BOUNDS ON A GENERALIZED ČEBYŠEV FUNCTIONAL FOR THE RIEMANN-STIELTJES INTEGRAL

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Abstract. In this paper we provide some bounds for the error in approximating the Riemann-Stieltjes integral $\int_a^b f(t)\, g(t)\, du(t)$ by the product

$$\frac{1}{b-a}\int_a^b f(t)\, dt \int_a^b g(t)\, du(t)$$

under various assumptions for the integrands f and g, and the integrator u for which the above integral exists.

1. Introduction

In 1998, S. S. Dragomir and I. Fedotov [17], in order to approximate the Riemann-Stieltjes integral $\int_a^b f(t)\, du(t)$ with the simpler expression

$$\frac{1}{b-a}\left[u(b)-u(a)\right]\int_a^b f(t)\, dt$$

introduced the following error functional

$$D(f,u;a,b) := \int_a^b f(t)\, du(t) - \frac{1}{b-a}\left[u(b)-u(a)\right]\int_a^b f(t)\, dt \tag{1.1}$$

provided that both the Riemann-Stieltjes integral $\int_a^b f(t)\, du(t)$ and the Riemann integral $\int_a^b f(t)\, dt$ exist.

Assume that in the Riemann-Stieltjes integral $\int_a^b f(t)\, du(t)$, the integrator u is L-*Lipschitzian*, i.e.,

$$|u(t)-u(s)| \le L|t-s| \qquad \text{for each } t,\ s\in[a,b], \tag{1.2}$$

where $L>0$ is given. It is well known that, in this case, the Riemann-Stieltjes integral $\int_a^b f(t)\, du(t)$ exists provided the integrand $f:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$.

Theorem 1 (Dragomir-Fedotov 1998, [17]). *If u is L-Lipschitzian on $[a,b]$ and f is Riemann integrable on $[a,b]$, then*

$$|D(f,u;a,b)| \le L\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\, ds\right| dt. \tag{1.3}$$

The inequality (1.3) is sharp.

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Moreover, if there exist the constants m, $M \in \mathbb{R}$ such that

$$m \leq f(t) \leq M \quad \text{for any} \quad t \in [a,b], \tag{1.4}$$

then

$$|D(f,u;a,b)| \leq \frac{1}{2} L(M-m)(b-a). \tag{1.5}$$

The constant $\frac{1}{2}$ is sharp in (1.5).

A function w is said to be of *bounded variation* if for any *division* I_n of $[a,b]$, $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, the variation of w on I_n is finite, which means that

$$\sum_{i=0}^{n-1} |w(x_{i+1}) - w(x_i)| < \infty. \tag{1.6}$$

The *total variation* of w on $[a,b]$ is denoted by $\bigvee_a^b (w)$, where

$$\bigvee_a^b (w) := \sup \left\{ \sum_{i=0}^{n-1} |w(x_{i+1}) - w(x_i)|, \ I_n \text{ is a division of } [a,b] \right\}. \tag{1.7}$$

Theorem 2 (Dragomir-Fedotov 2001, [18])**.** *If u is of bounded variation on $[a,b]$ and f is continuous on $[a,b]$, then*

$$|D(f,u;a,b)| \leq \max_{t\in[a,b]} \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\,ds \right| \bigvee_a^b (u). \tag{1.8}$$

The inequality (1.8) is sharp.

Moreover, if f is K-Lipschitzian, then

$$|D(f,u;a,b)| \leq \frac{1}{2} K(b-a) \bigvee_a^b (u). \tag{1.9}$$

The constant $\frac{1}{2}$ is best possible in (1.9).

For some related results for the functional $D(f,u;a,b)$, see [2]-[6], [10]-[14] and [17]-[19].

We can introduce the more general functional

$$D(f,g,u;a,b) := \int_a^b f(t)\,g(t)\,du(t) - \frac{1}{b-a} \int_a^b f(t)\,dt \int_a^b g(t)\,du(t), \tag{1.10}$$

provided that the involved Riemann-Stieltjes integrals exist. For $g(t) = 1$, $t \in [a,b]$, we get

$$D(f,1,u;a,b) = D(f,u;a,b).$$

Motivated by the above results, in this paper we establish some inequalities for the functional $D(f,g,u;a,b)$ under various assumptions of the integrands f and g and integrator u.

2. Inequalities for Bounded Variation Integrators

Assume that $u,\ f:[a,b]\to\mathbb{C}$. If the Riemann-Stieltjes integral $\int_a^b f(u)\,du(t)$ exists, we write for simplicity, like in [1, p. 142] that $f\in\mathcal{R}_{\mathbb{C}}(u,[a,b])$, or $\mathcal{R}_{\mathbb{C}}(u)$ when the interval is implicitely known. If the functions $u,\ f$ are real valued, then we write $f\in\mathcal{R}(u,[a,b])$, or $\mathcal{R}(u)$.

We start with the following simple fact:

Lemma 1. *Let $f,\ g,\ v:[a,b]\to\mathbb{C}$, $\lambda,\ \mu\in\mathbb{C}$ and $x\in[a,b]$. If $fg,\ g\in\mathcal{R}_{\mathbb{C}}(v,[a,x])\cap\mathcal{R}_{\mathbb{C}}(v,[x,b])$, then $fg,\ g\in\mathcal{R}_{\mathbb{C}}(v,[a,b])$ and*

$$\begin{aligned}\int_a^b f(t)\,g(t)\,dv(t) &= \lambda\int_a^x g(t)\,dv(t)+\mu\int_x^b g(t)\,dv(t)\\ &\quad+\int_a^x [f(t)-\lambda]\,g(t)\,dv(t)+\int_x^b [f(t)-\mu]\,g(t)\,dv(t)\\ &= \mu\int_a^b g(t)\,dv(t)+(\lambda-\mu)\int_a^x g(t)\,dv(t)\\ &\quad+\int_a^x [f(t)-\lambda]\,g(t)\,dv(t)+\int_x^b [f(t)-\mu]\,g(t)\,dv(t).\end{aligned}\tag{2.1}$$

In particular, for $\mu=\lambda$, we have

$$\begin{aligned}\int_a^b f(t)\,g(t)\,dv(t) &= \lambda\int_a^b g(t)\,dv(t)\\ &\quad+\int_a^x [f(t)-\lambda]\,g(t)\,dv(t)+\int_x^b [f(t)-\lambda]\,g(t)\,dv(t)\\ &= \lambda\int_a^b g(t)\,dv(t)+\int_a^b [f(t)-\lambda]\,g(t)\,dv(t).\end{aligned}\tag{2.2}$$

Proof. The integrability follows by Theorem 7. 4 from [1] which says that if a function is Riemann-Stieltjes integrable on the intervals $[a,x]$, $[x,b]$ with $x\in[a,b]$, then it is integrable on the whole interval $[a,b]$.

Using the properties of the Riemann-Stieltjes integral, we have

$$\begin{aligned}&\int_a^x [f(t)-\lambda]\,g(t)\,dv(t)+\int_x^b [f(t)-\mu]\,g(t)\,dv(t)\\ &=\int_a^x f(t)\,g(t)\,dv(t)-\lambda\int_a^x g(t)\,dv(t)+\int_x^b f(t)\,g(t)\,dv(t)-\mu\int_x^b g(t)\,dv(t)\\ &=\int_a^b f(t)\,g(t)\,dv(t)-\lambda\int_a^x g(t)\,dv(t)-\mu\int_x^b g(t)\,dv(t),\end{aligned}$$

which is equivalent to the first equality in (2.1).

The rest is obvious. □

Corollary 1. *Assume that $f,\ v:[a,b]\to\mathbb{C}$ and $x\in[a,b]$ are such that $f\in\mathcal{R}_{\mathbb{C}}(v,[a,x])\cap\mathcal{R}_{\mathbb{C}}(v,[x,b])$. Then for any $\lambda,\ \mu\in\mathbb{C}$ we have the equality*

$$\begin{aligned}\int_a^b f(t)\,dv(t) &= \lambda[v(x)-v(a)]+\mu[v(b)-v(x)]\\ &\quad+\int_a^x [f(t)-\lambda]\,dv(t)+\int_x^b [f(t)-\mu]\,dv(t).\end{aligned}\tag{2.3}$$

In particular, for $\mu = \lambda$, we have

$$\begin{aligned}(2.4)\qquad \int_a^b f(t)\,dv(t) &= \lambda\left[v(b) - v(a)\right] \\ &\quad + \int_a^x \left[f(t) - \lambda\right] dv(t) + \int_x^b \left[f(t) - \lambda\right] dv(t) \\ &= \lambda\left[v(b) - v(a)\right] + \int_a^b \left[f(t) - \lambda\right] dv(t).\end{aligned}$$

The proof follows by Lemma 1 for $g(t) = 1$, $t \in [a,b]$.

Remark 1. *We observe that, see* [1, Theorem 7.27], *if f, $g \in \mathcal{C}_{\mathbb{C}}[a,b]$, namely, are continuous on $[a,b]$ and $v \in \mathcal{BV}_{\mathbb{C}}[a,b]$, namely of bounded variation on $[a,b]$, then for any $x \in [a,b]$ the Riemann-Stieltjes integrals in Lemma 1 exist and the equalities (2.1) and (2.2) hold.*

If we use the equality (2.2) for $\lambda = \frac{1}{b-a}\int_a^b f(t)\,dt$, then we have

$$D(f,g,u;a,b) = \int_a^b \left[f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right] g(t)\,du(t). \tag{2.5}$$

In particular, for $g(t) = 1$, $t \in [a,b]$, we have

$$D(f,u;a,b) = \int_a^b \left[f(t) - \frac{1}{b-a}\int_a^b f(t)\,dt\right] du(t). \tag{2.6}$$

We have:

Theorem 3. *Assume that f, $g \in \mathcal{C}_{\mathbb{C}}[a,b]$ and $u \in \mathcal{BV}_{\mathbb{C}}[a,b]$, then*

$$(2.7)\qquad |D(f,g,u;a,b)| \le \begin{cases} \max_{t\in[a,b]} \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| \int_a^b |g(t)|\, d\left(\bigvee_a^t (u)\right) \\[2ex] \left(\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right|^p d\left(\bigvee_a^t (u)\right)\right)^{1/p} \\ \times \left(\int_a^b |g(t)|^q\, d\left(\bigvee_a^t (u)\right)\right)^{1/q} \\ \text{where } p,\ q, > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\[2ex] \max_{t\in[a,b]} |g(t)| \int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| d\left(\bigvee_a^t (u)\right). \end{cases}$$

Proof. It is well known that if $p \in \mathcal{R}(u,[a,b])$ where $u \in \mathcal{BV}_{\mathbb{C}}[a,b]$ then we have [1, p. 177]

$$\left|\int_a^b p(t)\,du(t)\right| \le \int_a^b |p(t)|\, d\left(\bigvee_a^t (u)\right) \le \sup_{t\in[a,b]} |p(t)| \bigvee_a^b (u). \tag{2.8}$$

Using the equality (2.5) we have

$$\left|\int_a^b f(t)\,g(t)\,du(t)-\frac{1}{b-a}\int_a^b f(t)\,dt\int_a^b g(t)\,du(t)\right| \leq \int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right||g(t)|\,d\left(\bigvee_a^t(u)\right). \tag{2.9}$$

Using Hölder's inequality for monotonic nondecreasing integrators, we have

$$\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right||g(t)|\,d\left(\bigvee_a^t(u)\right)$$
$$\leq\left(\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^p d\left(\bigvee_a^t(u)\right)\right)^{1/p}\left(\int_a^b|g(t)|^q\,d\left(\bigvee_a^t(u)\right)\right)^{1/q}$$

for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

By using (2.9) we then get the midle inequality in (2.7)

The rest is obvious. □

Remark 2. *If we take $g(t)=1,\ t\in[a,b]$, then by (2.7) we get*

$$|D(f,u;a,b)| \leq\begin{cases}\max_{t\in[a,b]}\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|\bigvee_a^b(u)\,, & \textit{see (1.8),}\\[2ex] \left(\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^p d\left(\bigvee_a^t(u)\right)\right)^{1/p}(u(b)-u(a))^{1/q} & \\ \textit{where } p,\ q,>1,\ \frac{1}{p}+\frac{1}{q}=1, & \\[2ex] \int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|d\left(\bigvee_a^t(u)\right). & \end{cases} \tag{2.10}$$

We have:

Theorem 4. *Assume that $f,\ g\in\mathcal{C}_{\mathbb{C}}[a,b]$ and $u\in\mathcal{BV}_{\mathbb{C}}[a,b]$. If $f\in\mathcal{BV}_{\mathbb{C}}[a,b]$, then*

$$|D(f,g,u;a,b)| \leq\max_{t\in[a,b]}|g(t)|\bigvee_a^b(f)\left[\bigvee_a^b(u)-\frac{1}{b-a}\int_a^b\left(\bigvee_a^t(u)\right)\operatorname{sgn}\left(t-\frac{a+b}{2}\right)dt\right] \leq\max_{t\in[a,b]}|g(t)|\bigvee_a^b(f)\bigvee_a^b(u)\,. \tag{2.11}$$

Proof. Using the Ostrowski type inequality for functions of bounded variation $g:[a,b]\to\mathbb{C}$, [7], [9] (or the survey [16])

$$\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|\leq\left[\frac{1}{2}+\left|\frac{t-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(g)\,,\ t\in[a,b]\,,$$

we have

$$\begin{aligned}(2.12)\quad &\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| |g(t)|\, d\left(\bigvee_a^t (u)\right) \\
&\leq \bigvee_a^b (f) \int_a^b \left[\frac{1}{2} + \left|\frac{t-\frac{a+b}{2}}{b-a}\right|\right] |g(t)|\, d\left(\bigvee_a^t (u)\right) \\
&\leq \max_{t\in[a,b]} |g(t)| \bigvee_a^b (f) \int_a^b \left[\frac{1}{2} + \left|\frac{t-\frac{a+b}{2}}{b-a}\right|\right] d\left(\bigvee_a^t (u)\right) \\
&= \max_{t\in[a,b]} |g(t)| \bigvee_a^b (f) \left[\frac{1}{2}\int_a^b d\left(\bigvee_a^t (u)\right) + \int_a^b \left|\frac{t-\frac{a+b}{2}}{b-a}\right| d\left(\bigvee_a^t (u)\right)\right] \\
&= \max_{t\in[a,b]} |g(t)| \bigvee_a^b (f) \left[\frac{1}{2}\bigvee_a^b (u) + \frac{1}{b-a}\int_a^b \left|t-\frac{a+b}{2}\right| d\left(\bigvee_a^t (u)\right)\right] = B(f,g,u).\end{aligned}$$

Using the integration by parts, we have

$$\begin{aligned}&\int_a^b \left|t-\frac{a+b}{2}\right| d\left(\bigvee_a^t (u)\right) \\
&= \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-t\right) d\left(\bigvee_a^t (u)\right) + \int_{\frac{a+b}{2}}^b \left(t-\frac{a+b}{2}\right) d\left(\bigvee_a^t (u)\right) \\
&= \left(\frac{a+b}{2}-t\right)\bigvee_a^t (u)\Bigg|_a^{\frac{a+b}{2}} + \int_a^{\frac{a+b}{2}} \left(\bigvee_a^t (u)\right) dt \\
&+ \left(t-\frac{a+b}{2}\right)\bigvee_a^t (u)\Bigg|_{\frac{a+b}{2}}^b - \int_{\frac{a+b}{2}}^b \left(\bigvee_a^t (u)\right) dt \\
&= \int_a^{\frac{a+b}{2}} \left(\bigvee_a^t (u)\right) dt + \frac{1}{2}(b-a)\bigvee_a^b (u) - \int_{\frac{a+b}{2}}^b \left(\bigvee_a^t (u)\right) dt \\
&= \frac{1}{2}(b-a)\bigvee_a^b (u) - \int_a^b \left(\bigvee_a^t (u)\right) \operatorname{sgn}\left(t-\frac{a+b}{2}\right) dt.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{1}{2}\bigvee_a^b (u) + \frac{1}{b-a}\int_a^b \left|t-\frac{a+b}{2}\right| d\left(\bigvee_a^t (u)\right) \\
&= \frac{1}{2}\bigvee_a^b (u) + \frac{1}{2}\bigvee_a^b (u) - \frac{1}{b-a}\int_a^b \left(\bigvee_a^t (u)\right) \operatorname{sgn}\left(t-\frac{a+b}{2}\right) dt \\
&= \bigvee_a^b (u) - \frac{1}{b-a}\int_a^b \left(\bigvee_a^t (u)\right) \operatorname{sgn}\left(t-\frac{a+b}{2}\right) dt,\end{aligned}$$

which implies that

$$B(f,g,u)=\max_{t\in[a,b]}|g(t)|\bigvee_a^b(f)\left[\bigvee_a^b(u)-\frac{1}{b-a}\int_a^b\left(\bigvee_a^t(u)\right)\operatorname{sgn}\left(t-\frac{a+b}{2}\right)dt\right]$$

that proves the first inequality in (2.11).

The functions $\bigvee_a^{\cdot}(u)$ and $\operatorname{sgn}\left(\cdot-\frac{a+b}{2}\right)$ are monotonic nondecreasing. By using *Čebyšev's inequality* for the same monotonicity functions, we have

$$\begin{aligned}&\frac{1}{b-a}\int_a^b\left(\bigvee_a^t(u)\right)\operatorname{sgn}\left(t-\frac{a+b}{2}\right)dt\\&\geq\frac{1}{b-a}\int_a^b\left(\bigvee_a^t(u)\right)dt\frac{1}{b-a}\int_a^b\operatorname{sgn}\left(t-\frac{a+b}{2}\right)dt\\&=0,\end{aligned}$$

which proves the last part of (2.11). □

Remark 3. *If we take $g(t)=1$, $t\in[a,b]$ in (2.11), then we get*

$$\begin{aligned}&|D(f,u;a,b)|\\&\leq\bigvee_a^b(f)\left[\bigvee_a^b(u)-\frac{1}{b-a}\int_a^b\left(\bigvee_a^t(u)\right)\operatorname{sgn}\left(t-\frac{a+b}{2}\right)dt\right]\leq\bigvee_a^b(f)\bigvee_a^b(u).\end{aligned}\tag{2.13}$$

3. Inequalities for Lipschitzian Integrators

We have:

Theorem 5. *Assume that f is Riemann-Integrable on $[a,b]$, $g\in\mathcal{C}_{\mathbb{C}}[a,b]$ and u is Lipschitzian with the constant $L>0$, namely $|u(t)-u(s)|\leq L|t-s|$ for any t, $s\in[a,b]$. Then*

$$\begin{aligned}&|D(f,g,u;a,b)|\\&\leq L\times\begin{cases}\sup_{t\in[a,b]}\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|\int_a^b|g(t)|\,dt,\\[2ex]\left(\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{1/p}\left(\int_a^b|g(t)|^q\,dt\right)^{1/q},\\p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex]\max_{t\in[a,b]}|g(t)|\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt.\end{cases}\end{aligned}\tag{3.1}$$

In particular, for $p=q=2$,

$$\begin{aligned}&|D(f,g,u;a,b)|\\&\leq L(b-a)\left[\frac{1}{b-a}\int_a^b|f(t)|^2\,dt-\left|\frac{1}{b-a}\int_a^b f(s)\,ds\right|^2\right]^{1/2}\left(\frac{1}{b-a}\int_a^b|g(t)|^2\,dt\right)^{1/2}.\end{aligned}\tag{3.2}$$

Proof. It is known that if $p:[c,d]\to\mathbb{C}$ is Riemann integrable and $u:[c,d]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, then the Riemann-Stieltjes integral $\int_a^b p(t)\,du(t)$ exists and

$$\left|\int_a^b p(t)\,du(t)\right|\le L\int_a^b |p(t)|\,dt.$$

Since $\in\mathcal{BV}_{\mathbb{C}}[a,b]$ and $g\in\mathcal{C}_{\mathbb{C}}[a,b]$, hence the Riemann-Stieltjes integral

$$\int_a^b\left[f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right]g(t)\,du(t)$$

exists and, by (2.5) we have

$$\left|\int_a^b f(t)\,g(t)\,du(t)-\frac{1}{b-a}\int_a^b f(t)\,dt\int_a^b g(t)\,du(t)\right| \le L\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right||g(t)|\,dt=:D(f,g). \tag{3.3}$$

Using Hölder's integral inequality, we have

$$\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right||g(t)|\,dt \le\begin{cases}\sup_{t\in[a,b]}\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|\int_a^b|g(t)|\,dt,\\[2ex] \left(\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{1/p}\left(\int_a^b|g(t)|^q\,dt\right)^{1/q},\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \max_{t\in[a,b]}|g(t)|\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt.\end{cases} \tag{3.4}$$

By making use of (3.3) and (3.4) we get

In particular, for $p=2$, we have

$$\begin{aligned}&\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^2 dt\\ &=(b-a)\left[\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^2 dt\right]\\ &=(b-a)\left[\frac{1}{b-a}\int_a^b|f(t)|^2\,dt-\left|\frac{1}{b-a}\int_a^b f(s)\,ds\right|^2\right],\end{aligned}$$

which proves (3.2). □

Remark 4. *If we take $g(t) = 1$, $t \in [a,b]$, in (3.1) and (3.2), then we get*

$$|D(f,u;a,b)| \leq L \times \begin{cases} (b-a)\sup_{t\in[a,b]} \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right|, \\ \\ (b-a)\left(\frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right|^p dt\right)^{1/p}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt. \end{cases} \tag{3.5}$$

and

$$|D(f,u;a,b)| \leq L(b-a)\left[\frac{1}{b-a}\int_a^b |f(t)|^2\,dt - \left|\frac{1}{b-a}\int_a^b f(s)\,ds\right|^2\right]^{1/2}. \tag{3.6}$$

Using a similar argument to the one in the proof of Theorem 4, we can state the following result as well:

Theorem 6. *Assume that $f \in \mathcal{BV}_{\mathbb{C}}[a,b]$, $g \in \mathcal{C}_{\mathbb{C}}[a,b]$ and u is Lipschitzian with the constant $L > 0$. Then*

$$|D(f,g,u;a,b)| \leq \frac{3}{4}(b-a)\max_{t\in[a,b]}|g(t)|\,L\bigvee_a^b(f). \tag{3.7}$$

Remark 5. *If in the inequality (3.7) we take $g(t) = 1$, $t \in [a,b]$, then we get, see also* [3],

$$|D(f,u;a,b)| \leq \frac{3}{4}(b-a)\,L\bigvee_a^b(f). \tag{3.8}$$

4. Some Further Bounds

In we proved amongst other that, if $f : [a,b] \to \mathbb{C}$ is of bounded variation, then

$$\left[\frac{1}{b-a}\int_a^b |f(t)|^2\,dt - \left|\frac{1}{b-a}\int_a^b f(s)\,ds\right|^2\right]^{1/2} \leq \frac{1}{2}\bigvee_a^b(f)$$

with the constant $\frac{1}{2}$ as best possible.

So, if f is of bounded variation on $[a,b]$, $g \in \mathcal{C}_{\mathbb{C}}[a,b]$ and u is Lipschitzian with the constant $L > 0$, then by (3.2) we get the simple bound:

$$|D(f,g,u;a,b)| \leq \frac{1}{2}L(b-a)\bigvee_a^b(f)\left(\frac{1}{b-a}\int_a^b |g(t)|^2\,dt\right)^{1/2}. \tag{4.1}$$

If in this inequality we take $g(t) = 1$, $t \in [a,b]$, then by (4.1) we have, see also [13],

$$|D(f,u;a,b)| \leq \frac{1}{2}L(b-a)\bigvee_a^b(f). \tag{4.2}$$

Now, for $\gamma,\ \Gamma\in\mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{[a,b]}(\gamma,\Gamma):=\left\{f:[a,b]\to\mathbb{C}|\operatorname{Re}\left[(\Gamma-f(t))\left(\overline{f(t)}-\overline{\gamma}\right)\right]\geq 0 \text{ for each } t\in[a,b]\right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma):=\left\{f:[a,b]\to\mathbb{C}|\ \left|f(t)-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma| \text{ for each } t\in[a,b]\right\}.$$

This family of functions is a particular case of the class introduced in [15]

$$\begin{aligned}&\bar{\Delta}_{[a,b],g}(\gamma,\Gamma)\\&:=\left\{f:[a,b]\to\mathbb{C}|\ \left|f(t)-\frac{\gamma+\Gamma}{2}g(t)\right|\leq\frac{1}{2}|\Gamma-\gamma||g(t)| \text{ for each } t\in[a,b]\right\},\end{aligned}$$

where $g:[a,b]\to\mathbb{C}$.

The following representation result may be stated.

Proposition 1. *For any $\gamma,\ \Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, we have that $\bar{U}_{[a,b]}(\gamma,\Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma)=\bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{4.3}$$

Proof. We observe that for any $z\in\mathbb{C}$ we have the equivalence

$$\left|z-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma|$$

if and only if

$$\operatorname{Re}\left[(\Gamma-z)(\bar{z}-\bar{\gamma})\right]\geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Gamma-\gamma|^2-\left|z-\frac{\gamma+\Gamma}{2}\right|^2=\operatorname{Re}\left[(\Gamma-z)(\bar{z}-\bar{\gamma})\right]$$

that holds for any $z\in\mathbb{C}$.

The equality (4.3) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 2. *For any $\gamma,\ \Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, we have that*

$$\begin{aligned}\bar{U}_{[a,b]}(\gamma,\Gamma)=\{f:[a,b]\to\mathbb{C}\ |\ &(\operatorname{Re}\Gamma-\operatorname{Re}f(t))(\operatorname{Re}f(t)-\operatorname{Re}\gamma)\\&+(\operatorname{Im}\Gamma-\operatorname{Im}f(t))(\operatorname{Im}f(t)-\operatorname{Im}\gamma)\geq 0 \textit{ for each } t\in[a,b]\}.\end{aligned} \tag{4.4}$$

Now, if we assume that $\operatorname{Re}(\Gamma)\geq\operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma)\geq\operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\begin{aligned}\bar{S}_{[a,b]}(\gamma,\Gamma):=\{f:[a,b]\to\mathbb{C}\ |\ &\operatorname{Re}(\Gamma)\geq\operatorname{Re}f(t)\geq\operatorname{Re}(\gamma)\\&\text{and }\operatorname{Im}(\Gamma)\geq\operatorname{Im}f(t)\geq\operatorname{Im}(\gamma)\text{ for each } t\in[a,b]\}.\end{aligned} \tag{4.5}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset\neq\bar{S}_{[a,b]}(\gamma,\Gamma)\subseteq\bar{U}_{[a,b]}(\gamma,\Gamma). \tag{4.6}$$

If $f \in \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ is square integrable on $[a,b]$, then we have the following Grüss type inequality (see for instance [8])

$$\left[\frac{1}{b-a}\int_a^b |f(t)|^2\,dt - \left|\frac{1}{b-a}\int_a^b f(s)\,ds\right|^2\right]^{1/2} \le \frac{1}{2}|\Gamma-\gamma|,$$

so if f is Riemann integrable on $[a,b]$, $g \in \mathcal{C}_{\mathbb{C}}[a,b]$ and u is Lipschitzian with the constant $L>0$, then by (3.2) we get the simple bound:

$$|D(f,g,u;a,b)| \le \frac{1}{2}|\Gamma-\gamma| L(b-a)\left(\frac{1}{b-a}\int_a^b |g(t)|^2\,dt\right)^{1/2}. \tag{4.7}$$

If in this inequality we take $g(t)=1$, $t\in[a,b]$, then by (4.7) we get

$$|D(f,g,u;a,b)| \le \frac{1}{2}|\Gamma-\gamma| L(b-a). \tag{4.8}$$

This inequality extends the result (1.5) from Introduction to complex valued functions $f \in \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$.

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