \\ &= m^4 - 2m^2 + 4m^2 + 1 \\ &= m^4 + 2m^2 + 1 \\ &= \left(m^2 + 1\right)^2 = h_3(m)^2 \end{aligned}$$

This proves that the triple (f_3, g_3, h_3) is a **Pythagorean triple**. Let's consider $m = 2, 3, 4, 5, 6, 7, 8, 9, 10$ in (9). This gives following triples:

$$\begin{aligned} (f_3(2), g_3(2), h_3(2)) &= (3, 4, 5) \\ (f_3(3), g_3(3), h_3(3)) &= (8, 6, 10) \\ (f_3(4), g_3(4), h_3(4)) &= (15, 8, 17) \\ (f_3(5), g_3(5), h_3(5)) &= (24, 10, 26) \\ (f_3(6), g_3(6), h_3(6)) &= (35, 12, 37) \\ (f_3(7), g_3(7), h_3(7)) &= (48, 14, 50) \\ (f_3(8), g_3(8), h_3(8)) &= (63, 16, 65) \\ (f_3(9), g_3(9), h_3(9)) &= (80, 18, 82) \\ (f_3(10), g_3(10), h_3(10)) &= (99, 20, 101) \end{aligned}$$

Remark 2. *Analysing the Procedures 2 and 3 given in (8) and (9) respectively, we observe that both the procedures given same results with difference of only one numbers, i.e.,*

$$(f_2(n), g_2(n), h_2(n)) = (g_3(n+1), f_3(n+1), h_3(n+1)).$$

In case of two variables both the procedures works separately (see [5]). Procedure 3 is very much known in the literature [1], but Procedure 2 seems to be new.

Examples of patterns in Pythagorean triples based on this procedure are given in subsection 3.3.

2.4 Procedure 4

Let's consider the following three functions:

$$\begin{aligned} f_4(m) &:= 4m^2 - 1 \\ g_4(m) &:= 4m \\ h_4(m) &:= 4m^2 + 1 \end{aligned} \tag{10}$$

Then we can easily check that

$$\begin{aligned} f_4(m)^2 + g_4(m)^2 &= \left(4m^2 - 1\right)^2) + (4m)^2 \\ &= 16m^4 - 8m^2 + 1 + 16m^2 \\ &= 16m^4 + 8m^2 + 1 \\ &= \left(4m^2 + 1\right)^2 = h_4(m)^2 \end{aligned}$$

This proves that the triple (f_4, g_4, h_4) is a **Pythagorean triple**. Let's consider $m = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ in (10). This gives following triples:

$$\begin{aligned} (f_4(1), g_4(1), h_4(1)) &= (3, 4, 5) \\ (f_4(2), g_4(2), h_4(2)) &= (15, 8, 17) \\ (f_4(3), g_4(3), h_4(3)) &= (35, 12, 37) \\ (f_4(4), g_4(4), h_4(4)) &= (63, 16, 65) \\ (f_4(5), g_4(5), h_4(5)) &= (99, 20, 101) \\ (f_4(6), g_4(6), h_4(6)) &= (143, 24, 145) \\ (f_4(7), g_4(7), h_4(7)) &= (195, 28, 197) \\ (f_4(8), g_4(8), h_4(8)) &= (255, 32, 257) \\ (f_4(9), g_4(9), h_4(9)) &= (323, 36, 325) \\ (f_4(10), g_4(10), h_4(10)) &= (399, 40, 401) \end{aligned}$$

The following property among Procedures 3 and 4 hold:

$$(f_4(m), g_4(m), h_4(m)) = (f_3(2m), g_3(2m), h_3(2m))$$

In particular, we have

$$\begin{aligned} (f_4(1), g_4(1), h_4(1)) = (f_3(2), g_3(2), h_3(2)) &= (3, 4, 5) \\ (f_4(2), g_4(2), h_4(2)) = (f_3(4), g_3(4), h_3(4)) &= (15, 8, 17) \\ (f_4(3), g_4(3), h_4(3)) = (f_3(6), g_3(6), h_3(6)) &= (35, 12, 37) \\ (f_4(4), g_4(4), h_4(4)) = (f_3(8), g_3(8), h_3(8)) &= (63, 16, 65) \\ (f_4(5), g_4(5), h_4(5)) = (f_3(10), g_3(10), h_3(10)) &= (99, 20, 101) \end{aligned}$$

Examples of patterns in Pythagorean triples based on this procedure are given in subsection 3.4.

2.5 Procedure 5

Let's consider the following three functions:

$$\begin{aligned} f_5(m) &:= 16m^2 - 1 \\ g_5(m) &:= 8m \\ h_5(m) &:= 16m^2 + 1 \end{aligned} \tag{11}$$

Then we can easily check that

$$\begin{aligned} f_5(m)^2 + g_5(m)^2 &= \left(16m^2 - 1\right)^2 + (8m)^2 \\ &= 256m^4 - 32m^2 + 1 + 64m^2 \\ &= 256m^4 + 32m^2 + 1 \\ &= \left(16m^2 + 1\right)^2 = h_5(m)^2. \end{aligned}$$

This proves that the triple (f_5, g_5, h_5) is a **Pythagorean triple**. Let's consider $m = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ in (11). This gives following triples:

$$\begin{aligned} (f_5(1), g_5(1), h_5(1)) &= (15, 8, 17) \\ (f_5(2), g_5(2), h_5(2)) &= (63, 16, 65) \\ (f_5(3), g_5(3), h_5(3)) &= (143, 24, 145) \\ (f_5(4), g_5(4), h_5(4)) &= (255, 32, 257) \\ (f_5(5), g_5(5), h_5(5)) &= (399, 40, 401) \\ (f_5(6), g_5(6), h_5(6)) &= (575, 48, 577) \\ (f_5(7), g_5(7), h_5(7)) &= (783, 56, 785) \\ (f_5(8), g_5(8), h_5(8)) &= (1023, 64, 1025) \\ (f_5(9), g_5(9), h_5(9)) &= (1295, 72, 1297) \\ (f_5(10), g_5(10), h_5(10)) &= (1599, 80, 1601) \end{aligned}$$

The following property among Procedures 4 and 5 hold:

$$(f_5(m), g_5(m), h_3(m)) = (f_4(2m), g_4(2m), h_4(2m))$$

In particular, we have

$$\begin{aligned} (f_5(1), g_5(1), h_5(1)) &= (f_4(2), g_4(2), h_4(2)) &&= (f_3(4), g_3(4), h_3(4)) &&= (15, 8, 17) \\ (f_5(2), g_5(2), h_5(2)) &= (f_4(4), g_4(4), h_4(4)) &&= (f_3(8), g_3(8), h_3(8)) &&= (63, 16, 65) \\ (f_5(3), g_5(3), h_5(3)) &= (f_4(6), g_4(6), h_4(6)) &&= (f_3(12), g_3(12), h_3(12)) &&= (143, 24, 145) \\ (f_5(4), g_5(4), h_5(4)) &= (f_4(8), g_4(8), h_4(8)) &&= (f_3(16), g_3(16), h_3(16)) &&= ((255, 32, 257) \\ (f_5(5), g_5(5), h_5(5)) &= (f_4(10), g_4(10), h_4(10)) &&= (f_3(12), g_3(20), h_3(20)) &&= (399, 40, 401)\,. \end{aligned}$$

Remark 3. *Procedures 1 and 4 are with primitive values, while procedures 2 and 3 are primitive and non-primitive values. The Procedure 5 again give all the triples primitive.*

Examples of patterns in Pythagorean triples based on this procedure are given in subsection 3.5.

3 Examples

This section brings examples of five procedures written above. These examples are patterned forms of Pythagorean triples.

3.1 Procedure 1

Below are some Pythagorean triples based on functional values given in (7).

- **For $n = 10, 100, 1000, ...$ in (7):**

$$\begin{aligned}
220^2 + 21^2 &= 221^2 && := 48841 \\
20200^2 + 201^2 &= 20201^2 && := 408080401 \\
2002000^2 + 2001^2 &= 2002001^2 && := 4008008004001 \\
200020000^2 + 20001^2 &= 200020001^2 && := 40008000800040001
\end{aligned} \tag{12}$$

- **For $n = 20, 200, 2000, ...$ in (7):**

$$\begin{aligned}
840^2 + 41^2 &= 841^2 \\
80400^2 + 401^2 &= 80401^2 && := 6464320801 \\
8004000^2 + 4001^2 &= 8004001^2 && := 64064032008001 \\
800040000^2 + 40001^2 &= 800040001^2 && := 640064003200080001
\end{aligned} \tag{13}$$

- **For $n = 30, 300, 3000, ...$ in (7):**

$$\begin{aligned}
1860^2 + 61^2 &= 1861^2 \\
180600^2 + 601^2 &= 180601^2 \\
18006000^2 + 6001^2 &= 18006100^2 && := 324216072012001 \\
1800060000^2 + 60001^2 &= 1800060001^2 && := 3240216007200120001 \\
180000600000^2 + 600001^2 &= 180000600001^2 && := 32400216000720001200001
\end{aligned} \tag{14}$$

- **For $n = 40, 400, 4000, ...$ in (7):**

$$\begin{aligned}
3280^2 + 81^2 &= 3281^2 \\
320800^2 + 801^2 &= 320801^2 \\
32008000^2 + 8001^2 &= 32008001^2 && := 1024512128016001 \\
3200080000^2 + 80001^2 &= 3200080001^2 && := 10240512012800160001 \\
320000800000^2 + 800001^2 &= 320000800001^2 && := 102400512001280001600001
\end{aligned} \tag{15}$$

- **For $n = 50, 500, 5000, ...$ in (7):**

$$\begin{aligned}
5100^2 + 101^2 &= 5101^2 \\
501000^2 + 1001^2 &= 501001^2 := 251002002001 \\
50010000^2 + 10001^2 &= 50010001^2 := 2501000200020001 \\
5000100000^2 + 100001^2 &= 5000100001^2 := 25001000020000200001
\end{aligned} \tag{16}$$

- **For $n = 600, 6000, 60000, ...$ in (7):**

$$\begin{aligned}
721200^2 + 1201^2 &= 721201^2 \\
72012000^2 + 12001^2 &= 72012001^2 \\
7200120000^2 + 120001^2 &= 7200120001^2 := 51841728028800240001 \\
720001200000^2 + 1200001^2 &= 720001200001^2 := 518401728002880002400001 \\
72000012000000^2 + 12000001^2 &= 72000012000001^2 := 5184001728000288000024000001
\end{aligned} \tag{17}$$

The first triple $(7320, 121, 7321)$ for $n = 60$ is not written above as it doesn't give good pattern.

- **For $n = 700, 7000, 70000, ...$ in (7):**

$$\begin{aligned}
981400^2 + 1401^2 &= 981401^2 \\
98014000^2 + 14001^2 &= 98014001^2 \\
9800140000^2 + 140001^2 &= 9800140001^2 := 96042744039200280001 \\
980001400000^2 + 1400001^2 &= 980001400001^2 := 960402744003920002800001 \\
98000014000000^2 + 14000001^2 &= 98000014000001^2 := 9604002744000392000028000001
\end{aligned} \tag{18}$$

The first triple $(9940, 141, 9941)$ for $m = 70$ is not written above as it doesn't give good pattern.

- **For $n = 800, 8000, 80000, ...$ in (7):**

$$\begin{aligned}
1281600^2 + 1601^2 &= 1281601^2 \\
128016000^2 + 16001^2 &= 128016001^2 \\
12800160000^2 + 160001^2 &= 12800160001^2 := 163844096051200320001 \\
1280001600000^2 + 1600001^2 &= 1280001600001^2 := 1638404096005120003200001 \\
128000016000000^2 + 16000001^2 &= 128000016000001^2 := 16384004096000512000032000001
\end{aligned} \tag{19}$$

The first triple $(12960, 161, 12961)$ for $m = 80$ is not written above as it doesn't give good pattern.

- **For $n = 900, 9000, 90000, ...$ in (7):**

$$\begin{aligned}
1621800^2 + 1801^2 &= 1621801^2 \\
162018000^2 + 18001^2 &= 162018001^2 \\
16200180000^2 + 180001^2 &= 16200180001^2 &&:= 262445832064800360001 \\
1620001800000^2 + 1800001^2 &= 1620001800001^2 &&:= 2624405832006480003600001 \\
162000018000000^2 + 18000001^2 &= 162000018000001^2 &&:= 26244005832000648000036000001
\end{aligned} \tag{20}$$

The first triple $(16380, 181, 16381)$ for $n = 90$ is not written above as it doesn't give good pattern.

- **For $n = 11, 101, 1001, 10001, ...$ in (7):**

$$\begin{aligned}
264^2 + 23^2 &= 265^2 \\
20604^2 + 203^2 &= 20605^2 &&:= 424566025 \\
2006004^2 + 2003^2 &= 2006005^2 &&:= 4024056060025 \\
200060004^2 + 20003^2 &= 200060005^2 &&:= 40024005600600025
\end{aligned} \tag{21}$$

- **For $n = 201, 2001, 20001, ...$ in (7):**

$$\begin{aligned}
81204^2 + 403^2 &= 81205^2 \\
8012004^2 + 4003^2 &= 8012005^2 &&:= 64192224120025 \\
800120004^2 + 40003^2 &= 800120005^2 &&:= 640192022401200025 \\
80001200004^2 + 400003^2 &= 80001200005^2 &&:= 6400192002240012000025
\end{aligned} \tag{22}$$

The first triple $(924, 43, 925)$ for $n = 21$ is not written above as it doesn't give good pattern.

- **For $n = 202, 2002, 20002, ...$ in (7):**

$$\begin{aligned}
82012^2 + 405^2 &= 82013^2 \\
8020012^2 + 4005^2 &= 8020013^2 &&:= 64320608520169 \\
800200012^2 + 40005^2 &= 800200013^2 &&:= 640320060805200169 \\
80002000012^2 + 400005^2 &= 80002000013^2 &&:= 6400320006080052000169
\end{aligned} \tag{23}$$

The first triple $(1012, 45, 1013)$ for $n = 22$ is not written above as it doesn't give good pattern.

- **For $n = 301, 3001, 30001, ...$ in (7):**

$$\begin{aligned}
181804^2 + 603^2 &= 181805^2 \\
18018004^2 + 6003^2 &= 18018005^2 &&:= 324648504180025 \\
1800180004^2 + 60003^2 &= 1800180005^2 &&:= 3240648050401800025 \\
180001800004^2 + 600003^2 &= 180001800005^2 &&:= 32400648005040018000025
\end{aligned} \tag{24}$$

The first triple $(1984, 63, 1985)$ for $n = 31$ is not written above as it doesn't give good pattern.

- **For $n = 302, 3002, 30002, ...$ in (7):**

$$\begin{aligned} 183012^2 + 605^2 &= 183013^2 \\ 18030012^2 + 6005^2 &= 18030013^2 \\ 1800300012^2 + 60005^2 &= 1800300013^2 &&:= 3241080136807800169 \\ 180003000012^2 + 600005^2 &= 180003000013^2 &&:= 32401080013680078000169 \\ 18000030000012^2 + 6000005^2 &= 18000030000013^2 &&:= 324001080001368000780000169 \end{aligned} \tag{25}$$

The first triple $(2112, 65, 2113)$ for $n = 32$ is not written above as it doesn't give good pattern.

- **For $n = 303, 3003, 30003, ...$ in (7):**

$$\begin{aligned} 184224^2 + 607^2 &= 184225^2 \\ 18042024^2 + 6007^2 &= 18042025^2 \\ 1800420024^2 + 60007^2 &= 1800420025^2 \\ 180004200024^2 + 600007^2 &= 180004200025^2 \end{aligned} \tag{26}$$

The first triple $(2244, 67, 2245)$ for $n = 33$ is not written above as it doesn't give good pattern.

- **For $n = 13, 133, 1333, 13333, ...$ in (7):**

$$\begin{aligned} 364^2 + 27^2 &= 365^2 \\ 35644^2 + 267^2 &= 35645^2 \\ 3556444^2 + 2667^2 &= 3556445^2 \\ 355564444^2 + 26667^2 &= 355564445^2 \end{aligned} \tag{27}$$

- **For $n = 1, 16, 166, 1666, ...$ in (7):**

$$\begin{aligned} 4^2 + 3^2 &= 5^2 \\ 544^2 + 33^2 &= 545^2 \\ 55444^2 + 333^2 &= 55445^2 \\ 5554444^2 + 3333^2 &= 5554445^2 \\ 555544444^2 + 33333^2 &= 555544445^2 \end{aligned} \tag{28}$$

- **For $n = 199, 1999, 19999, ...$ in (7):**

$$\begin{aligned} 79600^2 + 399^2 &= 79601^2 &&:= 6336319201 \\ 7996000^2 + 3999^2 &= 7996001^2 &&:= 63936031992001 \\ 799960000^2 + 39999^2 &= 799960001^2 &&:= 639936003199920001 \\ 79999600000^2 + 399999^2 &= 79999600001^2 &&:= 6399936000319999200001 \end{aligned} \tag{29}$$

The first triple $(760, 39, 761)$ for $n = 19$ is not written above as it doesn't give good pattern.

- **For $n = 29, 299, 2999, 29999, ...$ in (7):**

$$\begin{aligned} 1740^2 + 59^2 &= 1741^2 \\ 179400^2 + 599^2 &= 179401^2 &&:= 32184718801 \\ 17994000^2 + 5999^2 &= 17994001^2 &&:= 323784071988001 \\ 1799940000^2 + 59999^2 &= 1799940001^2 &&:= 3239784007199880001 \\ 179999400000^2 + 599999^2 &= 179999400001^2 &&:= 32399784000719998800001 \end{aligned} \tag{30}$$

- **For $n = 4, 49, 499, 4999, ...$ in (7):**

$$\begin{aligned} 40^2 + 9^2 &= 41^2 \\ 4900^2 + 99^2 &= 4901^2 &&:= 24019801 \\ 499000^2 + 999^2 &= 499001^2 &&:= 249001998001 \\ 49990000^2 + 9999^2 &= 49990001^2 &&:= 2499000199980001 \\ 4999900000^2 + 99999^2 &= 4999900001^2 &&:= 24999000019999800001 \end{aligned} \tag{31}$$

- **For $n = 1, 31, 331, 3331, 33331, ...$ in (7):**

$$\begin{aligned} 4^2 + 3^2 &= 5^2 \\ 1984^2 + 63^2 &= 1985^2 \\ 219784^2 + 663^2 &= 219785^2 \\ 22197784^2 + 6663^2 &= 22197785^2 \\ 2221977784^2 + 66663^2 &= 2221977785^2 \end{aligned} \tag{32}$$

- **For $n = 2, 32, 332, 3332, ...$ in (7):**

$$\begin{aligned} 12^2 + 5^2 &= 13^2 \\ 2112^2 + 65^2 &= 2113^2 \\ 221112^2 + 665^2 &= 221113^2 \\ 22211112^2 + 6665^2 &= 22211113^2 \\ 2222111112^2 + 66665^2 &= 2222111113^2 \end{aligned} \tag{33}$$

- **For $n = 3, 33, 333, 3333, ...$ in (7):**

$$\begin{aligned} 24^2 + 7^2 &= 25^2 \\ 2244^2 + 67^2 &= 2245^2 \\ 222444^2 + 667^2 &= 222445^2 \\ 22224444^2 + 6667^2 &= 22224445^2 \\ 2222244444^2 + 66667^2 &= 2222244445^2 \end{aligned} \tag{34}$$

- **For $n = 3, 36, 336, 3336, 33336, ...$ in (7):**

$$\begin{aligned}
24^2 + 7^2 &= 25^2 \\
2664^2 + 73^2 &= 2665^2 \\
226464^2 + 673^2 &= 226465^2 \\
22264464^2 + 6673^2 &= 22264465^2 \\
2222644464^2 + 66673^2 &= 2222644465^2
\end{aligned} \tag{35}$$

- **For $n = 339, 3339, 33339, ...$ in (7):**

$$\begin{aligned}
230520^2 + 679^2 &= 230521^2 \\
22304520^2 + 6679^2 &= 22304521^2 \\
2223044520^2 + 66679^2 &= 2223044521^2
\end{aligned} \tag{36}$$

The first two triples $(24, 7, 25)$ and $(3120, 79, 3121)$ for $n = 3$ and 39 are not written above as they don't give good pattern.

- **For $n = 1, 61, 661, 6661, 66661, ...$ in (7):**

$$\begin{aligned}
4^2 + 3^2 &= 5^2 \\
7564^2 + 123^2 &= 7565^2 \\
875164^2 + 1323^2 &= 875165^2 \\
88751164^2 + 13323^2 &= 88751165^2 \\
8887511164^2 + 133323^2 &= 8887511165^2
\end{aligned} \tag{37}$$

- **For $n = 2, 62, 662, 6662, 66662, ...$ in (7):**

$$\begin{aligned}
12^2 + 5^2 &= 13^2 \\
812^2 + 125^2 &= 7813^2 \\
877812^2 + 1325^2 &= 877813^2 \\
88777812^2 + 13325^2 &= 88777813^2 \\
8887777812^2 + 133325^2 &= 8887777813^2
\end{aligned} \tag{38}$$

- **For $n = 63, 663, 6663, 66663, ...$ in (7):**

$$\begin{aligned}
8064^2 + 127^2 &= 8065^2 \\
880464^2 + 1327^2 &= 880465^2 \\
88804464^2 + 13327^2 &= 88804465^2 \\
8888044464^2 + 133327^2 &= 8888044465^2
\end{aligned} \tag{39}$$

The first triple $(24, 7, 25)$ for $n = 3$ is not written above as it doesn't give good pattern.

- **For $n = 6, 66, 666, 6666, 66666, ...$ in (7):**

$$\begin{aligned} 84^2 + 13^2 &= 85^2 \\ 8844^2 + 133^2 &= 8845^2 \\ 888444^2 + 1333^2 &= 888445^2 \\ 88884444^2 + 13333^2 &= 88884445^2 \\ 8888844444^2 + 133333^2 &= 8888844445^2 \end{aligned} \tag{40}$$

- **For $n = 69, 669, 6669, 66669, ...$ in (7):**

$$\begin{aligned} 9660^2 + 139^2 &= 9661^2 \\ 896460^2 + 1339^2 &= 896461^2 \\ 88964460^2 + 13339^2 &= 88964461^2 \\ 8889644460^2 + 133339^2 &= 8889644461^2 \end{aligned} \tag{41}$$

The first triple $(84, 13, 85)$ for $n = 6$ is not written above as it doesn't give good pattern.

- **For $n = 699, 6999, 69999, 699999, ...$ in (7):**

$$\begin{aligned} 978600^2 + 1399^2 &= 978601 \\ 97986000^2 + 13999^2 &= 97986001 \\ 9799860000^2 + 139999^2 &= 9799860001 \\ 979998600000^2 + 1399999^2 &= 979998600001 \end{aligned} \tag{42}$$

The first two triples $(84, 13, 85)$ and $(9660, 139, 9661)$ for $n = 6$ and 69 are not written above as they don't give good pattern.

- **For $n = 991, 9991, 99991, ...$ in (7):**

$$\begin{aligned} 1966144^2 + 1983^2 &= 1966145^2 \\ 199660144^2 + 19983^2 &= 199660145^2 \\ 19996600144^2 + 199983^2 &= 19996600145^2 \end{aligned} \tag{43}$$

The first two triples $(4, 3, 5)$ and $(16744, 183, 16745)$ for $n = 1$ and 91 are not written above as they don't give good pattern.

- **For $n = 2, 92, 992, 9992, 99992, ...$ in (7):**

$$\begin{aligned} 12^2 + 5^2 &= 13^2 \\ 17112^2 + 185^2 &= 17113^2 \\ 1970112^2 + 1985^2 &= 1970113^2 \\ 199700112^2 + 19985^2 &= 199700113^2 \\ 19997000112^2 + 199985^2 &= 19997000113^2 \end{aligned} \tag{44}$$

- **For $n = 93, 993, 9993, 99993, ...$ in (7):**

$$\begin{aligned} 17484^2 + 187^2 &= 17485^2 \\ 1974084^2 + 1987^2 &= 1974085^2 \\ 199740084^2 + 19987^2 &= 199740085^2 \\ 19997400084^2 + 199987^2 &= 19997400085^2 \end{aligned} \tag{45}$$

The first triple $(24, 7, 25)$ for $n = 3$ is not written above as it doesn't give good pattern.

- **For $n = 96, 996, 9996, 99996, ...$ in (7):**

$$\begin{aligned} 18624^2 + 193^2 &= 18625^2 \\ 1986024^2 + 1993^2 &= 1986025^2 &&:= 3944295300625 \\ 199860024^2 + 19993^2 &= 199860025^2 &&:= 39944029593000625 \\ 19998600024^2 + 199993^2 &= 19998600025^2 &&:= 399944002959930000625 \end{aligned} \tag{46}$$

The first triple $(180, 19, 181)$ for $n = 9$ is not written above as it doesn't give good pattern.

- **For $n = 9, 99, 999, 9999, ...$ in (7):**

$$\begin{aligned} 180^2 + 19^2 &= 181^2 &&:= 32761 \\ 19800^2 + 199^2 &= 19801^2 &&:= 392079601 \\ 1998000^2 + 1999^2 &= 1998001^2 &&:= 3992007996001 \\ 199980000^2 + 19999^2 &= 199980001^2 &&:= 39992000799960001 \\ 19999800000^2 + 199999^2 &= 19999800001^2 &&:= 399992000079999600001 \end{aligned} \tag{47}$$

Remark 4. *Since out of three values in a triple, two are consecutive, obviously the patterns are with primitive values.*

3.2 Procedure 2

See below examples of patterns based on Procedure 2 given in subsection 2.2 and equation (8).

- **For $m = 10, 100, 1000, 10000, ...$ in (8):**

$$\begin{aligned} 120^2 + 22^2 &= 122^2 &&:= 14884 \\ 10200^2 + 202^2 &= 10202^2 &&:= 104080804 \\ 1002000^2 + 2002^2 &= 1002002^2 &&:= 1004008008004 \\ 100020000^2 + 20002^2 &= 100020002^2 &&:= 10004000800080004 \\ 10000200000^2 + 200002^2 &= 10000200002^2 &&:= 100004000080000800004 \\ 1000002000000^2 + 2000002^2 &= 1000002000002^2 &&:= 1000004000008000008000004 \end{aligned} \tag{48}$$

▶ **Division by 2**

$$\begin{aligned}
5100^2 + 101^2 &= 5101^2 \\
501000^2 + 1001^2 &= 501001^2 &:= 251002002001 \\
50010000^2 + 10001^2 &= 50010001^2 &:= 2501000200020001 \\
5000100000^2 + 100001^2 &= 5000100001^2 &:= 25001000020000200001
\end{aligned} \quad (49)$$

The first triple $(60, 11, 61)$ is not written above as it doesn't give good pattern.

- **For $m = 20, 200, 2000, 20000, ...$ in (8):**

$$\begin{aligned}
440^2 + 42^2 &= 442^2 \\
40400^2 + 402^2 &= 40402^2 &:= 1632321604 \\
4004000^2 + 4002^2 &= 4004002^2 &:= 16032032016004 \\
400040000^2 + 40002^2 &= 400040002^2 &:= 160032003200160004 \\
40000400000^2 + 400002^2 &= 40000400002^2 &:= 1600032000320001600004 \\
4000004000000^2 + 4000002^2 &= 4000004000002^2 &:= 16000032000032000016000004
\end{aligned} \quad (50)$$

▶ **Division by 2**

$$\begin{aligned}
220^2 + 21^2 &= 221^2 &:= 48841 \\
20200^2 + 201^2 &= 20201^2 &:= 408080401 \\
2002000^2 + 2001^2 &= 2002001^2 &:= 4008008004001 \\
200020000^2 + 20001^2 &= 200020001^2 &:= 40008000800040001 \\
20000200000^2 + 200001^2 &= 20000200001^2 &:= 400008000080000400001
\end{aligned} \quad (51)$$

- **For $m = 30, 300, 3000, 30000, ...$ in (8):**

$$\begin{aligned}
960^2 + 62^2 &= 962^2 \\
90600^2 + 602^2 &= 90602^2 \\
9006000^2 + 6002^2 &= 9006002^2 &:= 81108072024004 \\
900060000^2 + 60002^2 &= 900060002^2 &:= 810108007200240004 \\
90000600000^2 + 600002^2 &= 90000600002^2 &:= 8100108000720002400004 \\
9000006000000^2 + 6000002^2 &= 9000006000002^2 &:= 81000108000072000024000004
\end{aligned} \quad (52)$$

▶ **Division by 2**

$$\begin{aligned}
45300^2 + 301^2 &= 45301^2 \\
4503000^2 + 3001^2 &= 4503001^2 \\
450030000^2 + 30001^2 &= 450030001^2 &:= 202527001800060001 \\
45000300000^2 + 300001^2 &= 45000300001^2 &:= 2025027000180000600001
\end{aligned} \quad (53)$$

The first triple $(480, 31, 481)$ is not written above as it doesn't give good pattern.

- **For $m = 40, 400, 4000, 40000, \ldots$ in (8):**

$$\begin{aligned}
1680^2 + 82^2 &= 1682^2 \\
160800^2 + 802^2 &= 160802^2 \\
16008000^2 + 8002^2 &= 16008002^2 &&:= 256256128032004 \\
1600080000^2 + 80002^2 &= 1600080002^2 &&:= 2560256012800320004 \\
160000800000^2 + 800002^2 &= 160000800002^2 &&:= 25600256001280003200004 \\
16000008000000^2 + 8000002^2 &= 16000008000002^2 &&:= 256000256000128000032000004
\end{aligned} \tag{54}$$

▶ **Division by 2**

$$\begin{aligned}
840^2 + 41^2 &= 841^2 \\
80400^2 + 401^2 &= 80401^2 &&:= 6464320801 \\
8004000^2 + 4001^2 &= 8004001^2 &&:= 64064032008001 \\
800040000^2 + 40001^2 &= 800040001^2 &&:= 640064003200080001 \\
80000400000^2 + 400001^2 &= 80000400001^2 &&:= 6400064000320000800001
\end{aligned} \tag{55}$$

- **For $m = 500, 5000, 50000, \ldots$ in (8):**

$$\begin{aligned}
251000^2 + 1002^2 &= 251002^2 \\
25010000^2 + 10002^2 &= 25010002^2 &&:= 625500200040004 \\
2500100000^2 + 100002^2 &= 2500100002^2 &&:= 6250500020000400004 \\
250001000000^2 + 1000002^2 &= 250001000002^2 &&:= 62500500002000004000004 \\
25000010000000^2 + 10000002^2 &= 25000010000002^2 &&:= 625000500000200000040000004
\end{aligned} \tag{56}$$

The first triple $(2600, 102, 2602)$ for $m = 50$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
125500^2 + 501^2 &= 125501^2 \\
12505000^2 + 5001^2 &= 12505001^2 \\
1250050000^2 + 50001^2 &= 1250050001^2 \\
125000500000^2 + 500001^2 &= 125000500001^2
\end{aligned} \tag{57}$$

The first triple $(1300, 51, 1301)$ is not written above as it doesn't give good pattern.

- **For $m = 600, 6000, 60000, \ldots$ in (8):**

$$\begin{aligned}
361200^2 + 1202^2 &= 361202^2 \\
36012000^2 + 12002^2 &= 36012002^2 &&:= 1296864288048004 \\
3600120000^2 + 120002^2 &= 3600120002^2 &&:= 12960864028800480004 \\
360001200000^2 + 1200002^2 &= 360001200002^2 &&:= 129600864002880004800004
\end{aligned} \tag{58}$$

The first triple $(3720, 122, 3722)$ for $m = 60$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
1860^2 + 61^2 &= 1861^2 \\
180600^2 + 601^2 &= 180601^2 \\
18006000^2 + 6001^2 &= 18006001^2 &&:= 324216072012001 \\
1800060000^2 + 60001^2 &= 1800060001^2 &&:= 3240216007200120001 \\
180000600000^2 + 600001^2 &= 180000600001^2 &&:= 32400216000720001200001
\end{aligned} \tag{59}$$

- **For $m = 1, 11, 111, 1111, 11111, \ldots$ in (8):**

$$\begin{aligned}
3^2 + 4^2 &= 5^2 \\
143^2 + 24^2 &= 145^2 \\
12543^2 + 224^2 &= 12545^2 \\
1236543^2 + 2224^2 &= 1236545^2 \\
123476543^2 + 22224^2 &= 123476545^2 \\
12345876543^2 + 222224^2 &= 12345876545^2
\end{aligned} \tag{60}$$

- **For $m = 3, 33, 333, 3333, 3333, \ldots$ in (8):**

$$\begin{aligned}
15^2 + 8^2 &= 17^2 \\
1155^2 + 68^2 &= 1157^2 \\
111555^2 + 668^2 &= 111557^2 \\
11115555^2 + 6668^2 &= 11115557^2 \\
1111155555^2 + 66668^2 &= 1111155557^2 \\
111111555555^2 + 666668^2 &= 111111555557^2
\end{aligned} \tag{61}$$

- **For $m = 66, 666, 6666, 66666, ...$ in (8):**

$$\begin{aligned} 4488^2 + 134^2 &= 4490^2 \\ 444888^2 + 1334^2 &= 444890^2 \\ 44448888^2 + 13334^2 &= 44448890^2 \\ 4444488888^2 + 133334^2 &= 4444488890^2 \\ 444444888888^2 + 1333334^2 &= 444444888890^2 \end{aligned} \tag{62}$$

The first triple $(48, 14, 50)$ for $m = 6$ is not written above as it doesn't give good pattern.

- ▶ **Division by 2**

$$\begin{aligned} 24^2 + 7^2 &= 25^2 \\ 2244^2 + 67^2 &= 2245^2 \\ 222444^2 + 667^2 &= 222445^2 \\ 22224444^2 + 6667^2 &= 22224445^2 \\ 2222244444^2 + 66667^2 &= 2222244445^2 \\ 222222444444^2 + 666667^2 &= 222222444445^2 \\ 22222224444444^2 + 6666667^2 &= 22222224444445^2 \end{aligned} \tag{63}$$

- **For $m = 9, 99, 999, 9999, 99999, ...$ in (8):**

$$\begin{aligned} 99^2 + 20^2 &= 101^2 &&:= 10201 \\ 9999^2 + 200^2 &= 10001^2 &&:= 100020001 \\ 999999^2 + 2000^2 &= 1000001^2 &&:= 1000002000001 \\ 99999999^2 + 20000^2 &= 100000001^2 &&:= 10000000200000001 \\ 9999999999^2 + 200000^2 &= 10000000001^2 &&:= 100000000020000000001 \\ 999999999999^2 + 2000000^2 &= 1000000000001^2 &&:= 1000000000002000000000001 \end{aligned} \tag{64}$$

- **For $m = 1, 11, 101, 1001, 10001, ...$ in (8):**

$$\begin{aligned} 3^2 + 4^2 &= 5^2 \\ 143^2 + 24^2 &= 145^2 \\ 10403^2 + 204^2 &= 10405^2 &&:= 108264025 \\ 1004003^2 + 2004^2 &= 1004005^2 &&:= 1008026040025 \\ 100040003^2 + 20004^2 &= 100040005^2 &&:= 10008002600400025 \\ 10000400003^2 + 200004^2 &= 10000400005^2 &&:= 100008000260004000025 \end{aligned} \tag{65}$$

- **For $m = 202, 2002, 20002, ...$ in (8):**

$$\begin{aligned}
41208^2 + 406^2 &= 41210^2 \\
4012008^2 + 4006^2 &= 4012010^2 &:= 16096224240100 \\
400120008^2 + 40006^2 &= 400120010^2 &:= 160096022402400100 \\
40001200008^2 + 400006^2 &= 40001200010^2 &:= 1600096002240024000100
\end{aligned} \tag{66}$$

The first two triples $(8, 6, 10)$ and $(528, 46, 530)$ for $m = 2$ and 22 are not written above as they don't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
4^2 + 3^2 &= 5^2 \\
264^2 + 23^2 &= 265^2 \\
20604^2 + 203^2 &= 20605^2 &:= 424566025 \\
2006004^2 + 2003^2 &= 2006005^2 &:= 4024056060025 \\
200060004^2 + 20003^2 &= 200060005^2 &:= 40024005600600025 \\
20000600004^2 + 200003^2 &= 20000600005^2 &:= 400024000560006000025
\end{aligned} \tag{67}$$

- **For $m = 303, 3003, 30003, ...$ in (8):**

$$\begin{aligned}
92415^2 + 608^2 &= 92417^2 \\
9024015^2 + 6008^2 &= 9024017^2 \\
900240015^2 + 60008^2 &= 900240017^2 \\
90002400015^2 + 600008^2 &= 90002400017^2
\end{aligned} \tag{68}$$

The first two triples $(15, 8, 17)$ and $(1155, 68, 1157)$ for $m = 3$ and 33 are not written above as they don't give good pattern.

- **For $m = 606, 6006, 60006, ...$ in (8):**

$$\begin{aligned}
368448^2 + 1214^2 &= 368450^2 \\
36084048^2 + 12014^2 &= 36084050^2 \\
3600840048^2 + 120014^2 &= 3600840050^2 \\
360008400048^2 + 1200014^2 &= 360008400050^2
\end{aligned} \tag{69}$$

The first two triples $(48, 14, 50)$ and $(3720, 122, 3722)$ for $m = 6$ and 66 are not written above as they don't give good pattern.

▶ **Division by 2**

$$\begin{aligned}
184224^2 + 607^2 &= 184225^2 \\
18042024^2 + 6007^2 &= 18042025^2 \\
1800420024^2 + 60007^2 &= 1800420025^2 \\
180004200024^2 + 600007^2 &= 180004200025^2
\end{aligned} \tag{70}$$

- **For $m = 909, 9009, 90009, ...$ in (8):**

$$\begin{aligned}
828099^2 + 1820^2 &= 828101^2\\
81180099^2 + 18020^2 &= 81180101^2\\
8101800099^2 + 180020^2 &= 8101800101^2\\
810018000099^2 + 1800020^2 &= 810018000101^2\\
81000180000099^2 + 18000020^2 &= 81000180000101^2
\end{aligned} \tag{71}$$

The first two triples $(99, 20, 101)$ and $(9999, 200, 10001)$ for $m = 9$ and 99 are not written above as they don't give good pattern.

3.3 Procedure 3

Below are examples of patterns in Pythagorean triples based on Procedures 3 given in 2.3 and equation 9

- **For $m = 10, 100, 1000, 10000, ...$ in (9):**

$$\begin{aligned}
99^2 + 20^2 &= 101^2 := 10201\\
9999^2 + 200^2 &= 10001^2 := 100020001\\
999999^2 + 2000^2 &= 1000001^2 := 1000002000001\\
99999999^2 + 20000^2 &= 100000001^2 := 10000000200000001
\end{aligned} \tag{72}$$

- **For $m = 20, 200, 2000, 20000, ...$ in (9):**

$$\begin{aligned}
399^2 + 40^2 &= 401^2 := 160801\\
39999^2 + 400^2 &= 40001^2 := 1600080001\\
3999999^2 + 4000^2 &= 4000001^2 := 16000008000001\\
399999999^2 + 40000^2 &= 400000001^2 := 160000000800000001
\end{aligned} \tag{73}$$

The first triple $(3, 4, 5)$ for $m = 2$ is not written above as it doesn't give good pattern.

- **For $m = 30, 300, 3000, 30000, ...$ in (9):**

$$\begin{aligned}
899^2 + 60^2 &= 901^2 := 811801\\
89999^2 + 600^2 &= 90001^2 := 8100180001\\
8999999^2 + 6000^2 &= 9000001^2 := 81000018000001\\
899999999^2 + 60000^2 &= 900000001^2 := 810000001800000001
\end{aligned} \tag{74}$$

The first triple $(8, 6, 10)$ for $m = 3$ is not written above as it doesn't give good pattern.

- **For $m = 40, 400, 4000, 40000, ...$ in (9):**

$$
\begin{aligned}
1599^2 + 80^2 &= 1601^2 &&:= 2563201\\
159999^2 + 800^2 &= 160001^2 &&:= 25600320001\\
15999999^2 + 8000^2 &= 16000001^2 &&:= 256000032000001\\
1599999999^2 + 80000^2 &= 1600000001^2 &&:= 2560000003200000001
\end{aligned}
\tag{75}
$$

The first triple $(15, 8, 17)$ for $m = 4$ is not written above as it doesn't give good pattern.

- **For $m = 50, 500, 5000, 50000, ...$ in (9):**

$$
\begin{aligned}
2499^2 + 100^2 &= 2501^2 &&:= 6255001\\
249999^2 + 1000^2 &= 250001^2 &&:= 62500500001\\
24999999^2 + 10000^2 &= 25000001^2 &&:= 625000050000001\\
2499999999^2 + 100000^2 &= 2500000001^2 &&:= 6250000005000000001
\end{aligned}
\tag{76}
$$

The first triple $(24, 10, 26)$ for $m = 5$ is not written above as it doesn't give good pattern.

- **For $m = 60, 600, 6000, 60000, ...$ in (9):**

$$
\begin{aligned}
3599^2 + 120^2 &= 3601^2 &&:= 12967201\\
359999^2 + 1200^2 &= 360001^2 &&:= 129600720001\\
35999999^2 + 12000^2 &= 36000001^2 &&:= 1296000072000001\\
3599999999^2 + 120000^2 &= 3600000001^2 &&:= 12960000007200000001
\end{aligned}
\tag{77}
$$

The first triple $(35, 12, 37)$ for $m = 6$ is not written above as it doesn't give good pattern.

- **For $m = 70, 700, 7000, 70000, ...$ in (9):**

$$
\begin{aligned}
4899^2 + 140^2 &= 4901^2 &&:= 24019801\\
489999^2 + 1400^2 &= 490001^2 &&:= 240100980001\\
48999999^2 + 14000^2 &= 49000001^2 &&:= 2401000098000001\\
4899999999^2 + 140000^2 &= 4900000001^2 &&:= 24010000009800000001
\end{aligned}
\tag{78}
$$

The first triple $(48, 14, 50)$ for $m = 7$ is not written above as it doesn't give good pattern.

- **For $m = 80, 800, 8000, 80000, ...$ in (9):**

$$
\begin{aligned}
6399^2 + 160^2 &= 6401^2 &&\\
639999^2 + 1600^2 &= 640001^2 &&:= 409601280001\\
63999999^2 + 16000^2 &= 64000001^2 &&:= 4096000128000001\\
6399999999^2 + 160000^2 &= 6400000001^2 &&:= 40960000012800000001
\end{aligned}
\tag{79}
$$

The first triple $(63, 16, 65)$ for $m = 8$ is not written above as it doesn't give good pattern.

- **For $m = 90, 900, 9000, 90000, \ldots$ in (9):**

$$\begin{aligned}
8099^2 + 180^2 &= 8101^2 \\
809999^2 + 1800^2 &= 810001^2 &&:= 656101620001 \\
80999999^2 + 18000^2 &= 81000001^2 &&:= 6561000162000001 \\
8099999999^2 + 180000^2 &= 8100000001^2 &&:= 65610000016200000001
\end{aligned} \tag{80}$$

The first triple $(80, 18, 82)$ for $m = 9$ is not written above as it doesn't give good pattern.

- **For $m = 11, 111, 1111, 11111, \ldots$ in (9):**

$$\begin{aligned}
\mathbf{1}\,20^2 + 22^2 &= \mathbf{1}\,22^2 \\
\mathbf{123}\,20^2 + 222^2 &= \mathbf{123}\,22^2 \\
\mathbf{12343}\,20^2 + 2222^2 &= \mathbf{12343}\,22^2 \\
\mathbf{1234543}\,20^2 + 22222^2 &= \mathbf{1234543}\,22^2 \\
\mathbf{123456543}\,20^2 + 222222^2 &= \mathbf{123456543}\,22^2 \\
\mathbf{12345676543}\,20^2 + 2222222^2 &= \mathbf{12345676543}\,22^2 \\
\mathbf{1234567876543}\,20^2 + 22222222^2 &= \mathbf{1234567876543}\,22^2 \\
\mathbf{123456789876543}\,20^2 + 222222222^2 &= \mathbf{123456789876543}\,22^2 \\
\mathbf{12345679009876543}\,20^2 + 2222222222^2 &= \mathbf{12345679009876543}\,22^2
\end{aligned} \tag{81}$$

▶ **Division by 2**

$$\begin{aligned}
60^2 + 11^2 &= 61^2 \\
6160^2 + 111^2 &= 6161^2 \\
617160^2 + 1111^2 &= 617161^2 \\
61727160^2 + 11111^2 &= 61727161^2 \\
6172827160^2 + 111111^2 &= 6172827161^2
\end{aligned} \tag{82}$$

Here the division by 2 doesn't give good pattern as original one. For systematic example of this type see subsection **??**.

- **For $m = 3, 33, 333, 3333, 33333, \ldots$ in (9):**

$$\begin{aligned}
8^2 + 6^2 &= 10^2 \\
1088^2 + 66^2 &= 1090^2 \\
110888^2 + 666^2 &= 110890^2 \\
11108888^2 + 6666^2 &= 11108890^2 \\
1111088888^2 + 66666^2 &= 1111088890^2 \\
111110888888^2 + 666666^2 &= 111110888890^2
\end{aligned} \tag{83}$$

▶ **Division by 2**

$$\begin{aligned}5^2+3^2 &= 5^2\\ 544^2+33^2 &= 545^2\\ 55444^2+333^2 &= 55445^2\\ 5554444^2+3333^2 &= 5554445^2\\ 555544444^2+33333^2 &= 555544445^2\\ 55555444444^2+333333^2 &= 55555444445^2\end{aligned} \tag{84}$$

- **For $m = 6, 66, 666, 6666, 66666, ...$ in (9):**

$$\begin{aligned}35^2+12^2 &= 37^2\\ 4355^2+132^2 &= 4357^2\\ 443555^2+1332^2 &= 443557^2\\ 44435555^2+13332^2 &= 44435557^2\\ 4444355555^2+133332^2 &= 4444355557^2\end{aligned} \tag{85}$$

- **For $m = 9, 99, 999, 9999, 99999, ...$ in (9):**

$$\begin{aligned}80^2+18^2 &= 82^2\\ 9800^2+198^2 &= 9802^2 &&:= 96079204\\ 998000^2+1998^2 &= 998002^2 &&:= 996007992004\\ 99980000^2+19998^2 &= 99980002^2 &&:= 9996000799920004\\ 9999800000^2+199998^2 &= 9999800002^2 &&:= 99996000079999200004\\ 999998000000^2+1999998^2 &= 999998000002^2 &&:= 999996000007999992000004\end{aligned} \tag{86}$$

▶ **Division by 2**

$$\begin{aligned}40^2+9^2 &= 41^2\\ 4900^2+99^2 &= 4901^2 &&:= 24019801\\ 499000^2+999^2 &= 499001^2 &&:= 249001998001\\ 49990000^2+9999^2 &= 49990001^2 &&:= 2499000199980001\\ 4999900000^2+99999^2 &= 4999900001^2 &&:= 24999000019999800001\\ 499999000000^2+999999^2 &= 499999000001^2 &&:= 249999000001999998000001\end{aligned} \tag{87}$$

- **For $m = 11, 101, 1001, 10001, ...$ in (9):**

$$\begin{aligned}
120^2 + 22^2 &= 122^2 &&:= 14884\\
10200^2 + 202^2 &= 10202^2 &&:= 104080804\\
1002000^2 + 2002^2 &= 1002002^2 &&:= 1004008008004\\
100020000^2 + 20002^2 &= 100020002^2 &&:= 10004000800080004\\
10000200000^2 + 200002^2 &= 10000200002^2 &&:= 100004000080000800004\\
1000002000000^2 + 2000002^2 &= 1000002000002^2 &&:= 1000004000008000008000004
\end{aligned} \tag{88}$$

▶ **Division by 2**

$$\begin{aligned}
5100^2 + 101^2 &= 5101^2 &&\\
501000^2 + 1001^2 &= 501001^2 &&:= 251002002001\\
50010000^2 + 10001^2 &= 50010001^2 &&:= 2501000200020001\\
5000100000^2 + 100001^2 &= 5000100001^2 &&:= 25001000020000200001\\
500001000000^2 + 1000001^2 &= 500001000001^2 &&:= 250001000002000002000001
\end{aligned} \tag{89}$$

The first triple $(60, 11, 61)$ for $m = 11$ is not written above as it doesn't give good pattern.

- **For $m = 22, 202, 2002, 20002, ...$ in (9):**

$$\begin{aligned}
483^2 + 44^2 &= 485^2 &&\\
40803^2 + 404^2 &= 40805^2 &&\\
4008003^2 + 4004^2 &= 4008005^2 &&:= 16064104080025\\
400080003^2 + 40004^2 &= 400080005^2 &&:= 160064010400800025\\
40000800003^2 + 400004^2 &= 40000800005^2 &&:= 1600064001040008000025
\end{aligned} \tag{90}$$

- **For $m = 3, 33, 303, 3003, 30003, 300003, ...$ in (9):**

$$\begin{aligned}
8^2 + 6^2 &= 10^2 &&\\
1088^2 + 66^2 &= 1090^2 &&\\
91808^2 + 606^2 &= 91810^2 &&\\
9018008^2 + 6006^2 &= 9018010^2 &&:= 81324504360100\\
900180008^2 + 60006^2 &= 900180010^2 &&:= 810324050403600100\\
90001800008^2 + 600006^2 &= 90001800010^2 &&:= 8100324005040036000100\\
9000018000008^2 + 6000006^2 &= 9000018000010^2 &&:= 81000324000504000360000100
\end{aligned} \tag{91}$$

▶ **Division by 2**

$$
\begin{aligned}
4^2+3^2 &= 5^2\\
544^2+33^2 &= 545^2\\
45904^2+303^2 &= 45905^2\\
4509004^2+3003^2 &= 4509005^2\\
450090004^2+30003^2 &= 450090005^2 := 202581012600900025\\
45000900004^2+300003^2 &= 45000900005^2 := 2025081001260009000025\\
4500009000004^2+3000003^2 &= 4500009000005^2 := 20250081000126000090000025
\end{aligned} \tag{92}
$$

- **For $m = 404, 4004, 40004, 400004, ...$ in (9):**

$$
\begin{aligned}
163215^2+808^2 &= 163217^2\\
16032015^2+8008^2 &= 16032017^2\\
1600320015^2+80008^2 &= 1600320017^2 := 2561024156810880289\\
160003200015^2+800008^2 &= 160003200017^2 := 25601024015680108800289
\end{aligned} \tag{93}
$$

The first triple $(1935, 88, 1937)$ for $m = 44$ is not written above as it doesn't give good pattern.

- **For $m = 505, 5005, 50005, 500005, ...$ in (9):**

$$
\begin{aligned}
255024^2+1010^2 &= 255026^2\\
25050024^2+10010^2 &= 25050026^2\\
2500500024^2+100010^2 &= 2500500026^2 := 6252500380026000676\\
250005000024^2+1000010^2 &= 250005000026^2 := 62502500038000260000676\\
25000050000024^2+10000010^2 &= 25000050000026^2 := 625002500003800002600000676
\end{aligned} \tag{94}
$$

The first triple $(3024, 110, 3026)$ for $m = 55$ is not written above as it doesn't give good pattern.

► **Division by 2**

$$
\begin{aligned}
12525012^2+5005^2 &= 12525013^2\\
1250250012^2+50005^2 &= 1250250013^2\\
125002500012^2+500005^2 &= 125002500013^2 := 15625625009500065000169\\
12500025000012^2+5000005^2 &= 12500025000013^2 := 156250625000950000650000169
\end{aligned} \tag{95}
$$

The first triple $(127512, 505, 127513)$ is not written above as it doesn't give good pattern.

- **For $m = 606, 6006, 60006, 600006, ...$ in (9):**

$$
\begin{aligned}
367235^2+1212^2 &= 367237^2\\
36072035^2+12012^2 &= 36072037^2\\
3600720035^2+120012^2 &= 3600720037^2 := 12965184784853281369\\
360007200035^2+1200012^2 &= 360007200037^2 := 129605184078480532801369
\end{aligned} \tag{96}
$$

The first triple $(4355, 132, 4357)$ for $m = 66$ is not written above as it doesn't give good pattern.

- **For $m = 707, 7007, 70007, 700007, ...$ in (9):**

$$\begin{aligned} 499848^2 + 1414^2 &= 499850^2 \\ 49098048^2 + 14014^2 &= 49098050^2 \\ 4900980048^2 + 140014^2 &= 4900980050^2 \\ 490009800048^2 + 1400014^2 &= 490009800050^2 \\ 49000098000048^2 + 14000014^2 &= 49000098000050^2 \end{aligned} \tag{97}$$

The first triple $(5928, 154, 5930)$ for $m = 77$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned} 24549024^2 + 7007^2 &= 24549025^2 \\ 2450490024^2 + 70007^2 &= 2450490025^2 \\ 245004900024^2 + 700007^2 &= 245004900025^2 \\ 24500049000024^2 + 7000007^2 &= 24500049000025^2 \end{aligned} \tag{98}$$

The first triple $(249924, 707, 249925)$ for $m = 707$ is not written above as it doesn't give good pattern.

- **For $m = 8008, 80008, 800008, ...$ in (9):**

$$\begin{aligned} 64128063^2 + 16016^2 &= 64128065^2 \\ 6401280063^2 + 160016^2 &= 6401280065^2 \\ 640012800063^2 + 1600016^2 &= 640012800065^2 := 409616384247041664004225 \\ 64000128000063^2 + 16000016^2 &= 64000128000065^2 := 4096016384024704016640004225 \end{aligned} \tag{99}$$

The first two triples $(7743, 176, 7745)$ and $(652863, 1616, 652865)$ for $m = 88$ and 808 are not written above as they don't give good pattern.

- **For $m = 9009, 90009, 900009, ...$ in (9):**

$$\begin{aligned} 81162080^2 + 18018^2 &= 81162082^2 \\ 8101620080^2 + 180018^2 &= 8101620082^2 \\ 810016200080^2 + 1800018^2 &= 810016200082^2 := 656126244395282656806724 \\ 81000162000080^2 + 18000018^2 &= 81000162000082^2 := 6561026244039528026568006724 \end{aligned} \tag{100}$$

The first two triples $(9800, 198, 9802)$ and $(826280, 1818, 826282)$ for $m = 99$ and 909 are not written above as they don't give good pattern.

▶ **Division by 2**

$$\begin{aligned} 40581040^2 + 9009^2 &= 40581041^2 \\ 4050810040^2 + 90009^2 &= 4050810041^2 \\ 405008100040^2 + 900009^2 &= 405008100041^2 \\ 40500081000040^2 + 9000009^2 &= 40500081000041^2 \end{aligned} \tag{101}$$

- **For $m = 10, 110, 1110, 11110, ...$ in (9):**

$$\begin{aligned} 99^2 + 20^2 &= 101^2 \\ 12099^2 + 220^2 &= 12101^2 \\ 1232099^2 + 2220^2 &= 1232101^2 \\ 123432099^2 + 22220^2 &= 123432101^2 \end{aligned} \tag{102}$$

- **For $m = 3, 13, 133, 1333, 13333, 133333, ...$ in (9):**

$$\begin{aligned} 8^2 + 6^2 &= 10^2 \\ 168^2 + 26^2 &= 170^2 \\ 17688^2 + 266^2 &= 17690^2 \\ 1776888^2 + 2666^2 &= 1776890^2 \\ 177768888^2 + 26666^2 &= 177768890^2 \\ 17777688888^2 + 266666^2 &= 17777688890^2 \end{aligned} \tag{103}$$

▶ **Division by 2**

$$\begin{aligned} 4^2 + 3^2 &= 5^2 \\ 84^2 + 13^2 &= 85^2 \\ 8844^2 + 133^2 &= 8845^2 \\ 888444^2 + 1333^2 &= 888445^2 \\ 88884444^2 + 13333^2 &= 88884445^2 \\ 8888844444^2 + 133333^2 &= 8888844445^2 \\ 888888444444^2 + 1333333^2 &= 888888444445^2 \end{aligned} \tag{104}$$

- **For $m = 233, 2333, 23333, 233333, ...$ in (9):**

$$\begin{aligned} 54288^2 + 466^2 &= 54290^2 \\ 5442888^2 + 4666^2 &= 5442890^2 \\ 544428888^2 + 46666^2 &= 544428890^2 \\ 54444288888^2 + 466666^2 &= 54444288890^2 \end{aligned} \tag{105}$$

The first triple $(528, 46, 530)$ for $m = 23$ is not written above as it doesn't give good pattern.

▶ **Division by 2**

$$\begin{aligned} 27144^2 + 233^2 &= 27145^2 \\ 2721444^2 + 2333^2 &= 2721445^2 \\ 272214444^2 + 23333^2 &= 272214445^2 \\ 27222144444^2 + 233333^2 &= 27222144445^2 \\ 2722221444444^2 + 2333333^2 &= 2722221444445^2 \end{aligned} \tag{106}$$

- **For $m = 16, 166, 1666, 16666, 166666, ...$ in (9):**

$$\begin{aligned} 255^2 + 32^2 &= 257^2 \\ 27555^2 + 332^2 &= 27557^2 \\ 2775555^2 + 3332^2 &= 2775557^2 \\ 277755555^2 + 33332^2 &= 277755557^2 \end{aligned} \tag{107}$$

- **For $m = 266, 2666, 26666, 266666, ...$ in (9):**

$$\begin{aligned} 70755^2 + 532^2 &= 70757^2 \\ 7107555^2 + 5332^2 &= 7107557^2 \\ 711075555^2 + 53332^2 &= 711075557^2 \\ 71110755555^2 + 533332^2 &= 71110755557^2 \end{aligned} \tag{108}$$

The first triple $(675, 52, 677)$ for $m = 26$ is not written above as it doesn't give good pattern.

- **For $m = 19, 199, 1999, 19999, 199999, ...$ in (9):**

$$\begin{aligned} 360^2 + 38^2 &= 362 \\ 39600^2 + 398^2 &= 39602^2 &&:= 1568318404 \\ 3996000^2 + 3998^2 &= 3996002^2 &&:= 15968031984004 \\ 399960000^2 + 39998^2 &= 399960002^2 &&:= 159968003199840004 \\ 39999600000^2 + 399998^2 &= 39999600002^2 &&:= 1599968000319998400004 \end{aligned} \tag{109}$$

▶ **Division by 2**

$$\begin{aligned} 180^2 + 19^2 &= 181^2 \\ 19800^2 + 199^2 &= 19801^2 &&:= 392079601 \\ 1998000^2 + 1999^2 &= 1998001^2 &&:= 3992007996001 \\ 199980000^2 + 19999^2 &= 199980001^2 &&:= 39992000799960001 \\ 19999800000^2 + 199999^2 &= 19999800001^2 &&:= 399992000079999600001 \end{aligned} \tag{110}$$

- **For $m = 29, 299, 2999, 29999, 299999, ...$ in (9):**

$$\begin{aligned} 840^2 + 58^2 &= 842^2 \\ 89400^2 + 598^2 &= 89402^2 \\ 8994000^2 + 5998^2 &= 8994002^2 &&:= 80892071976004 \\ 899940000^2 + 59998^2 &= 899940002^2 &&:= 809892007199760004 \\ 89999400000^2 + 599998^2 &= 89999400002^2 &&:= 8099892000719997600004 \end{aligned} \tag{111}$$

▶ **Division by 2**

$$
\begin{aligned}
44700^2+299^2 &= 44701^2 \\
4497000^2+2999^2 &= 4497001^2 \\
449970000^2+29999^2 &= 449970001^2 := 202473001799940001 \\
44999700000^2+299999^2 &= 44999700001^2 := 2024973000179999400001
\end{aligned} \tag{112}
$$

The first triple $(420, 29, 421)$ for $m = 29$ is not written above as it doesn't give good pattern.

- **For $m = 39, 399, 3999, 39999, 399999, ...$ in (9):**

$$
\begin{aligned}
1520^2+78^2 &= 1522^2 \\
159200^2+798^2 &= 159202^2 \\
15992000^2+7998^2 &= 15992002^2 := 255744127968004 \\
1599920000^2+79998^2 &= 1599920002^2 := 2559744012799680004 \\
159999200000^2+799998^2 &= 159999200002^2 := 25599744001279996800004
\end{aligned} \tag{113}
$$

▶ **Division by 2**

$$
\begin{aligned}
760^2+39^2 &= 761^2 \\
79600^2+399^2 &= 79601^2 := 6336319201 \\
7996000^2+3999^2 &= 7996001^2 := 63936031992001 \\
799960000^2+39999^2 &= 799960001^2 := 639936003199920001 \\
79999600000^2+399999^2 &= 79999600001^2 := 6399936000319999200001 \\
7999996000000^2+3999999^2 &= 7999996000001^2 := 63999936000031999992000001
\end{aligned} \tag{114}
$$

3.4 Procedure 4

Below are some patterns obtained by use of formulas (10).

- **For $m = 1, 16, 166, 1666, 16666, ...$ in (10):**

$$
\begin{aligned}
3^2+4^2 &= 5^2 \\
1023^2+64^2 &= 1025^2 \\
110223^2+664^2 &= 110225^2 \\
11102223^2+6664^2 &= 11102225^2 \\
1111022223^2+66664^2 &= 1111022225^2
\end{aligned} \tag{115}
$$

- **For $m = 366, 3666, 36666, ...$ in (10):**

$$\begin{aligned} 535823^2 + 1464^2 &= 535825^2 \\ 53758223^2 + 14664^2 &= 53758225^2 \\ 5377582223^2 + 146664^2 &= 5377582225^2 \\ 537775822223^2 + 1466664^2 &= 537775822225^2 \end{aligned} \tag{116}$$

The first two triples $(35, 12, 37)$ and $(5183, 144, 5185)$ for $m = 3$ and 36 are not written above as they don't give good pattern.

- **For $m = 633, 6333, 63333, 633333, ...$ in (10):**

$$\begin{aligned} 1602755^2 + 2532^2 &= 1602757^2 \\ 160427555^2 + 25332^2 &= 160427557^2 \\ 16044275555^2 + 253332^2 &= 16044275557^2 \\ 1604442755555^2 + 2533332^2 &= 1604442755557^2 \end{aligned} \tag{117}$$

The first two triples $(143, 24, 145)$ and $(15875, 252, 15877)$ for $m = 6$ and 63 are not written above as they don't give good pattern.

- **For $m = 6, 66, 666, 6666, ...$ in (10):**

$$\begin{aligned} 143^2 + 24^2 &= 145^2 \\ 17423^2 + 264^2 &= 17425^2 \\ 1774223^2 + 2664^2 &= 1774225^2 \\ 177742223^2 + 26664^2 &= 177742225^2 \\ 17777422223^2 + 266664^2 &= 17777422225^2 \end{aligned} \tag{118}$$

- **For $m = 699, 6999, 69999, 699999, ...$ in (10):**

$$\begin{aligned} 1954403^2 + 2796^2 &= 1954405^2 \\ 195944003^2 + 27996^2 &= 195944005^2 \\ 19599440003^2 + 279996^2 &= 19599440005^2 \\ 1959994400003^2 + 2799996^2 &= 1959994400005^2 \end{aligned} \tag{119}$$

The first two triples $(143, 24, 145)$ and $(19043, 276, 19045)$ for $m = 6$ and 69 are not written above as they don't give good pattern.

- **For $m = 39, 399, 3999, 39999, ...$ in (10):**

$$\begin{aligned} 6083^2 + 156^2 &= 6085^2 \\ 636803^2 + 1596^2 &= 636805^2 \\ 63968003^2 + 15996^2 &= 63968005^2 \\ 6399680003^2 + 159996^2 &= 6399680005^2 \end{aligned} \tag{120}$$

The first triple $(35, 12, 37)$ for $m = 3$ is not written above as it doesn't give good pattern.

- **For $m = 933, 9333, 93333, 933333, ...$ in (10):**

$$\begin{aligned} 3481955^2 + 3732^2 &= 3481957^2 \\ 348419555^2 + 37332^2 &= 348419557^2 \\ 34844195555^2 + 373332^2 &= 34844195557^2 \\ 3484441955555^2 + 3733332^2 &= 3484441955557^2 \end{aligned} \tag{121}$$

The first two triples $(323, 36, 325)$ and $(34595, 373, 34597)$ for $m = 9$ and 93 are not written above as they don't give good pattern.

- **For $m = 966, 9666, 96666, 966666, ...$ in (10):**

$$\begin{aligned} 3732623^2 + 3864^2 &= 3732625^2 \\ 373726223^2 + 38664^2 &= 373726225^2 \\ 37377262223^2 + 386664^2 &= 37377262225^2 \\ 3737772622223^2 + 3866664^2 &= 3737772622225^2 \end{aligned} \tag{122}$$

The first two triples $(323, 36, 325)$ and $(36863, 384, 36865)$ for $m = 9$ and 96 are not written above as they don't give good pattern.

- **For $m = 9, 99, 999, 9999, ...$ in (10):**

$$\begin{aligned} 323^2 + 36^2 &= 325^2 \\ 39203^2 + 396^2 &= 39205^2 \\ 3992003^2 + 3996^2 &= 3992005^2 && := 15936103920025 \\ 399920003^2 + 39996^2 &= 399920005^2 && := 1599360103999200025 \\ 39999200003^2 + 399996^2 &= 39999200005^2 && := 15999360010399992000025 \end{aligned} \tag{123}$$

First two expressions don't give a good pattern with final sums. good pattern.

3.5 Procedure 5

Below are examples of patterns in Pythagorean triples based on Procedures 5 given in 2.5 and equation 11.

- **For $m = 10, 100, 1000, 10000, 10000, ...$ in (11):**

$$\begin{aligned} 1599^2 + 80^2 &= 1601^2 &&:= 2563201\\ 159999^2 + 800^2 &= 160001^2 &&:= 25600320001\\ 15999999^2 + 8000^2 &= 16000001^2 &&:= 256000032000001\\ 1599999999^2 + 80000^2 &= 1600000001^2 &&:= 2560000003200000001\\ 159999999999^2 + 800000^2 &= 160000000001^2 &&:= 25600000000320000000001 \end{aligned} \quad (124)$$

- **For $m = 20, 200, 2000, 20000, 20000, ...$ in (11):**

$$\begin{aligned} 6399^2 + 160^2 &= 6401^2\\ 639999^2 + 1600^2 &= 640001^2 &&:= 409601280001\\ 63999999^2 + 16000^2 &= 64000001^2 &&:= 4096000128000001\\ 6399999999^2 + 160000^2 &= 6400000001^2 &&:= 40960000012800000001\\ 639999999999^2 + 1600000^2 &= 640000000001^2 &&:= 409600000001280000000001 \end{aligned} \quad (125)$$

- **For $m = 30, 300, 3000, 30000, 30000, ...$ in (11):**

$$\begin{aligned} 14399^2 + 240^2 &= 14401^2\\ 1439999^2 + 2400^2 &= 1440001^2 &&:= 2073602880001\\ 143999999^2 + 24000^2 &= 144000001^2 &&:= 20736000288000001\\ 14399999999^2 + 240000^2 &= 14400000001^2 &&:= 207360000028800000001\\ 1439999999999^2 + 2400000^2 &= 1440000000001^2 &&:= 2073600000002880000000001 \end{aligned} \quad (126)$$

- **For $m = 40, 400, 4000, 40000, 40000, ...$ in (11):**

$$\begin{aligned} 25599^2 + 320^2 &= 25601^2\\ 2559999^2 + 3200^2 &= 2560001^2 &&:= 6553605120001\\ 255999999^2 + 32000^2 &= 256000001^2 &&:= 65536000512000001\\ 25599999999^2 + 320000^2 &= 25600000001^2 &&:= 655360000051200000001\\ 2559999999999^2 + 3200000^2 &= 2560000000001^2 &&:= 6553600000005120000000001 \end{aligned} \quad (127)$$

- **For $m = 5, 50, 500, 5000, 50000, 50000, ...$ in (11):**

$$\begin{aligned} 399^2 + 40^2 &= 401^2 &&:= 160801\\ 39999^2 + 400^2 &= 40001^2 &&:= 1600080001\\ 3999999^2 + 4000^2 &= 4000001^2 &&:= 16000008000001\\ 399999999^2 + 40000^2 &= 400000001^2 &&:= 160000000800000001\\ 39999999999^2 + 400000^2 &= 40000000001^2 &&:= 1600000000080000000001\\ 3999999999999^2 + 4000000^2 &= 4000000000001^2 &&:= 16000000000008000000000001 \end{aligned} \quad (128)$$

- **For $m = 60, 600, 6000, 60000, 60000, ...$ in (11):**

$$\begin{aligned}
57599^2 + 480^2 &= 57601^2 \\
5759999^2 + 4800^2 &= 5760001^2 &&:= 33177611520001 \\
575999999^2 + 48000^2 &= 576000001^2 &&:= 331776001152000001 \\
57599999999^2 + 480000^2 &= 57600000001^2 &&:= 3317760000115200000001 \\
5759999999999^2 + 4800000^2 &= 5760000000001^2 &&:= 33177600000011520000000001
\end{aligned} \tag{129}$$

- **For $m = 70, 700, 7000, 70000, 70000, ...$ in (11):**

$$\begin{aligned}
78399^2 + 560^2 &= 78401^2 \\
7839999^2 + 5600^2 &= 7840001^2 &&:= 61465615680001 \\
783999999^2 + 56000^2 &= 784000001^2 &&:= 614656001568000001 \\
78399999999^2 + 560000^2 &= 78400000001^2 &&:= 6146560000156800000001 \\
7839999999999^2 + 5600000^2 &= 7840000000001^2 &&:= 61465600000015680000000001
\end{aligned} \tag{130}$$

- **For $m = 80, 800, 8000, 80000, 80000, ...$ in (11):**

$$\begin{aligned}
102399^2 + 640^2 &= 102401^2 \\
10239999^2 + 6400^2 &= 10240001^2 &&:= 104857620480001 \\
1023999999^2 + 64000^2 &= 1024000001^2 &&:= 1048576002048000001 \\
102399999999^2 + 640000^2 &= 102400000001^2 &&:= 10485760000204800000001 \\
10239999999999^2 + 6400000^2 &= 10240000000001^2 &&:= 104857600000020480000000001
\end{aligned} \tag{131}$$

- **For $m = 90, 900, 9000, 90000, 90000, ...$ in (11):**

$$\begin{aligned}
129599^2 + 720^2 &= 129601^2 \\
12959999^2 + 7200^2 &= 12960001^2 &&:= 167961625920001 \\
1295999999^2 + 72000^2 &= 1296000001^2 &&:= 1679616002592000001 \\
129599999999^2 + 720000^2 &= 129600000001^2 &&:= 16796160000259200000001 \\
12959999999999^2 + 7200000^2 &= 12960000000001^2 &&:= 167961600000025920000000001
\end{aligned} \tag{132}$$

- **For $m = 101, 1001, 10001, 100001, ...$ in (11):**

$$\begin{aligned}
163215^2 + 808^2 &= 163217^2 \\
16032015^2 + 8008^2 &= 16032017^2 \\
1600320015^2 + 80008^2 &= 1600320017^2 &&:= 2561024156810880289 \\
160003200015^2 + 800008^2 &= 160003200017^2 &&:= 25601024015680108800289 \\
16000032000015^2 + 8000008^2 &= 16000032000017^2 &&:= 256001024001568001088000289
\end{aligned} \tag{133}$$

- **For $m = 202, 2002, 20002, 200002, ...$ in (11):**

$$\begin{aligned}
652863^2 + 1616^2 &= 652865^2 \\
64128063^2 + 16016^2 &= 64128065^2 \\
6401280063^2 + 160016^2 &= 6401280065^2 \\
640012800063^2 + 1600016^2 &= 640012800065^2 &&:= 409616384247041664004225 \\
64000128000063^2 + 16000016^2 &= 64000128000065^2 &&:= 4096016384024704016640004225
\end{aligned} \tag{134}$$

- **For $m = 303, 3003, 30003, 300003, ...$ in (11):**

$$\begin{aligned}
1468943^2 + 2424^2 &= 1468945^2 \\
144288143^2 + 24024^2 &= 144288145^2 \\
14402880143^2 + 240024^2 &= 14402880145^2 \\
1440028800143^2 + 2400024^2 &= 1440028800145^2 \\
144000288000143^2 + 24000024^2 &= 144000288000145^2
\end{aligned} \tag{135}$$

- **For $m = 5, 55, 505, 5005, 50005, 500005, ...$ in (11):**

$$\begin{aligned}
399^2 + 40^2 &= 401^2 \\
48399^2 + 440^2 &= 48401^2 \\
4080399^2 + 4040^2 &= 4080401^2 \\
400800399^2 + 40040^2 &= 400800401^2 \\
40008000399^2 + 400040^2 &= 40008000401^2 &&:= 1600640096086416160801 \\
4000080000399^2 + 4000040^2 &= 4000080000401^2 &&:= 16000640009608064160160801 \\
400000800000399^2 + 40000040^2 &= 400000800000401^2 &&:= 160000640000960800641600160801
\end{aligned} \tag{136}$$

- **For $m = 808, 8008, 80008, 800008, ...$ in (11):**

$$\begin{aligned}
10445823^2 + 6464^2 &= 10445825^2 \\
1026049023^2 + 64064^2 &= 1026049025^2 \\
102420481023^2 + 640064^2 &= 102420481025^2 \\
10240204801023^2 + 6400064^2 &= 10240204801025^2 \\
1024002048001023^2 + 64000064^2 &= 1024002048001025^2
\end{aligned} \tag{137}$$

4 Pandigital Palindromic-Type Patterns

Below are some examples of **palindormic-type pandigial patterns**. These are of three types with different possibilities.

▶ First-Type: Part 1

- **For $m = 10, 110, 1110, 11110, ...$ in (9): or**
- **For $m = 5, 55, 555, 5555, 55555, ...$ in (10): or**
- **For $m = 5/2, 55/2, 555/2, 5555/2, 55555/2, ...$ in (11):**

$$\begin{aligned}
099^2 + 20^2 &= \mathbf{1}\,01^2 \\
\mathbf{12}\,099^2 + 220^2 &= \mathbf{121}\,01^2 \\
\mathbf{1232}\,099^2 + 2220^2 &= \mathbf{12321}\,01^2 \\
\mathbf{123432}\,099^2 + 22220^2 &= \mathbf{1234321}\,01^2 \\
\mathbf{12345432}\,099^2 + 222220^2 &= \mathbf{123454321}\,01^2 \\
\mathbf{1234565432}\,099^2 + 2222220^2 &= \mathbf{12345654321}\,01^2 \\
\mathbf{123456765432}\,099^2 + 22222220^2 &= \mathbf{1234567654321}\,01^2 \\
\mathbf{12345678765432}\,099^2 + 222222220^2 &= \mathbf{123456787654321}\,01^2 \\
\mathbf{1234567898765432}\,099^2 + 2222222220^2 &= \mathbf{12345678987654321}\,01^2
\end{aligned} \tag{138}$$

▶ First-Type: Part 2

- **For $m = 100, 1100, 11100, 111100, ...$ in (9): or**
- **For $m = 50, 550, 5550, 55550, 555550, ...$ in (10): or**
- **For $m = 50/2, 550/2, 5550/2, 55550/2, 555550/2, ...$ in (11):**

$$\begin{aligned}
09999^2 + 200^2 &= 1\,0001^2\\
12\,09999^2 + 2200^2 &= 121\,0001^2\\
1232\,09999^2 + 22200^2 &= 12321\,0001^2\\
123432\,09999^2 + 222200^2 &= 1234321\,0001^2\\
12345432\,09999^2 + 2222200^2 &= 123454321\,0001^2\\
1234565432\,09999^2 + 22222200^2 &= 12345654321\,0001^2\\
123456765432\,09999^2 + 222222200^2 &= 1234567654321\,0001^2\\
12345678765432\,09999^2 + 2222222200^2 &= 123456787654321\,0001^2\\
1234567898765432\,09999^2 + 22222222200^2 &= 12345678987654321\,0001^2 \qquad (139)
\end{aligned}$$

► First-Type: Part 3

- **For $m = 1000, 11000, 111000, 1111000, ...$ in (9): or**
- **For $m = 1000/2, 11000/2, 111000/2, 1111000/2, ...$ in (10): or**
- **For $m = 1000/4, 11000/4, 111000/4, 1111000/4, ...$ in (11):**

$$\begin{aligned}
0999999^2 + 2000^2 &= 1\,000001^2\\
12\,0999999^2 + 22000^2 &= 121\,000001^2\\
1232\,0999999^2 + 222000^2 &= 12321\,000001^2\\
123432\,0999999^2 + 2222000^2 &= 1234321\,000001^2\\
12345432\,0999999^2 + 22222000^2 &= 123454321\,000001^2\\
1234565432\,0999999^2 + 222222000^2 &= 12345654321\,000001^2\\
123456765432\,0999999^2 + 2222222000^2 &= 1234567654321\,000001^2\\
12345678765432\,0999999^2 + 22222222000^2 &= 123456787654321\,000001^2\\
1234567898765432\,0999999^2 + 222222222000^2 &= 12345678987654321\,000001^2 \qquad (140)
\end{aligned}$$

Remark 5. *Extending the above study for two variables, the first case lead us to 9 values, the second case lead us to 99 values and the third case lead us to 999 values. This has been done separately in Taneja [6].*

► Second-Type: Part 1

- **For $m = 10, 1010, 101010, 10101010, ...$ in (9): or**
- **For $m = 10/2, 1010/2, 101010/2, 10101010/2, ...$ in (10): or**
- **For $m = 10/4, 1010/4, 101010/4, 10101010/4, ...$ in (11):**

$$
\begin{aligned}
0099^2 + 20^2 &= \mathbf{1}\,01^2\\
\mathbf{102}\,0099^2 + 2020^2 &= \mathbf{10201}\,01^2\\
\mathbf{1020302}\,0099^2 + 202020^2 &= \mathbf{102030201}\,01^2\\
\mathbf{10203040302}\,0099^2 + 20202020^2 &= \mathbf{1020304030201}\,01^2\\
\mathbf{102030405040302}\,0099^2 + 2020202020^2 &= \mathbf{10203040504030201}\,01^2\\
\mathbf{1020304050605040302}\,0099^2 + 202020202020^2 &= \mathbf{102030405060504030201}\,01^2\\
\mathbf{10203040506070605040302}\,0099^2 + 20202020202020^2 &= \mathbf{1020304050607060504030201}\,01^2\\
\mathbf{102030405060708070605040302}\,0099^2 + 2020202020202020^2 &= \mathbf{10203040506070807060504030201}\,01^2\\
\mathbf{1020304050607080908070605040302}\,0099^2 + 202020202020202020^2 &= \mathbf{102030405060708090807060504030201}\,01^2
\end{aligned} \tag{141}
$$

► Second-Type: Part 2

- **For $m = 100, 10100, 1010100, 101010100, 10101010100, \ldots$ in (9): or**
- **For $m = 50, 5050, 505050, 50505050, 5050505050, \ldots$ in (10): or**
- **For $m = 25, 2525, 252525, 25252525, \ldots$ in (11):**

$$
\begin{aligned}
009999^2 + 200^2 &= \mathbf{1}\,0001^2\\
\mathbf{102}\,009999^2 + 20200^2 &= \mathbf{10201}\,0001^2\\
\mathbf{1020302}\,009999^2 + 2020200^2 &= \mathbf{102030201}\,0001^2\\
\mathbf{10203040302}\,009999^2 + 202020200^2 &= \mathbf{1020304030201}\,0001^2\\
\mathbf{102030405040302}\,009999^2 + 20202020200^2 &= \mathbf{10203040504030201}\,0001^2\\
\mathbf{1020304050605040302}\,009999^2 + 2020202020200^2 &= \mathbf{102030405060504030201}\,0001^2\\
\mathbf{10203040506070605040302}\,009999^2 + 202020202020200^2 &= \mathbf{1020304050607060504030201}\,0001^2\\
\mathbf{102030405060708070605040302}\,009999^2 + 20202020202020200^2 &= \mathbf{10203040506070807060504030201}\,0001^2\\
\mathbf{1020304050607080908070605040302}\,009999^2 + 2020202020202020200^2 &= \mathbf{102030405060708090807060504030201}\,0001^2
\end{aligned} \tag{142}
$$

► Second-Type: Part 3

- **For $m = 1000, 101000, 10101000, 1010101000, 101010101000, \ldots$ in (9): or**
- **For $m = 1000/2, 101000/2, 10101000/2, 1010101000/2, 101010101000/2, \ldots$ in (10): or**
- **For $m = 1000/4, 101000/4, 10101000/4, 1010101000/4, 101010101000/4, \ldots$ in (11):**

$$
\begin{aligned}
00999999^2 + 2000^2 &= \mathbf{1}\,000001^2\\
\mathbf{102}\,00999999^2 + 202000^2 &= \mathbf{10201}\,000001^2\\
\mathbf{1020302}\,00999999^2 + 20202000^2 &= \mathbf{102030201}\,000001^2\\
\mathbf{10203040302}\,00999999^2 + 2020202000^2 &= \mathbf{1020304030201}\,000001^2\\
\mathbf{102030405040302}\,00999999^2 + 202020202000^2 &= \mathbf{10203040504030201}\,000001^2\\
\mathbf{1020304050605040302}\,00999999^2 + 20202020202000^2 &= \mathbf{102030405060504030201}\,000001^2\\
\mathbf{10203040506070605040302}\,00999999^2 + 2020202020202000^2 &= \mathbf{1020304050607060504030201}\,000001^2\\
\mathbf{102030405060708070605040302}\,00999999^2 + 202020202020202000^2 &= \mathbf{10203040506070807060504030201}\,000001^2\\
\mathbf{1020304050607080908070605040302}\,00999999^2 + 20202020202020202000^2 &= \mathbf{102030405060708090807060504030201}\,0001^2
\end{aligned} \tag{143}
$$

Remark 6. *Extending the above study for two variables, the first case lead us to 9 values, the second case lead us to 99 values and the third case lead us to 999 values. This has been done separately in Taneja [7].*

▶ Third-Type: Part 1

- **For $m = 100, 100100, 100100100, 100100100100, ...$ in (9): or**
- **For $m = 100/2, 100100/2, 100100100/2, 100100100100/2, ...$ in (10): or**
- **For $m = 100/4, 100100/4, 100100100/4, 100100100100/4, ...$ in (11):**

$$\begin{aligned}
0009999^2 + 200^2 &= \mathbf{1}\,0001^2\\
\mathbf{1002}\,0009999^2 + 200200^2 &= \mathbf{1002001}\,0001^2\\
\mathbf{1002003002}\,0009999^2 + 200200200^2 &= \mathbf{1002003002001}\,0001^2\\
\mathbf{1002003004003002}\,0009999^2 + 200200200200^2 &= \mathbf{1002003004003002001}\,0001^2\\
\mathbf{1002003004005004003002}\,0009999^2 + 200200200200200^2 &= \mathbf{1002003004005004003002001}\,0001^2\\
\mathbf{1002003004005006005004003002}\,0009999^2 + 200200200200200200^2 &= \mathbf{1002003004005006005004003002001}\,0001^2\\
\mathbf{1002003004005006007006005004003002}\,0009999^2 + 200200200200200200200^2 &= \mathbf{1002003004005006007006005004003002001}\,0001^2\\
\mathbf{1002003004005006007008007006005004003002}\,0009999^2 + 200200200200200200200200^2 &= \mathbf{1002003004005006007008007006005004003002001}\,0001^2\\
\mathbf{1002003004005006007008009008007006005004003002}\,0009999^2 + 200200200200200200200200200^2 &= \mathbf{1002003004005006007008009008007006005004003002001}\,0001^2
\end{aligned} \tag{144}$$

▶ Third-Type: Part 2

- **For $m = 500, 500500, 500500500, 500500500500, ...$ in (10): or**
- **For $m = 250, 250250, 250250250, 250250250250, ...$ in (11):**

$$\begin{aligned}
000999999^2 + 2000^2 &= \mathbf{1}\,000001^2\\
\mathbf{1002}\,000999999^2 + 2002000^2 &= \mathbf{1002001}\,000001^2\\
\mathbf{1002003002}\,000999999^2 + 2002002000^2 &= \mathbf{1002003002001}\,000001^2\\
\mathbf{1002003004003002}\,000999999^2 + 2002002002000^2 &= \mathbf{1002003004003002001}\,000001^2\\
\mathbf{1002003004005004003002}\,000999999^2 + 2002002002002000^2 &= \mathbf{1002003004005004003002001}\,000001^2\\
\mathbf{1002003004005006005004003002}\,000999999^2 + 2002002002002002000^2 &= \mathbf{1002003004005006005004003002001}\,000001^2\\
\mathbf{1002003004005006007006005004003002}\,000999999^2 + 2002002002002002002000^2 &= \mathbf{1002003004005006007006005004003002001}\,000001^2\\
\mathbf{1002003004005006007008007006005004003002}\,000999999^2 + 2002002002002002002002000^2 &= \mathbf{1002003004005006007008007006005004003002001}\,000001^2\\
\mathbf{1002003004005006007008009008007006005004003002}\,000999999^2 + 2002002002002002002002002000^2 &= \mathbf{1002003004005006007008009008007006005004003002001}\,000001^2
\end{aligned} \tag{145}$$

5 Final Comments

This work brings five procedures to get patterns in Pythagorean triples. In some cases, they give same results depending on the variations of variables. Extensions to two variables, the Procedures 1 and 2 are combined in [5]. Two variables extensions of the Procedures 3, 4 and 5 are done separately for each case in [5]. From the section 4, we observe that the **pandigital palindromic-type patterns** can be obtained from the Procedures 3, 4 or 5 by considering different values of the variables. This study is extended for two variables in [6, 7], where the **pandigital palindromic-type**

patterns are obtained with more possibilities. Moreover, we observe that the results appearing in section 4 are based on the following pattern:

$$\begin{aligned}
99^2 + 20^2 &= 101^2 && := 10201 \\
9999^2 + 200^2 &= 10001^2 && := 100020001 \\
999999^2 + 2000^2 &= 1000001^2 && := 1000002000001 \\
99999999^2 + 20000^2 &= 100000001^2 && := 10000000200000001 \\
9999999999^2 + 200000^2 &= 10000000001^2 && := 100000000020000000001 \\
999999999999^2 + 2000000^2 &= 1000000000001^2 && := 1000000000002000000000001
\end{aligned}$$

This is the same pattern as given in (64). It is good with final sum too, but the results given in section 4 aren't good with final sum. Moreover, this is also good for magic squares with perfect square sum [3]

The Procedure 5 can be extended further to Procedure 6 given by

$$\begin{aligned}
f_6(m) &:= 64m^2 - 1 \\
g_6(m) &:= 16m \\
h_6(m) &:= 64m^2 + 1 \qquad (146)
\end{aligned}$$

There is not very much advantage in extending it as the same results can be obtained from the procedures Procedures 3, 4 or 5.

References

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[2] Pythagorean triple - Wikipedia, *https://goo.gl/Zwu7sC*

[3] I.J. TANEJA, Magic Squares with Perfect Square Number Sums, Research Report Collection, **20**(2017), Article 11, pp.1-24, *http://rgmia.org/papers/v20/v20a11.pdf*.

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[7] I.J. TANEJA, Pandigital Palindromic-Type Patterns in Pythagorean Triples - II, RGMIA Research Report Collection, **21**(2018), *http://rgmia.org/v21.php*