SOME JENSEN'S TYPE INEQUALITIES FOR CONVEX FUNCTIONS AND AN INTEGRAL OPERATOR

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we establish some Jensen's type inequalities for convex functions and the integral operator

$$D_{a+,b-}g(x) := \frac{1}{2}\left[\frac{1}{x-a}\int_a^x g(t)\,dt + \frac{1}{b-x}\int_x^b g(t)\,dt\right],\ x\in(a,b)$$

defined for integrable functions $g:[a,b]\to\mathbb{R}$. Various Hermite-Hadamard type inequalities improving some classical results are also provided. Some examples for logarithm and power function are given.

1. Introduction

The following integral inequality

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(t)\,dt\le\frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for any convex function $f:[a,b]\to\mathbb{R}$, is well known in the literature as the *Hermite-Hadamard inequality*.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, from which we would like to refer the reader to the monograph [10], the recent survey paper [6], the research papers [1]-[2], [12]-[20] and the references therein.

Assume that the function $f:(a,b)\to\mathbb{C}$ is Lebesgue integrable on (a,b). We consider the following operator, see also [7]

$$D_{a+,b-}f(x) := \frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt + \frac{1}{b-x}\int_x^b f(t)\,dt\right],\ x\in(a,b). \tag{1.2}$$

We observe that if we take $x=\frac{a+b}{2}$, then we have

$$D_{a+,b-}f\left(\frac{a+b}{2}\right)=\frac{1}{b-a}\int_a^b f(t)\,dt.$$

Moreover, if $f(a+) := \lim_{x\to a+} f(x)$ exists and is finite, then we have

$$\lim_{x\to a+} D_{a+,b-}f(x)=\frac{1}{2}\left[f(a+)+\frac{1}{b-a}\int_a^b f(t)\,dt\right]$$

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and if $f(b-) := \lim_{x\to b-} f(x)$ exists and is finite, then we have

$$\lim_{x\to b-} D_{a+,b-}f(x) = \frac{1}{2}\left[f(b-) + \frac{1}{b-a}\int_a^b f(t)\,dt\right].$$

So, if $f : [a,b] \to \mathbb{C}$ is Lebesgue integrable on $[a,b]$ and continuous at right in a and at left in b, then we can extend the operator on the whole interval by putting

$$D_{a+,b-}f(a) := \frac{1}{2}\left[f(a) + \frac{1}{b-a}\int_a^b f(t)\,dt\right]$$

and

$$D_{a+,b-}f(b) = \frac{1}{2}\left[f(b) + \frac{1}{b-a}\int_a^b f(t)\,dt\right].$$

We have the following lower and upper bounds for $D_{a+,b-}f$, see [8]:

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$. Then for any $x \in (a,b)$ we have*

$$\frac{1}{2}\left[f\left(\frac{a+x}{2}\right) + f\left(\frac{x+b}{2}\right)\right] \le D_{a+,b-}f(x) \le \frac{1}{2}\left[f(x) + \frac{f(a)+f(b)}{2}\right]. \tag{1.3}$$

We can state now the following result that provides a refinement of the second Hermite-Hadamard inequality, see [8]:

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$. Then*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,dx &\le \frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\,dx \\ &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx + \frac{f(a)+f(b)}{2}\right]\left(\le \frac{f(a)+f(b)}{2}\right).\end{aligned} \tag{1.4}$$

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \ge 0$ for μ -a.e. (almost every) $x \in \Omega$, consider the Lebesgue space

$$L_w(\Omega,\mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(x)|\,w(x)\,d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$.

The following general Hermite-Hadamard type inequalities hold [5]:

Theorem 3. *Let $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on $[m,M]$ and $g : \Omega \to [m,M]$ so that $\Phi \circ g,\ g \in L_w(\Omega,\mu)$, where $w \ge 0$ μ-a.e. on Ω with $\int_\Omega w d\mu = 1$.*

Then we have the inequalities:

$$\Phi\left(\frac{m+M}{2}\right)+\varphi\left(\frac{m+M}{2}\right)\int_{\Omega}\left(g-\frac{m+M}{2}\right)wd\mu \tag{1.5}$$
$$\leq\int_{\Omega}(\Phi\circ g)\,wd\mu$$
$$\leq\frac{\Phi(m)+\Phi(M)}{2}+\frac{\Phi(M)-\Phi(m)}{M-m}\int_{\Omega}\left(g-\frac{m+M}{2}\right)wd\mu,$$

where $\varphi\left(\frac{m+M}{2}\right)\in\left[\Phi'_{-}\left(\frac{m+M}{2}\right),\Phi'_{+}\left(\frac{m+M}{2}\right)\right]$.

If

$$\int_{\Omega}\left(g-\frac{m+M}{2}\right)wd\mu=0,$$

then we have the general Fejér type [12] *inequalities*

$$\Phi\left(\frac{m+M}{2}\right)\leq\int_{\Omega}(\Phi\circ g)\,wd\mu\leq\frac{\Phi(m)+\Phi(M)}{2}. \tag{1.6}$$

Motivated by the above results, in this paper we establish some Jensen's type inequalities for convex functions and the integral operator $D_{a+,b-}$ defined above in (1.2). Various Hermite-Hadamard type inequalities improving some classical results are also provided. Some examples for logarithm and power function are given.

2. Main Results

We have:

Theorem 4. *Let* $\Phi: I\subset\mathbb{R}\to\mathbb{R}$ *be a convex function and* $g:[a,b]\subset\mathbb{R}\to I$ *an integrable function such that* $\Phi\circ g$ *is also integrable on* $[a,b]$. *Then for any* $x\in(a,b)$ *we have*

$$\Phi\left(D_{a+,b-}g(x)\right) \tag{2.1}$$
$$\leq\frac{1}{2}\left[\Phi\left(\frac{\int_a^x g(t)\,dt}{x-a}\right)+\Phi\left(\frac{\int_x^b g(t)\,dt}{b-x}\right)\right]\leq D_{a+,b-}(\Phi\circ g)(x).$$

Proof. Using Jensen's inequality for the convex function $\Phi: I\subset\mathbb{R}\to\mathbb{R}$, we have

$$\Phi\left(\frac{\int_a^x g(t)\,dt}{x-a}\right)\leq\frac{\int_a^x(\Phi\circ g)(t)\,dt}{x-a} \tag{2.2}$$

and

$$\Phi\left(\frac{\int_x^b g(t)\,dt}{b-x}\right)\leq\frac{\int_x^b(\Phi\circ g)(t)\,dt}{b-x}, \tag{2.3}$$

where $x\in(a,b)$.

If we add these two inequalities and divide by 2 we get

$$\frac{1}{2}\left[\Phi\left(\frac{\int_a^x g(t)\,dt}{x-a}\right)+\Phi\left(\frac{\int_x^b g(t)\,dt}{b-x}\right)\right]$$
$$\leq\frac{1}{x-a}\int_a^x(\Phi\circ g)(t)\,dt+\frac{1}{b-x}\int_x^b(\Phi\circ g)(t)\,dt=D_{a+,b-}(\Phi\circ g)(x),$$

where $x\in(a,b)$, which proves the second inequality in (2.1).

By the convexity of Φ we also have

$$\Phi\left(D_{a+,b-}g\left(x\right)\right)=\Phi\left(\frac{1}{2}\left[\frac{\int_a^x g\left(t\right)dt}{x-a}+\frac{\int_x^b g\left(t\right)dt}{b-x}\right]\right)$$
$$\leq\frac{1}{2}\left[\Phi\left(\frac{\int_a^x g\left(t\right)dt}{x-a}\right)+\Phi\left(\frac{\int_x^b g\left(t\right)dt}{b-x}\right)\right],$$

where $x\in\left(a,b\right)$, which proves the first part of (2.1). □

Remark 1. *If we take $\Phi=f$, a convex function on $[a,b]$ and $g=\ell$, where $\ell\left(t\right)=t$, then by (2.1) we get*

$$f\left[\frac{1}{2}\left(x+\frac{a+b}{2}\right)\right]\leq\frac{1}{2}\left[f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)\right]\leq D_{a+,b-}\left(f\right)\left(x\right)\tag{2.4}$$

for all $x\in\left(a,b\right)$.

The following lemma is of interest in itself:

Lemma 1. *Assume that the function $f:\left(a,b\right)\to\mathbb{C}$ is Lebesgue integrable on $\left(a,b\right)$ and $f\left(a+\right)$, $f\left(b-\right)$ exists and are finite. Then we have*

$$\int_a^b D_{a+,b-}f\left(x\right)dx=\int_a^b\ln\left(\frac{b-a}{\sqrt{\left(x-a\right)\left(b-x\right)}}\right)f\left(x\right)dx.\tag{2.5}$$

Proof. We have

$$\int_a^b D_{a+,b-}f\left(x\right)dx$$
$$=\frac{1}{2}\left[\int_a^b\left(\frac{1}{x-a}\int_a^x f\left(t\right)dt\right)dx+\int_a^b\left(\frac{1}{b-x}\int_x^b f\left(t\right)dt\right)dx\right].$$

Observe that, integrating by parts, we have

$$\int_a^b\left(\frac{1}{x-a}\int_a^x f\left(t\right)dt\right)dx=\int_a^b\left(\int_a^x f\left(t\right)dt\right)d\left(\ln\left(x-a\right)\right)$$
$$=\ln\left(x-a\right)\left(\int_a^x f\left(t\right)dt\right)\Big|_{a+}^{b}-\int_a^b\ln\left(x-a\right)f\left(x\right)dx$$
$$=\ln\left(b-a\right)\left(\int_a^b f\left(t\right)dt\right)-\lim_{x\to a+}\left[\ln\left(x-a\right)\left(\int_a^x f\left(t\right)dt\right)\right]-\int_a^b\ln\left(x-a\right)f\left(x\right)dx.$$

Since

$$\lim_{x\to a+}\left[\ln\left(x-a\right)\left(\int_a^x f\left(t\right)dt\right)\right]=\lim_{x\to a+}\left[\left(x-a\right)\ln\left(x-a\right)\left(\frac{1}{x-a}\int_a^x f\left(t\right)dt\right)\right]$$
$$=\lim_{x\to a+}\left[\left(x-a\right)\ln\left(x-a\right)\right]\lim_{x\to a+}\left(\frac{1}{x-a}\int_a^x f\left(t\right)dt\right)=0f\left(a+\right)=0,$$

hence

$$\int_a^b \left(\frac{1}{x-a}\int_a^x f(t)\,dt\right)dx = \ln(b-a)\left(\int_a^b f(t)\,dt\right) - \int_a^b \ln(x-a) f(x)\,dx$$
$$= \int_a^b [\ln(b-a) - \ln(x-a)] f(x)\,dx = \int_a^b \ln\left(\frac{b-a}{x-a}\right) f(x)\,dx$$

Also, integrating by parts, we have

$$\int_a^b \left(\frac{1}{b-x}\int_x^b f(t)\,dt\right)dx = -\int_a^b \left(\int_x^b f(t)\,dt\right) d(\ln(b-x))$$
$$= -\ln(b-x)\left(\int_x^b f(t)\,dt\right)\Bigg|_a^{b-} + \int_a^b \ln(b-x)\,d\left(\int_x^b f(t)\,dt\right)$$
$$= -\lim_{x\to b-}\left[\ln(b-x)\left(\int_x^b f(t)\,dt\right)\right] + \ln(b-a)\left(\int_a^b f(t)\,dt\right)$$
$$- \int_a^b \ln(b-x) f(x)\,dx$$
$$= \ln(b-a)\left(\int_a^b f(t)\,dt\right) - \int_a^b \ln(b-x) f(x)\,dx = \int_a^b \ln\left(\frac{b-a}{b-x}\right) f(x)\,dx.$$

Therefore

$$\int_a^b D_{a+,b-} f(x)\,dx = \frac{1}{2}\left[\int_a^b \ln\left(\frac{b-a}{x-a}\right) f(x)\,dx + \int_a^b \ln\left(\frac{b-a}{b-x}\right) f(x)\,dx\right]$$
$$= \frac{1}{2}\int_a^b \ln\left[\frac{(b-a)^2}{(x-a)(b-x)}\right] f(x)\,dx = \int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\,dx$$

and the equality (2.5) is obtained. □

Remark 2. *If we take $f = \ell$ in (2.5), then we get*

$$\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) x dx$$
$$= \int_a^b D_{a+,b-}\ell(x)\,dx = \frac{1}{2}\int_a^b \left[\frac{1}{x-a}\int_a^x tdt + \frac{1}{b-x}\int_x^b tdt\right] dx = (b-a)\frac{a+b}{2}.$$

We also have:

Theorem 5. *Let $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $g : [a,b] \subset \mathbb{R} \to I$ an integrable function such that $\Phi \circ D_{a+,b-}g$ is also integrable on $[a,b]$. Then we have*

$$\begin{aligned}(2.6)\quad & \Phi\left(\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) g(x)\,dx\right) \\ &\le \frac{1}{b-a}\int_a^b \Phi\left(D_{a+,b-}g(x)\right)dx \\ &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b \Phi\left(\frac{\int_a^x g(t)\,dt}{x-a}\right)dx + \frac{1}{b-a}\int_a^b \Phi\left(\frac{\int_x^b g(t)\,dt}{b-x}\right)dx\right] \\ &\le \frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)(\Phi\circ g)(x)\,dx.\end{aligned}$$

Proof. If we take the integral mean $\frac{1}{b-a}\int_a^b$ in (2.1), we get

$$\begin{aligned}(2.7)\quad & \frac{1}{b-a}\int_a^b \Phi\left(D_{a+,b-}g(x)\right)dx \\ &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b \Phi\left(\frac{\int_a^x g(t)\,dt}{x-a}\right)dx + \frac{1}{b-a}\int_a^b \Phi\left(\frac{\int_x^b g(t)\,dt}{b-x}\right)dx\right] \\ &\le \frac{1}{b-a}\int_a^b D_{a+,b-}(\Phi\circ g)(x)\,dx.\end{aligned}$$

By using Lemma 1 for the function $\Phi\circ g$, we get

$$\frac{1}{b-a}\int_a^b D_{a+,b-}(\Phi\circ g)(x)\,dx = \frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)(\Phi\circ g)(x)\,dx$$

and the second and third inequalities in (2.6) are proved.

By Jensen's integral inequality for the convex function Φ we also have

$$(2.8)\quad \Phi\left(\frac{1}{b-a}\int_a^b D_{a+,b-}g(x)\,dx\right) \le \frac{1}{b-a}\int_a^b \Phi\left(D_{a+,b-}g(x)\right)dx$$

and since, by Lemma 1 for the function g, we have

$$\frac{1}{b-a}\int_a^b D_{a+,b-}g(x)\,dx = \frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) g(x)\,dx$$

then by (2.8) we get the first inequality in (2.6). □

Remark 3. *If we take $\Phi = f$, a convex function on $[a,b]$ and $g = \ell$, then by (2.6) we get*

$$\begin{aligned}(2.9)\quad & f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f\left(\frac{1}{2}\left(x+\frac{a+b}{2}\right)\right)dx \\ &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(\frac{a+x}{2}\right)dx + \frac{1}{b-a}\int_a^b f\left(\frac{b+x}{2}\right)dx\right] \\ &\le \frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\,dx.\end{aligned}$$

Since, by changing the variables,

$$\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(\frac{a+x}{2}\right)dx+\frac{1}{b-a}\int_a^b f\left(\frac{b+x}{2}\right)dx\right]=\frac{1}{b-a}\int_a^b f(x)\,dx,$$

hence by (2.9) we get

$$\begin{aligned}(2.10)\quad f\left(\frac{a+b}{2}\right)&\le\frac{1}{b-a}\int_a^b f\left(\frac{1}{2}\left(x+\frac{a+b}{2}\right)\right)dx\\&\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{1}{b-a}\int_a^b\ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)f(x)\,dx.\end{aligned}$$

We can give now some reverse inequalities that are similar to ones from (1.5):

Theorem 6. *Let $\Phi:I\subset\mathbb{R}\to\mathbb{R}$ be a convex function and $g:[a,b]\subset\mathbb{R}\to[m,M]\subset I$ an integrable function such that $\Phi\circ g$ is also integrable on $[a,b]$. Then for any $x\in(a,b)$ we have*

$$\begin{aligned}(2.11)\quad &D_{a+,b-}(\Phi\circ g)(x)\\&\le\frac{\Phi(m)+\Phi(M)}{2}+\frac{\Phi(M)-\Phi(m)}{M-m}\left(D_{a+,b-}g(x)-\frac{m+M}{2}\right)\end{aligned}$$

and

$$\begin{aligned}(2.12)\quad &\frac{1}{b-a}\int_a^b\ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)(\Phi\circ g)(x)\,dx\le\frac{\Phi(m)+\Phi(M)}{2}\\&+\frac{\Phi(M)-\Phi(m)}{M-m}\left(\frac{1}{b-a}\int_a^b\ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)g(x)\,dx-\frac{m+M}{2}\right).\end{aligned}$$

Proof. By the convexity of Φ we have

$$\begin{aligned}\Phi(s)&=\Phi\left(\frac{M-s}{M-m}m+\frac{s-m}{M-m}M\right)\\&\le\frac{M-s}{M-m}\Phi(m)+\frac{s-m}{M-m}\Phi(M)\\&=\frac{\Phi(m)+\Phi(M)}{2}+\left(\frac{M-s}{M-m}-\frac{1}{2}\right)\Phi(m)+\left(\frac{s-m}{M-m}-\frac{1}{2}\right)\Phi(M)\\&=\frac{\Phi(m)+\Phi(M)}{2}-\Phi(m)\left(\frac{s-\frac{m+M}{2}}{M-m}\right)+\Phi(M)\left(\frac{s-\frac{m+M}{2}}{M-m}\right)\\&=\frac{\Phi(m)+\Phi(M)}{2}+\frac{\Phi(M)-\Phi(m)}{M-m}\left(s-\frac{m+M}{2}\right)\end{aligned}$$

for any $s\in[m,M]$.

This inequality implies that

$$(2.13)\qquad \Phi(g(t))\le\frac{\Phi(m)+\Phi(M)}{2}+\frac{\Phi(M)-\Phi(m)}{M-m}\left(g(t)-\frac{m+M}{2}\right)$$

for any $t\in[a,b]$.

Let $x\in(a,b)$. By taking the integral mean on $[a,x]$ of the inequality (2.13) we get

$$\frac{1}{x-a}\int_a^x(\Phi\circ g)(t)\,dt\leq\frac{\Phi(m)+\Phi(M)}{2}+\frac{\Phi(M)-\Phi(m)}{M-m}\left(\frac{1}{x-a}\int_a^x g(t)\,dt-\frac{m+M}{2}\right),\tag{2.14}$$

while by taking the integral mean on $[x,b]$, we get

$$\frac{1}{b-x}\int_x^b(\Phi\circ g)(t)\,dt\leq\frac{\Phi(m)+\Phi(M)}{2}+\frac{\Phi(M)-\Phi(m)}{M-m}\left(\frac{1}{b-x}\int_x^b g(t)\,dt-\frac{m+M}{2}\right).\tag{2.15}$$

If we add (2.14) and (2.15) and divide by 2 we get (2.11).

If we take the integral mean on $[a,b]$ in the inequality (2.11), we get

$$\begin{aligned}&\frac{1}{b-a}\int_a^b D_{a+,b-}(\Phi\circ g)(x)\,dx\\&\leq\frac{\Phi(m)+\Phi(M)}{2}+\frac{\Phi(M)-\Phi(m)}{M-m}\left(\frac{1}{b-a}\int_a^b D_{a+,b-}g(x)\,dx-\frac{m+M}{2}\right)\end{aligned}\tag{2.16}$$

and by using Lemma 1 we get the desired inequality (2.12). □

Remark 4. *If we take $\Phi=f$, a convex function on $[a,b]$ and $g=\ell$, then by (2.11) we get*

$$D_{a+,b-}(f)(x)\leq\frac{f(a)+f(b)}{2}+\frac{1}{2}\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\tag{2.17}$$

for any $x\in(a,b)$.

From (2.12) we obtain

$$\frac{1}{b-a}\int_a^b\ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)f(x)\,dx\leq\frac{f(a)+f(b)}{2}.\tag{2.18}$$

3. Applications

Assume that $\Psi:I\to(0,\infty)$ is *log-convex* on I, namely $\Phi=\ln\Psi$ is convex on I, then for $g:[a,b]\subset\mathbb{R}\to I$ an integrable function such that $\ln\Psi\circ g$ is also integrable on $[a,b]$, we have

$$\begin{aligned}\Psi(D_{a+,b-}g(x))&\leq\sqrt{\Psi\left(\frac{\int_a^x g(t)\,dt}{x-a}\right)\Psi\left(\frac{\int_x^b g(t)\,dt}{b-x}\right)}\\&\leq\exp\left[D_{a+,b-}(\ln\Psi\circ g)(x)\right]\end{aligned}\tag{3.1}$$

for any $x\in(a,b)$.

If we write the inequality (2.6) for $\Phi = \ln \Psi$, then we get

$$
\begin{aligned}
(3.2)\quad & \Psi\left(\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) g(x)\,dx\right) \\
&\leq \exp\left[\frac{1}{b-a}\int_a^b \ln \Psi\left(D_{a+,b-}g(x)\right)dx\right] \\
&\leq \exp\left[\frac{1}{b-a}\int_a^b \ln\left(\sqrt{\Psi\left(\frac{\int_a^x g(t)\,dt}{x-a}\right)\Psi\left(\frac{\int_x^b g(t)\,dt}{b-x}\right)}\right)dx\right] \\
&\leq \exp\left[\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)(\ln \Psi \circ g)(x)\,dx\right].
\end{aligned}
$$

Let $\Psi : I \to (0,\infty)$ be *log-convex* on I and $g : [a,b] \subset \mathbb{R} \to [m,M] \subset I$ an integrable function such that $(\ln \Phi) \circ g$ is also integrable on $[a,b]$. Then for any $x \in (a,b)$ we have

$$
\begin{aligned}
(3.3)\quad & \exp\left[D_{a+,b-}(\ln \Psi \circ g)(x)\right] \\
&\leq \sqrt{\Psi(m)\Psi(M)}\left(\frac{\Psi(M)}{\Psi(m)}\right)^{\frac{1}{M-m}\left(D_{a+,b-}g(x)-\frac{m+M}{2}\right)}
\end{aligned}
$$

and

$$
\begin{aligned}
(3.4)\quad & \exp\left[\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)(\ln \Psi \circ g)(x)\,dx\right] \\
&\leq \sqrt{\Psi(m)\Psi(M)}\left(\frac{\Psi(M)}{\Psi(m)}\right)^{\frac{1}{M-m}\left(\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)g(x)dx-\frac{m+M}{2}\right)}.
\end{aligned}
$$

If we take $\Psi(t) = t^{-1}$, $t > 0$, which is a log-convex function, then from (3.1)-(3.4) we can state some simple inequalities as well. The details are left to the interested reader.

If we consider the function $\Phi : (0,\infty) \to (0,\infty)$, $\Phi(t) = t^p$, $p \in (-\infty,0) \cup [1,\infty)$, then by (2.1) we have for integrable function $g : [a,b] \to (0,\infty)$ that

$$
\begin{aligned}
(3.5)\quad & \left(D_{a+,b-}g(x)\right)^p \\
&\leq \frac{1}{2}\left[\left(\frac{\int_a^x g(t)\,dt}{x-a}\right)^p + \left(\frac{\int_x^b g(t)\,dt}{b-x}\right)^p\right] \leq D_{a+,b-}(g^p)(x),
\end{aligned}
$$

for $x \in (a,b)$.

If we employ the inequality (2.6) for the same power function, then we have

$$\left(\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) g(x)\,dx\right)^p \tag{3.6}$$
$$\leq \frac{1}{b-a}\int_a^b \left(D_{a+,b-}g(x)\right)^p dx$$
$$\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b \left(\frac{\int_a^x g(t)\,dt}{x-a}\right)^p dx + \frac{1}{b-a}\int_a^b \left(\frac{\int_x^b g(t)\,dt}{b-x}\right)^p dx\right]$$
$$\leq \frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) g^p(x)\,dx,$$

for integrable function $g:[a,b]\to(0,\infty)$.

If $g:[a,b]\subset\mathbb{R}\to[m,M]\subset(0,\infty)$ is integrable, then by Theorem 6 for the function $\Phi:(0,\infty)\to(0,\infty)$, $\Phi(t)=t^p$, $p\in(-\infty,0)\cup[1,\infty)$ we get

$$D_{a+,b-}(g^p)(x) \leq \frac{m^p+M^p}{2} + \frac{M^p-m^p}{M-m}\left(D_{a+,b-}g(x) - \frac{m+M}{2}\right) \tag{3.7}$$

for $x\in(a,b)$ and

$$\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) g^p(x)\,dx \leq \frac{m^p+M^p}{2} \tag{3.8}$$
$$+\frac{M^p-m^p}{M-m}\left(\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) g(x)\,dx - \frac{m+M}{2}\right).$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa