RIEMANN-STIELTJES INTEGRAL INEQUALITIES OF MODIFIED TRAPEZOID TYPE

SILVESTRU SEVER DRAGOMIR

Abstract. In this paper we provide some bounds for the error in approximating the Riemann-Stieltjes integral $\int_a^b f(t)\,g(t)\,du(t)$ by the modified trapezoidal rule

$$f(a)\int_a^x g(t)\,du(t)+f(b)\int_x^b g(t)\,du(t)$$

under various assumptions for the integrands f and g, and the integrator u for which the above integral exists. Applications for continuous functions of selfadjoint operators in Hilbert spaces are provided as well.

1. Introduction

For two functions $f,\ u:[a,b]\to\mathbb{R}$ and $x\in[a,b]$, consider the *generalized Ostrowski functional* (see, [5]):

$$\theta(f,u;a,x,b):=[u(b)-u(a)]\,f(x)-\int_a^b f(t)\,du(t), \tag{1.1}$$

where the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ is assumed to exist.

In [7], the author proved the following inequality

$$|\theta(f,u;a,x,b)|\le H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(u) \tag{1.2}$$

for all $x\in[a,b]$, where $f:[a,b]\to\mathbb{R}$ is of r-H-Hölder type, u a function of bounded variation and $\bigvee_a^b(u)$ is its total variation on $[a,b]$. We recall this to mean,

$$|f(x)-f(y)|\le H|x-y|^r \quad\text{for any}\quad x,\ y\in[a,b]; \tag{1.3}$$

where $H>0$, $r\in(0,1]$ are given. He has shown also that the constant $\frac{1}{2}$, the coefficient of $(b-a)$, is the best possible for all $r\in(0,1]$.

In [5], by the use of a different technique, the author has proved the following complementary result

$$|\theta(f,u;a,x,b)|\le H\left[(x-a)^r\bigvee_a^x(f)+(b-x)^r\bigvee_x^b(f)\right] \tag{1.4}$$

1991 *Mathematics Subject Classification.* 26D15, 41A55, 47A63.

Key words and phrases. Riemann-Stieltjes integral, Trapezoidal Quadrature Rule, Selfadjoint operators, Functions of Selfadjoint operators, Spectral representation, Inequalities for selfadjoint operators.

RGMIA Res. Rep. Coll. 21 (2018), Art. 87, 16 pp. Received 26/07/18

$$\leq \begin{cases} H\left[(x-a)^r+(b-x)^r\right]\left[\dfrac{1}{2}\bigvee_a^b(f)+\dfrac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right], \\ H\left[(x-a)^{qr}+(b-x)^{qr}\right]^{\frac{1}{q}}\left[\left(\bigvee_a^x(f)\right)^p+\left(\bigvee_x^b(f)\right)^p\right]^{\frac{1}{p}} \\ \qquad\qquad \text{where } \ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(f), \end{cases}$$

provided f is a mapping of bounded variation and u is of Hölder type with the constant $r\in(0,1]$ and $H>0$.

In 2000, Dragomir et al. [10] have also considered another approach in approximating the Riemann-Stieltjes integral. Namely, they introduced the *generalized trapezoid functional*

$$GT(f,u;a,x,b):=[u(x)-u(a)]f(a)+[u(b)-u(x)]f(b)-\int_a^b f(t)\,du(t) \tag{1.5}$$

and using the identity

$$GT(f,u;a,x,b)=\int_a^b (u(t)-u(x))\,df(t) \tag{1.6}$$

they proved the result

$$|GT(f,u;a,x,b)|\leq H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(f), \tag{1.7}$$

provided u is of r-H-Hölder type ($H>0,\ r\in(0,1]$) and f is of bounded variation. Here the constant $\frac{1}{2}$, the coefficient of $(b-a)$, is also sharp. A partitioning of the interval of integration allowed the estimation of the error to be determined, enabling *a priory* knowledge for a desired accuracy.

In [4], Cerone and Dragomir obtained the following result for the *generalized trapezoid functional*:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a function of r-H-Hölder type and $u:[a,b]\to\mathbb{R}$ a function of bounded variation on $[a,b]$. Then*

$$\begin{aligned} &|GT(f;u,a,x,b)| \\ &\leq H\left[(x-a)^r\bigvee_a^x(u)+(b-x)^r\bigvee_x^b(u)\right] \\ &\leq \begin{cases} H\bigvee_a^b(u)\left[\frac{1}{2}+\left|\dfrac{x-\frac{a+b}{2}}{b-a}\right|\right]^r(b-a)^r, \\ H\left[\left(\bigvee_a^x(u)\right)^\beta+\left(\bigvee_x^b(u)\right)^\beta\right]^{\frac{1}{\beta}}\left[\left(\frac{x-a}{b-a}\right)^{\alpha r}+\left(\frac{b-x}{b-a}\right)^{\alpha r}\right]^{\frac{1}{\alpha}}(b-a)^r \\ \qquad\qquad \textit{where } \ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ H\left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right]\left[\left(\frac{x-a}{b-a}\right)^r+\left(\frac{b-x}{b-a}\right)^r\right](b-a)^r, \end{cases} \end{aligned} \tag{1.8}$$

for all $x\in[a,b]$.

For some trapezoid type inequalities, see [2]-[10] and [12]-[13].

We consider now the more general functional

$$GT(f,g,u;a,x,b) := f(a)\int_a^x g(t)\,du(t) + f(b)\int_x^b g(t)\,du(t) - \int_a^b f(t)\,g(t)\,du(t)$$

for functions for which the Riemann-Stieltjes integral exist and $x\in[a,b]$.

In particular, for $g\equiv 1$, we have

$$GT(f,1,u;a,x,b) = GT(f;u,a,x,b).$$

Motivated by the above results, in this paper we establish some inequalities for the quantity $GT(f,g,u;a,x,b)$ under various assumptions for the functions involved. Applications for continuous functions of selfadjoint operators in Hilbert spaces are provided as well.

2. Inequalities for Integrands of Bounded Variation

Assume that $u,\ f:[a,b]\to\mathbb{C}$. If the Riemann-Stieltjes integral $\int_a^b f(u)\,du(t)$ exists, we write for simplicity, like in [1, p. 142] that $f\in\mathcal{R}_{\mathbb{C}}(u,[a,b])$, or $\mathcal{R}_{\mathbb{C}}(u)$ when the interval is implicitly known. If the functions $u,\ f$ are real valued, then we write $f\in\mathcal{R}(u,[a,b])$, or $\mathcal{R}(u)$.

We start with the following simple fact, see also :

Lemma 1. *Let $f,\ g,\ v:[a,b]\to\mathbb{C}$, $\lambda,\ \mu\in\mathbb{C}$ and $x\in[a,b]$. If $fg,\ g\in\mathcal{R}_{\mathbb{C}}(v,[a,x])\cap\mathcal{R}_{\mathbb{C}}(v,[x,b])$, then $fg,\ g\in\mathcal{R}_{\mathbb{C}}(v,[a,b])$ and*

$$\begin{aligned}\int_a^b f(t)\,g(t)\,dv(t) &= \lambda\int_a^x g(t)\,dv(t) + \mu\int_x^b g(t)\,dv(t)\\ &\quad+\int_a^x [f(t)-\lambda]\,g(t)\,dv(t) + \int_x^b [f(t)-\mu]\,g(t)\,dv(t)\\ &= \mu\int_a^b g(t)\,dv(t) + (\lambda-\mu)\int_a^x g(t)\,dv(t)\\ &\quad+\int_a^x [f(t)-\lambda]\,g(t)\,dv(t) + \int_x^b [f(t)-\mu]\,g(t)\,dv(t).\end{aligned} \tag{2.1}$$

In particular, for $\mu=\lambda$, we have

$$\begin{aligned}\int_a^b f(t)\,g(t)\,dv(t) &= \lambda\int_a^b g(t)\,dv(t)\\ &\quad+\int_a^x [f(t)-\lambda]\,g(t)\,dv(t) + \int_x^b [f(t)-\lambda]\,g(t)\,dv(t)\\ &= \lambda\int_a^b g(t)\,dv(t) + \int_a^b [f(t)-\lambda]\,g(t)\,dv(t).\end{aligned} \tag{2.2}$$

Proof. The integrability follows by Theorem 7. 4 from [1] which says that if a function is Riemann-Stieltjes integrable on the intervals $[a,x]$, $[x,b]$ with $x\in[a,b]$, then it is integrable on the whole interval $[a,b]$.

Using the properties of the Riemann-Stieltjes integral, we have

$$\begin{aligned} &\int_a^x [f(t)-\lambda] g(t)\,dv(t) + \int_x^b [f(t)-\mu] g(t)\,dv(t) \\ &= \int_a^x f(t) g(t)\,dv(t) - \lambda \int_a^x g(t)\,dv(t) + \int_x^b f(t) g(t)\,dv(t) - \mu \int_x^b g(t)\,dv(t) \\ &= \int_a^b f(t) g(t)\,dv(t) - \lambda \int_a^x g(t)\,dv(t) - \mu \int_x^b g(t)\,dv(t), \end{aligned}$$

which is equivalent to the first equality in (2.1).

The rest is obvious. □

Corollary 1. *Assume that* $f,\ v : [a,b] \to \mathbb{C}$ *and* $x \in [a,b]$ *are such that* $f \in \mathcal{R}_{\mathbb{C}}(v,[a,x]) \cap \mathcal{R}_{\mathbb{C}}(v,[x,b])$. *Then for any* $\lambda,\ \mu \in \mathbb{C}$ *we have the equality*

$$\begin{aligned} \int_a^b f(t)\,dv(t) &= \lambda [v(x)-v(a)] + \mu [v(b)-v(x)] \\ &\quad + \int_a^x [f(t)-\lambda]\,dv(t) + \int_x^b [f(t)-\mu]\,dv(t). \end{aligned} \tag{2.3}$$

In particular, for $\mu = \lambda$, *we have*

$$\begin{aligned} \int_a^b f(t)\,dv(t) &= \lambda [v(b)-v(a)] \\ &\quad + \int_a^x [f(t)-\lambda]\,dv(t) + \int_x^b [f(t)-\lambda]\,dv(t) \\ &= \lambda [v(b)-v(a)] + \int_a^b [f(t)-\lambda]\,dv(t). \end{aligned} \tag{2.4}$$

The proof follows by Lemma 1 for $g(t) = 1$, $t \in [a,b]$.

Remark 1. *We observe that, see* [1, Theorem 7.27], *if* $f,\ g \in \mathcal{C}_{\mathbb{C}}[a,b]$, *namely, are continuous on* $[a,b]$ *and* $v \in \mathcal{BV}_{\mathbb{C}}[a,b]$, *namely of bounded variation on* $[a,b]$, *then for any* $x \in [a,b]$ *the Riemann-Stieltjes integrals in Lemma 1 exist and the equalities (2.1) and (2.2) hold.*

If we take $\lambda = f(a)$ and $\mu = f(b)$ in (2.1) we get for $x \in [a,b]$ that

$$\begin{aligned} \int_a^b f(t) g(t)\,du(t) &= f(a) \int_a^x g(t)\,du(t) + f(b) \int_x^b g(t)\,du(t) \\ &\quad + \int_a^x [f(t)-f(a)] g(t)\,du(t) + \int_x^b [f(t)-f(b)] g(t)\,du(t). \end{aligned} \tag{2.5}$$

In particular, for $g(t) = 1$, $t \in [a,b]$, we have for $x \in [a,b]$ that

$$\begin{aligned} \int_a^b f(t)\,du(t) &= [u(x)-u(a)] f(a) + [u(b)-u(x)] f(b) \\ &\quad + \int_a^x [f(t)-f(a)]\,du(t) + \int_x^b [f(t)-f(b)]\,du(t). \end{aligned} \tag{2.6}$$

We have:

Theorem 2. *Assume that* $f,\ g\in\mathcal{C}_{\mathbb{C}}[a,b]$ *and* $u\in\mathcal{BV}_{\mathbb{C}}[a,b]$. *If* $f\in\mathcal{BV}_{\mathbb{C}}[a,b]$, *then*

$$
\begin{aligned}
(2.7)\quad & |GT(f,g,u;a,x,b)| \\
&\le \bigvee_a^x (f)\int_a^x |g(t)|\,d\left(\bigvee_a^t (u)\right)+\bigvee_x^b (f)\int_x^b |g(t)|\,d\left(\bigvee_a^t (u)\right)\\
&\le \frac{1}{2}\left(\bigvee_a^b (f)+\left|\bigvee_a^x (f)-\bigvee_x^b (f)\right|\right)\int_a^b |g(t)|\,d\left(\bigvee_a^t (u)\right)\\
&\le \frac{1}{2}\left(\bigvee_a^b (f)+\left|\bigvee_a^x (f)-\bigvee_x^b (f)\right|\right)\max_{t\in[a,b]}|g(t)|\bigvee_a^b (u)
\end{aligned}
$$

for all $x\in[a,b]$.

Proof. It is well known that if $p\in\mathcal{R}(u,[a,b])$ where $u\in\mathcal{BV}_{\mathbb{C}}[a,b]$ then we have [1, p. 177]

$$
(2.8)\qquad \left|\int_a^b p(t)\,du(t)\right|\le\int_a^b |p(t)|\,d\left(\bigvee_a^t (u)\right)\le\sup_{t\in[a,b]}|p(t)|\bigvee_a^b (u).
$$

Using the representation (2.5) we have

$$
\begin{aligned}
(2.9)\quad & \left|\int_a^b f(t)\,g(t)\,du(t)-f(a)\int_a^x g(t)\,du(t)-f(b)\int_x^b g(t)\,du(t)\right|\\
&\le\left|\int_a^x [f(t)-f(a)]\,g(t)\,du(t)\right|+\left|\int_x^b [f(t)-f(b)]\,g(t)\,du(t)\right|\\
&\le\int_a^x |f(t)-f(a)|\,|g(t)|\,d\left(\bigvee_a^t (u)\right)+\int_x^b |f(t)-f(b)|\,|g(t)|\,d\left(\bigvee_a^t (u)\right)\\
&=:B(f,g,u;x)
\end{aligned}
$$

for $x\in(a,b)$.

Since f is of bounded variation on $[a,b]$, hence

$$
|f(t)-f(a)|\le\bigvee_a^t (f)\ \text{ for } t\in[a,x]
$$

and

$$
|f(t)-f(b)|\le\bigvee_t^b (f)\ \text{ for } t\in[x,b],
$$

which gives

$$\begin{aligned}
&B(f,g,u;x)\\
&\leq \int_a^x |g(t)| \left(\bigvee_a^t (f)\right) d\left(\bigvee_a^t (u)\right) + \int_x^b |g(t)| \bigvee_t^b (f)\, d\left(\bigvee_a^t (u)\right)\\
&\leq \bigvee_a^x (f) \int_a^x |g(t)|\, d\left(\bigvee_a^t (u)\right) + \bigvee_x^b (f) \int_x^b |g(t)|\, d\left(\bigvee_a^t (u)\right)\\
&\leq \max\left\{\bigvee_a^x (f), \bigvee_x^b (f)\right\} \int_a^b |g(t)|\, d\left(\bigvee_a^t (u)\right)\\
&= \left(\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right) \int_a^b |g(t)|\, d\left(\bigvee_a^t (u)\right).
\end{aligned}$$

This proves the first and second inequality in (2.7).

The last part of (2.7) is obvious. □

Corollary 2. *If $p \in [a,b]$ is such that $\bigvee_a^p (f) = \bigvee_p^b (f)$, then*

$$\begin{aligned}
&|GT(f,g,u;a,p,b)| \\
&\qquad \leq \frac{1}{2}\bigvee_a^b (f) \int_a^b |g(t)|\, d\left(\bigvee_a^t (u)\right) \leq \frac{1}{2}\max_{t\in[a,b]} |g(t)| \bigvee_a^b (f) \bigvee_a^b (u).
\end{aligned} \tag{2.10}$$

Remark 2. *In particular, for $g(t) = 1$, $t \in [a,b]$, we have for $x \in [a,b]$ that*

$$\begin{aligned}
|GT(f;u,a,x,b)| &\leq \bigvee_a^x (f) \bigvee_a^x (u) + \bigvee_x^b (f) \bigvee_x^b (u)\\
&\leq \begin{cases} \frac{1}{2}\left(\bigvee_a^b (f) + \left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right) \bigvee_a^b (u) \\ \\ \frac{1}{2}\left(\bigvee_a^b (u) + \left|\bigvee_a^x (u) - \bigvee_x^b (u)\right|\right) \bigvee_a^b (f) \end{cases} \leq \bigvee_a^b (f) \bigvee_a^b (u).
\end{aligned} \tag{2.11}$$

If $p \in [a,b]$ is such that $\bigvee_a^p (f) = \bigvee_p^b (f)$, then

$$|GT(f;u,a,p,b)| \leq \frac{1}{2}\bigvee_a^b (f) \bigvee_a^b (u). \tag{2.12}$$

If $m \in [a,b]$ is such that $\bigvee_a^m (u) = \bigvee_m^b (u)$, then

$$|GT(f;u,a,m,b)| \leq \frac{1}{2}\bigvee_a^b (f) \bigvee_a^b (u). \tag{2.13}$$

Corollary 3. *Assume that* $f \in \mathcal{C}_{\mathbb{C}}[a,b] \cap \mathcal{BV}_{\mathbb{C}}[a,b]$ *and* $u \in \mathcal{BV}_{\mathbb{C}}[a,b]$. *If* g *is such that* $|g|$ *is convex on* $[a,b]$, *then for all* $x \in [a,b]$

$$|GT(f,g,u;a,x,b)| \le \frac{1}{2(b-a)}\left(\bigvee_a^b(f) + \left|\bigvee_a^x(f) - \bigvee_x^b(f)\right|\right) \times \left[|g(a)|\int_a^b\left(\bigvee_a^t(u)\right)dt + |g(b)|\int_a^b\left(\bigvee_t^b(u)\right)dt\right] \le \frac{|g(a)|+|g(b)|}{2}\left(\bigvee_a^b(f) + \left|\bigvee_a^x(f) - \bigvee_x^b(f)\right|\right)\bigvee_a^b(u). \tag{2.14}$$

Proof. Since $|g|$ is convex on $[a,b]$, then

$$|g(t)| = \left|g\left(\frac{(b-t)a+(t-a)b}{b-a}\right)\right| \le \frac{(b-t)|g(a)|+(t-a)|g(b)|}{b-a}$$

for $t \in [a,b]$.

Since $\bigvee_a^{\cdot}(u)$ is monotonic nondecreasing, then

$$\int_a^b |g(t)|\, d\left(\bigvee_a^t(u)\right) \le \int_a^b\left[\frac{(b-t)|g(a)|+(t-a)|g(b)|}{b-a}\right]d\left(\bigvee_a^t(u)\right) = \frac{|g(a)|}{b-a}\int_a^b(b-t)\,d\left(\bigvee_a^t(u)\right) + \frac{|g(b)|}{b-a}\int_a^b(t-a)\,d\left(\bigvee_a^t(u)\right). \tag{2.15}$$

Using the integration by parts formula, we have

$$\int_a^b(b-t)\,d\left(\bigvee_a^t(u)\right) = (b-t)\bigvee_a^t(u)\Big|_a^b + \int_a^b\left(\bigvee_a^t(u)\right)dt = \int_a^b\left(\bigvee_a^t(u)\right)dt$$

and

$$\int_a^b(t-a)\,d\left(\bigvee_a^t(u)\right) = (t-a)\bigvee_a^t(u)\Big|_a^b - \int_a^b\left(\bigvee_a^t(u)\right)dt = (b-a)\bigvee_a^b(u) - \int_a^b\left(\bigvee_a^t(u)\right)dt = \int_a^b\left(\bigvee_a^b(u) - \bigvee_a^t(u)\right)dt = \int_a^b\left(\bigvee_t^b(u)\right)dt.$$

By using the second inequality in (2.7) we get the first inequality in (2.14).

Also, observe that

$$\int_a^b\left(\bigvee_a^t(u)\right)dt \le (b-a)\bigvee_a^b(u) \text{ and } \int_a^b\left(\bigvee_t^b(u)\right)dt \le (b-a)\bigvee_a^b(u),$$

which proves the last part of (2.14). □

Remark 3. *If* $p \in [a,b]$ *is such that* $\bigvee_a^p (f) = \bigvee_p^b (f)$, *then*

$$|GT(f,g,u;a,p,b)| \leq \frac{1}{2(b-a)} \bigvee_a^b (f) \left[|g(a)| \int_a^b \left(\bigvee_a^t (u) \right) dt + |g(b)| \int_a^b \left(\bigvee_t^b (u) \right) dt \right] \leq \frac{|g(a)| + |g(b)|}{2} \bigvee_a^b (f) \bigvee_a^b (u). \tag{2.16}$$

3. Inequalities for Lipschitzian Integrands

The following result also holds:

Theorem 3. *Assume that* f *satisfies the end-point Lipschitzian conditions*

$$|f(t) - f(a)| \leq L_a (t-a)^\alpha \text{ and } |f(b) - f(t)| \leq L_b (b-t)^\beta \tag{3.1}$$

for any $t \in (a,b)$ *where the constants* L_a, $L_b > 0$ *and* α, $\beta > 0$ *are given. If* $g \in \mathcal{C}_{\mathbb{C}}[a,b]$ *and* $u \in \mathcal{BV}_{\mathbb{C}}[a,b]$, *then for any* $x \in [a,b]$

$$\begin{aligned} |GT(f,g,u;a,x,b)| &\leq L_a \int_a^x (t-a)^\alpha |g(t)| \, d\left(\bigvee_a^t (u) \right) + L_b \int_x^b (b-t)^\beta |g(t)| \, d\left(\bigvee_a^t (u) \right) \\ &\leq \alpha L_a \max_{t\in[a,x]} |g(t)| \int_a^x (t-a)^{\alpha-1} \left(\bigvee_t^x (u) \right) dt \\ &\quad + \beta L_b \max_{t\in[x,b]} |g(t)| \int_x^b (b-t)^{\beta-1} \left(\bigvee_x^t (u) \right) dt \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\leq \max_{t\in[a,b]} |g(t)| \left[\alpha L_a \int_a^x (t-a)^{\alpha-1} \left(\bigvee_t^x (u) \right) dt + \beta L_b \int_x^b (b-t)^{\beta-1} \left(\bigvee_x^t (u) \right) dt \right] \\ &\leq \max_{t\in[a,b]} |g(t)| \left[(x-a)^\alpha L_a \bigvee_a^x (u) + (b-x)^\beta L_b \bigvee_x^b (u) \right] \\ &\leq \frac{1}{2} \max_{t\in[a,b]} |g(t)| \left[L_a (x-a)^\alpha + L_b (b-x)^\beta \right] \left(\bigvee_a^b (u) + \left| \bigvee_a^x (u) - \bigvee_x^b (u) \right| \right). \end{aligned}$$

Proof. Since f satisfies the condition (3.1) on $[a,b]$, hence

$$\begin{aligned}
&\left|\int_a^b f(t)\,g(t)\,du(t) - f(a)\int_a^x g(t)\,du(t) - f(b)\int_x^b g(t)\,du(t)\right| \qquad (3.3)\\
&\le \left|\int_a^x [f(t)-f(a)]\,g(t)\,du(t)\right| + \left|\int_x^b [f(t)-f(b)]\,g(t)\,du(t)\right|\\
&\le \int_a^x |f(t)-f(a)|\,|g(t)|\,d\left(\bigvee_a^t(u)\right) + \int_x^b |f(t)-f(b)|\,|g(t)|\,d\left(\bigvee_a^t(u)\right)\\
&\le L_a\int_a^x (t-a)^\alpha\,|g(t)|\,d\left(\bigvee_a^t(u)\right) + L_b\int_x^b (b-t)^\beta\,|g(t)|\,d\left(\bigvee_a^t(u)\right)\\
&\le L_a \max_{t\in[a,x]}|g(t)|\int_a^x (t-a)^\alpha\,d\left(\bigvee_a^t(u)\right)\\
&\quad + L_b \max_{t\in[x,b]}|g(t)|\int_x^b (b-t)^\beta\,d\left(\bigvee_a^t(u)\right) =: C(g,u;x),
\end{aligned}$$

for $x\in(a,b)$.

Using the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\begin{aligned}
&\int_a^x (t-a)^\alpha\,d\left(\bigvee_a^t(u)\right)\\
&= (t-a)^\alpha \bigvee_a^t(u)\Big|_a^x - \alpha\int_a^x (t-a)^{\alpha-1}\left(\bigvee_a^t(u)\right)dt\\
&= (x-a)^\alpha \bigvee_a^x(u) - \alpha\int_a^x (t-a)^{\alpha-1}\left(\bigvee_a^t(u)\right)dt\\
&= \alpha\bigvee_a^x(u)\int_a^x (t-a)^{\alpha-1}\,dt - \alpha\int_a^x (t-a)^{\alpha-1}\left(\bigvee_a^t(u)\right)dt\\
&= \alpha\int_a^x (t-a)^{\alpha-1}\left(\bigvee_a^x(u)-\bigvee_a^t(u)\right)dt = \alpha\int_a^x (t-a)^{\alpha-1}\left(\bigvee_t^x(u)\right)dt
\end{aligned}$$

and

$$\begin{aligned}
\int_x^b (b-t)^\beta\,d\left(\bigvee_x^t(u)\right) &= (b-t)^\beta\bigvee_x^t(u)\Big|_x^b + \beta\int_x^b (b-t)^{\beta-1}\left(\bigvee_x^t(u)\right)dt\\
&= \beta\int_x^b (b-t)^{\beta-1}\left(\bigvee_x^t(u)\right)dt,
\end{aligned}$$

which implies that

$$C(g,u;x) \leq \alpha L_a \max_{t\in[a,x]} |g(t)| \int_a^x (t-a)^{\alpha-1} \left(\bigvee_t^x (u)\right) dt + \beta L_b \max_{t\in[x,b]} |g(t)| \int_x^b (b-t)^{\beta-1} \left(\bigvee_x^t (u)\right) dt$$

for $x \in (a,b)$.

The last part of (3.2) is obvious. □

Corollary 4. *With the assumptions of Theorem 3 and if $m \in [a,b]$ is such that $\bigvee_a^m (u) = \bigvee_m^b (u)$, then*

$$\begin{aligned} (3.4)\quad & |GT(f,g,u;a,m,b)| \\ & \leq L_a \int_a^m (t-a)^{\alpha} |g(t)| \, d\left(\bigvee_a^t (u)\right) + L_b \int_m^b (b-t)^{\beta} |g(t)| \, d\left(\bigvee_a^t (u)\right) \\ & \leq \alpha L_a \max_{t\in[a,m]} |g(t)| \int_a^m (t-a)^{\alpha-1} \left(\bigvee_t^m (u)\right) dt \\ & \quad + \beta L_b \max_{t\in[m,b]} |g(t)| \int_m^b (b-t)^{\beta-1} \left(\bigvee_m^t (u)\right) dt \\ & \leq \max_{t\in[a,b]} |g(t)| \left[\alpha L_a \int_a^m (t-a)^{\alpha-1} \left(\bigvee_t^m (u)\right) dt + \beta L_b \int_m^b (b-t)^{\beta-1} \left(\bigvee_m^t (u)\right) dt\right] \\ & \leq \frac{1}{2} \max_{t\in[a,b]} |g(t)| \left[L_a (m-a)^{\alpha} + L_b (b-m)^{\beta}\right] \bigvee_a^b (u). \end{aligned}$$

Remark 4. *If we take $g(t) = 1$ in (3.2), then we get*

$$\begin{aligned} (3.5)\quad & |GT(f;u,a,x,b)| \\ & \leq L_a \int_a^x (t-a)^{\alpha} \, d\left(\bigvee_a^t (u)\right) + L_b \int_x^b (b-t)^{\beta} \, d\left(\bigvee_a^t (u)\right) \\ & \leq \alpha L_a \int_a^x (t-a)^{\alpha-1} \left(\bigvee_t^x (u)\right) dt + \beta L_b \int_x^b (b-t)^{\beta-1} \left(\bigvee_x^t (u)\right) dt \\ & \leq \alpha L_a \int_a^x (t-a)^{\alpha-1} \left(\bigvee_t^x (u)\right) dt + \beta L_b \int_x^b (b-t)^{\beta-1} \left(\bigvee_x^t (u)\right) dt \\ & \leq (x-a)^{\alpha} L_a \bigvee_a^x (u) + (b-x)^{\beta} L_b \bigvee_x^b (u) \\ & \leq \frac{1}{2} \left[L_a (x-a)^{\alpha} + L_b (b-x)^{\beta}\right] \left(\bigvee_a^b (u) + \left|\bigvee_a^x (u) - \bigvee_x^b (u)\right|\right). \end{aligned}$$

If $m \in [a,b]$ *is such that* $\bigvee_a^m (u) = \bigvee_m^b (u)$, *then*

$$
\begin{aligned}
(3.6)\quad & |GT(f;u,a,m,b)| \\
&\le L_a \int_a^m (t-a)^{\alpha} d\left(\bigvee_a^t (u)\right) + L_b \int_m^b (b-t)^{\beta} d\left(\bigvee_a^t (u)\right) \\
&\le \alpha L_a \int_a^m (t-a)^{\alpha-1}\left(\bigvee_t^m (u)\right) dt + \beta L_b \int_m^b (b-t)^{\beta-1}\left(\bigvee_m^t (u)\right) dt
\end{aligned}
$$

$$
\begin{aligned}
&\le \alpha L_a \int_a^m (t-a)^{\alpha-1}\left(\bigvee_t^m (u)\right) dt + \beta L_b \int_m^b (b-t)^{\beta-1}\left(\bigvee_m^t (u)\right) dt \\
&\qquad \le \frac{1}{2}\left[L_a (m-a)^{\alpha} + L_b (b-m)^{\beta}\right] \bigvee_a^b (u).
\end{aligned}
$$

Corollary 5. *Assume that* f *satisfies the Lipschitzian condition*

$$|f(t) - f(s)| \le L|t-s|$$

for any $t,\ s \in (a,b)$ *where the constant* $L > 0$. *If* $g \in \mathcal{C}_{\mathbb{C}}[a,b]$ *and* $u \in \mathcal{BV}_{\mathbb{C}}[a,b]$, *then for any* $x \in [a,b]$

$$
\begin{aligned}
(3.7)\quad & |GT(f,g,u;a,x,b)| \\
&\le L\left[\int_a^x (t-a)|g(t)| d\left(\bigvee_a^t (u)\right) + \int_x^b (b-t)|g(t)| d\left(\bigvee_a^t (u)\right)\right] \\
&\le L\left[\max_{t\in[a,x]} |g(t)| \int_a^x \left(\bigvee_t^x (u)\right) dt + \max_{t\in[x,b]} |g(t)| \int_x^b \left(\bigvee_x^t (u)\right) dt\right]
\end{aligned}
$$

$$
\begin{aligned}
&\le L \max_{t\in[a,b]} |g(t)| \left[\int_a^x \left(\bigvee_t^x (u)\right) dt + \int_x^b \left(\bigvee_x^t (u)\right) dt\right] \\
&\qquad \le L \max_{t\in[a,b]} |g(t)| \left[(x-a)\bigvee_a^x (u) + (b-x)\bigvee_x^b (u)\right] \\
&\qquad\qquad \le \frac{1}{2} L \max_{t\in[a,b]} |g(t)| (b-a)\left(\bigvee_a^b (u) + \left|\bigvee_a^x (u) - \bigvee_x^b (u)\right|\right).
\end{aligned}
$$

In particular, if $m \in [a,b]$ *is such that* $\bigvee_a^m(u) = \bigvee_m^b(u)$, *then*

$$
\begin{aligned}
(3.8)\quad & |GT(f,g,u;a,m,b)| \\
& \leq L\left[\int_a^m (t-a)\,|g(t)|\,d\left(\bigvee_a^t(u)\right) + \int_m^b (b-t)\,|g(t)|\,d\left(\bigvee_a^t(u)\right)\right] \\
& \leq L\left[\max_{t\in[a,m]}|g(t)|\int_a^m\left(\bigvee_t^m(u)\right)dt + \max_{t\in[m,b]}|g(t)|\int_m^b\left(\bigvee_m^t(u)\right)dt\right] \\
& \leq L\max_{t\in[a,b]}|g(t)|\left[\int_a^m\left(\bigvee_t^m(u)\right)dt + \int_m^b\left(\bigvee_m^t(u)\right)dt\right] \\
& \leq \frac{1}{2}L\max_{t\in[a,b]}|g(t)|\,(b-a)\bigvee_a^b(u).
\end{aligned}
$$

Remark 5. *We the assumptions of Corollary 5 and if* $g(t) = 1$, $t \in [a,b]$, *then by (3.7) we get*

$$
\begin{aligned}
(3.9)\quad & |GT(f;u,a,x,b)| \\
& \leq L\left[\int_a^x (t-a)\,d\left(\bigvee_a^t(u)\right) + \int_x^b (b-t)\,d\left(\bigvee_a^t(u)\right)\right] \\
& \leq L\left[\int_a^x\left(\bigvee_t^x(u)\right)dt + \int_x^b\left(\bigvee_x^t(u)\right)dt\right] \\
& \leq L\left[(x-a)\bigvee_a^x(u) + (b-x)\bigvee_x^b(u)\right] \\
& \leq \frac{1}{2}L(b-a)\left(\bigvee_a^b(u) + \left|\bigvee_a^x(u) - \bigvee_x^b(u)\right|\right)
\end{aligned}
$$

for $x \in [a,b]$, *while if* $m \in [a,b]$ *is such that* $\bigvee_a^m(u) = \bigvee_m^b(u)$, *then by (3.8)*

$$
\begin{aligned}
(3.10)\quad & |GT(f;u,a,m,b)| \\
& \leq L\left[\int_a^m (t-a)\,d\left(\bigvee_a^t(u)\right) + \int_m^b (b-t)\,d\left(\bigvee_a^t(u)\right)\right] \\
& \leq L\left[\int_a^m\left(\bigvee_t^m(u)\right)dt + \int_m^b\left(\bigvee_m^t(u)\right)dt\right] \leq \frac{1}{2}L(b-a)\bigvee_a^b(u).
\end{aligned}
$$

4. Applications for Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Let $A \in \mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda \in \mathbb{R}$ as follows

$$
\varphi_\lambda(s) := \begin{cases} 1, \text{ for } -\infty < s \leq \lambda, \\ \\ 0, \text{ for } \lambda < s < +\infty. \end{cases}
$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$E_{\lambda} := \varphi_{\lambda}(A) \tag{4.1}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [11, p. 256]:

Theorem 4 (Spectral Representation Theorem)**.** *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a = \min\{\lambda \,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $b = \max\{\lambda \,|\lambda \in Sp(A)\} =: \max Sp(A)$. Then there exists a family of projections $\{E_{\lambda}\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_{\lambda} \le E_{\lambda'}$ *for* $\lambda \le \lambda'$;
b) $E_{a-0} = 0, E_b = I$ *and* $E_{\lambda+0} = E_{\lambda}$ *for all* $\lambda \in \mathbb{R}$;
c) *We have the representation*

$$A = \int_{a-0}^{b} \lambda dE_{\lambda}.$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ there exists a unique operator $\varphi(A) \in \mathcal{B}(H)$ such that for every $\varepsilon > 0$ there exists a $\delta > 0$ satisfying the inequality

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi(\lambda_k') \left[E_{\lambda_k} - E_{\lambda_{k-1}}\right] \right\| \le \varepsilon$$

whenever

$$\begin{cases} \lambda_0 < a = \lambda_1 < \dots < \lambda_{n-1} < \lambda_n = b, \\ \lambda_k - \lambda_{k-1} \le \delta \text{ for } 1 \le k \le n, \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \le k \le n \end{cases}$$

this means that

$$\varphi(A) = \int_{a-0}^{b} \varphi(\lambda)\, dE_{\lambda}, \tag{4.2}$$

where the integral is of Riemann-Stieltjes type.

Corollary 6. *With the assumptions of Theorem 4 for A, E_{λ} and φ we have the representations*

$$\varphi(A)\, x = \int_{a-0}^{b} \varphi(\lambda)\, dE_{\lambda} x \quad \text{for all } x \in H$$

and

$$\langle \varphi(A)\, x, y \rangle = \int_{a-0}^{b} \varphi(\lambda)\, d\langle E_{\lambda} x, y \rangle \quad \text{for all } x,\ y \in H. \tag{4.3}$$

In particular,

$$\langle \varphi(A)\, x, x \rangle = \int_{a-0}^{b} \varphi(\lambda)\, d\langle E_{\lambda} x, x \rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(A)x\|^2 = \int_{a-0}^{b} |\varphi(\lambda)|^2 \, d\|E_\lambda x\|^2 \quad \textit{for all } x \in H.$$

We need the following result that provides an upper bound for the total variation of the function $\mathbb{R} \ni \lambda \mapsto \langle E_\lambda x, y\rangle \in \mathbb{C}$ on an interval $[\alpha, \beta]$, see [**?**].

Lemma 2. *Let $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A. Then for any x, $y \in H$ and $\alpha < \beta$ we have the inequality*

$$\left[\bigvee_{\alpha}^{\beta}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right]^2 \le \langle (E_\beta - E_\alpha) x, x\rangle \langle (E_\beta - E_\alpha) y, y\rangle, \tag{4.4}$$

where $\bigvee_{\alpha}^{\beta}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$ denotes the total variation of the function $\left\langle E_{(\cdot)}x, y\right\rangle$ on $[\alpha, \beta]$.

Remark 6. *For $\alpha = a - \varepsilon$ with $\varepsilon > 0$ and $\beta = b$ we get from (4.4) the inequality*

$$\bigvee_{a-\varepsilon}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \le \langle (I - E_{a-\varepsilon}) x, x\rangle^{1/2} \langle (I - E_{a-\varepsilon}) y, y\rangle^{1/2} \tag{4.5}$$

for any x, $y \in H$.

This implies, for any x, $y \in H$, that

$$\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \le \|x\| \|y\|, \tag{4.6}$$

where $\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$ denotes the limit $\lim_{\varepsilon \to 0+}\left[\bigvee_{a-\varepsilon}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right]$.

We can state the following result for functions of selfadjoint operators:

Theorem 5. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a = \min\{\lambda | \lambda \in Sp(A)\} =: \min Sp(A)$ and $b = \max\{\lambda | \lambda \in Sp(A)\} =: \max Sp(A)$. Also, assume that $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A and $f : I \to \mathbb{C}$ is continuous on I, $[a, b] \subset \mathring{I}$ (the interior of I) with f of locally bounded variation on I. Then for any x, $y \in H$ and $s \in (a, b)$*

$$\begin{aligned} &|\langle f(A)x, y\rangle - \langle E_s x, y\rangle f(a) - \langle (1_H - E_s) x, y\rangle f(b)| \\ &\quad \le \frac{1}{2}\left(\bigvee_{a}^{b}(f) + \left|\bigvee_{a}^{s}(f) - \bigvee_{s}^{b}(f)\right|\right) \bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \\ &\quad \le \frac{1}{2}\left(\bigvee_{a}^{b}(f) + \left|\bigvee_{a}^{s}(f) - \bigvee_{s}^{b}(f)\right|\right) \|x\| \|y\|. \end{aligned} \tag{4.7}$$

Proof. If we use the inequality (2.11), we have for small $\varepsilon > 0$ and for any $x, y \in H$ that

$$[\langle E_s x, y\rangle - \langle E_{a-\varepsilon} x, y\rangle] f(a-\varepsilon) + [\langle E_b x, y\rangle - \langle E_s x, y\rangle] f(b) - \int_{a-\varepsilon}^{b} f(t)\, d\langle E_t x, y\rangle$$
$$\leq \frac{1}{2}\left(\bigvee_{a-\varepsilon}^{b}(f) + \left|\bigvee_{a-\varepsilon}^{x}(f) - \bigvee_{x}^{b}(f)\right|\right)\bigvee_{a-\varepsilon}^{b}\left(\langle E_{(\cdot)} x, y\rangle\right).$$

Taking the limit over $\varepsilon \to 0+$ and using the continuity of f, g and the Spectral Representation Theorem, we deduce the desired result (4.7). □

Remark 7. *If $p \in [a,b]$ is such that $\bigvee_a^p(f) = \bigvee_p^b(f)$, then we obtain from (4.7) that*

$$\text{(4.8)} \quad |\langle f(A) x, y\rangle - \langle E_p x, y\rangle f(a) - \langle (1_H - E_p) x, y\rangle f(b)|$$
$$\leq \frac{1}{2}\bigvee_{a}^{b}(f)\bigvee_{a-0}^{b}\left(\langle E_{(\cdot)} x, y\rangle\right) \leq \frac{1}{2}\bigvee_{a}^{b}(f)\,\|x\|\,\|y\|.$$

If we use Theorem 1 we can state the following result as well:

Theorem 6. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a = \min\{\lambda \,|\, \lambda \in Sp(A)\} =: \min Sp(A)$ and $b = \max\{\lambda \,|\, \lambda \in Sp(A)\} =: \max Sp(A)$. Also, assume that $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A and $f : I \to \mathbb{C}$ of r-H-Hölder type on I, $[a,b] \subset \mathring{I}$ (the interior of I). Then for any $x, y \in H$ and $s \in (a,b)$ we have*

$$\text{(4.9)} \quad |\langle f(A) x, y\rangle - \langle E_s x, y\rangle f(a) - \langle (1_H - E_s) x, y\rangle f(b)|$$
$$\leq H\left[\frac{1}{2} + \left|\frac{s - \frac{a+b}{2}}{b-a}\right|\right]^r (b-a)^r \bigvee_{a-0}^{b}\left(\langle E_{(\cdot)} x, y\rangle\right)$$
$$\leq H\left[\frac{1}{2} + \left|\frac{s - \frac{a+b}{2}}{b-a}\right|\right]^r (b-a)^r \|x\|\,\|y\|.$$

In particular, we have

$$\text{(4.10)} \quad \left|\langle f(A) x, y\rangle - \left\langle E_{\frac{a+b}{2}} x, y\right\rangle f(a) - \left\langle \left(1_H - E_{\frac{a+b}{2}}\right) x, y\right\rangle f(b)\right|$$
$$\leq \frac{1}{2^r} H (b-a)^r \bigvee_{a-0}^{b}\left(\langle E_{(\cdot)} x, y\rangle\right) \leq \frac{1}{2^r} H (b-a)^r \|x\|\,\|y\|,$$

for any $x, y \in H$.

If we take $f(t) = \ln t$, and $[a,b] \subset (0,\infty)$, then for (4.7) we get

$$\text{(4.11)} \quad |\langle \ln A x, y\rangle - \langle E_s x, y\rangle \ln a - \langle (1_H - E_s) x, y\rangle \ln b|$$
$$\leq \frac{1}{2}\left(\ln\left(\frac{b}{a}\right) + \left|\ln\left(\frac{s^2}{ab}\right)\right|\right)\bigvee_{a-0}^{b}\left(\langle E_{(\cdot)} x, y\rangle\right)$$
$$\leq \frac{1}{2}\left(\ln\left(\frac{b}{a}\right) + \left|\ln\left(\frac{s^2}{ab}\right)\right|\right)\|x\|\,\|y\|$$

for any $s \in (a, b)$ and $x, y \in H$.

If we take $s = \sqrt{ab}$ in (4.11), then we get

$$\left|\langle \ln Ax, y\rangle - \left\langle E_{\sqrt{ab}} x, y\right\rangle \ln a - \left\langle \left(1_H - E_{\sqrt{ab}}\right) x, y\right\rangle \ln b\right| \leq \frac{1}{2} \ln\left(\frac{b}{a}\right) \bigvee_{a-0}^{b} \left(\left\langle E_{(\cdot)} x, y\right\rangle\right) \leq \frac{1}{2} \ln\left(\frac{b}{a}\right) \|x\| \|y\| \tag{4.12}$$

for any $x, y \in H$.

Similar inequalities may be obtained for other examples of continuous functions f. The details are left to the interested reader.

References

[1] T. M. Apostol, *Mathematical Analysis,* Addison-Wesley Publishing Company, Second Edition, 1981.

[2] N. S. Barnett, W. S. Cheung, S. S. Dragomir, A. Sofo, Ostrowski and trapezoid type inequalities for the Stieltjes integral with Lipschitzian integrands or integrators, *Computers & Mathematics with Applications*, **57** (2009), Issue 2, 195-201.

[3] P. Cerone, W. S. Cheung, S. S. Dragomir, On Ostrowski type inequalities for Stieltjes integrals with absolutely continuous integrands and integrators of bounded variation, *Computers & Mathematics with Applications*, **54** (2007), Issue 2, 183-191.

[4] P. Cerone and S. S. Dragomir, New bounds for the three-point rule involving the Riemann-Stieltjes integral. (English summary) *Advances in statistics, combinatorics and related areas,* 53–62, World Sci. Publ., River Edge, NJ, 2002.

[5] S. S. Dragomir, , On the Ostrowski's inequality for Riemann-Stieltjes integral and applications, *Korean J. Comput. & Appl. Math.*, **7** (3) (2000), 477-485.

[6] S. S. Dragomir, Some inequalities for Riemann-Stieltjes integral and applications, in: A. Rubinov and B. Glover (eds.), *Optimization and Related Topics*, 197-235, Kluwer Academic Publishers, 2001.

[7] S. S. Dragomir, On the Ostrowski inequality for the Riemann-Stieltjes integral $\int_a^b f(u)\,du(t)$, where f is of Hölder type and u is of bounded variation and applications, *J. KSIAM*, **5** (1) (2001), 35-45.

[8] S. S. Dragomir, Some inequalities of midpoint and trapezoid type for the Riemann–Stieltjes integral, *Nonlinear Analysis*, **47** (2001), Issue 4, 2333-2340.

[9] S. S. Dragomir, Approximating the Riemann–Stieltjes integral in terms of generalised trapezoidal rules, *Nonlinear Analysis: Theory, Methods & Applications*, **71** (2009), e62-e72.

[10] S. S. Dragomir, C. Buşe, M. V. Boldea and L. Braescu, A generalisation of the trapezoidal rule for the Riemann-Stieltjes integral and applications, Nonlinear Analysis Forum, **6** (2) (2001), 337-351. Preprint *RGMIA Res. Rep. Coll.*, **3** (4), Article 2.

[11] G. Helmberg, *Introduction to Spectral Theory in Hilbert Space*, John Wiley & Sons, Inc. -New York, 1969.

[12] P. R. Mercer, Hadamard's inequality and trapezoid rules for the Riemann–Stieltjes integral, *J. Math. Anal. Appl.* **344** (2008) 921–926.

[13] M. Munteanu, Quadrature formulas for the generalized Riemann-Stieltjes integral. *Bull. Braz. Math. Soc.* (N.S.) **38** (2007), no. 1, 39–50.

Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir/

School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag-3, Wits-2050, Johannesburg, South Africa