SOME MIXED MID-POINT AND TRAPEZOID TYPE INEQUALITIES FOR RIEMANN-STIELTJES INTEGRAL

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Abstract. In this paper we provide some mixed mid-point and trapezoid type inequalities to approximate the Riemann-Stieltjes integral of a product of two functions $\int_a^b f(t) g(t) dv(t)$. Applications for continuous functions of selfadjoint operators on Hilbert spaces are also given.

1. Introduction

One can approximate the *Stieltjes integral* $\int_a^b f(t) du(t)$ with the following simpler quantities:

$$\frac{1}{b-a}[u(b)-u(a)]\int_a^b f(t) dt \qquad ([25], [26]) \tag{1.1}$$

$$f(x)[u(b)-u(a)] \qquad ([15], [16]) \tag{1.2}$$

or with

$$[u(b)-u(x)] f(b)+[u(x)-u(a)] f(a) \qquad ([24]), \tag{1.3}$$

where $x \in [a,b]$.

In order to provide *a priory* sharp bounds for the *approximation error*, consider the functionals:

$$D(f,u;a,b) := \int_a^b f(t) du(t) - \frac{1}{b-a}[u(b)-u(a)] \cdot \int_a^b f(t) dt,$$

$$\Theta(f,u;a,b,x) := \int_a^b f(t) du(t) - f(x)[u(b)-u(a)]$$

and

$$T(f,u;a,b,x) := \int_a^b f(t) du(t) - [u(b)-u(x)] f(b) - [u(x)-u(a)] f(a).$$

If the *integrand* f is *Riemann integrable* on $[a,b]$ and the *integrator* $u:[a,b]\to\mathbb{R}$ is $L-$*Lipschitzian*, i.e.,

$$|u(t)-u(s)| \le L|t-s| \qquad \text{for each } \ t,s \in [a,b], \tag{1.4}$$

then the Stieltjes integral $\int_a^b f(t) du(t)$ exists and, as pointed out in [25],

$$|D(f,u;a,b)| \le L\int_a^b \left| f(t) - \int_a^b \frac{1}{b-a} f(s) ds \right| dt. \tag{1.5}$$

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The inequality (1.5) is sharp in the sense that the multiplicative constant $C=1$ in front of L cannot be replaced by a smaller quantity. Moreover, if there exists the constants $m,\ M\in\mathbb{R}$ such that $m\leq f(t)\leq M$ for a.e. $t\in[a,b]$, then [25]

$$|D(f,u;a,b)|\leq\frac{1}{2}L(M-m)(b-a). \tag{1.6}$$

The constant $\frac{1}{2}$ is best possible in (1.6).

A different approach in the case of integrands of bounded variation were considered by the same authors in 2001, [26], where they showed that

$$|D(f,u;a,b)|\leq\max_{t\in[a,b]}\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|\bigvee_a^b(u), \tag{1.7}$$

provided that f is continuous and u is of bounded variation. Here $\bigvee_a^b(u)$ denotes the total variation of u on $[a,b]$. The inequality (1.7) is sharp.

If we assume that f is K-Lipschitzian, then [26]

$$|D(f,u;a,b)|\leq\frac{1}{2}K(b-a)\bigvee_a^b(u), \tag{1.8}$$

with $\frac{1}{2}$ the best possible constant in (1.8).

For various bounds on the error functional $D(f,u;a,b)$ where f and u belong to different classes of function for which the Stieltjes integral exists, see [21], [20], [19], and [8] and the references therein.

The error functional $T(f,u;a,b,x)$ satisfies similar bounds, see [24], [8], [3] and [2] and the details are omitted.

We consider the *mixed mid-point functional*

$$\begin{aligned}F(f,g,u;a,x,b):=&\int_a^b f(t)g(t)\,du(t)\\&-f\left(\frac{a+x}{2}\right)\int_a^x g(t)\,du(t)-f\left(\frac{x+b}{2}\right)\int_x^b g(t)\,du(t),\end{aligned} \tag{1.9}$$

where $x\in[a,b]$, which for $g(t)=1$, reduces to

$$\begin{aligned}F(f,u;a,x,b):=F(f,1,u;a,x,b)=&\int_a^b f(t)\,du(t)\\&-f\left(\frac{a+x}{2}\right)[u(x)-u(a)]-f\left(\frac{x+b}{2}\right)[u(b)-u(x)].\end{aligned} \tag{1.10}$$

Also, consider the *mixed trapezoid functional*

$$\begin{aligned}\Upsilon(f,g,u;a,x,b):=&\int_a^b f(t)g(t)\,du(t)\\&-\frac{f(a)+f(x)}{2}\int_a^x g(t)\,du(t)-\frac{f(x)+f(b)}{2}\int_x^b g(t)\,du(t),\end{aligned} \tag{1.11}$$

where $x \in [a,b]$, which for $g(t) = 1$, reduces to

$$\Upsilon(f,u;a,x,b) := \Upsilon(f,1,u;a,x,b) = \int_a^b f(t)\,du(t) - \frac{f(a)+f(x)}{2}[u(x)-u(a)] - \frac{f(x)+f(b)}{2}[u(b)-u(x)]. \tag{1.12}$$

In this paper we establish some bounds for the magnitude of the functionals $F(f,g,u;a,x,b)$ and $\Upsilon(f,g,u;a,x,b)$ under various assumptions for the functions f, g, u involved and such that the Riemann-Stieltjes integrals under consideration exist. Applications for continuous functions of selfadjoint operators on Hilbert spaces are also given.

2. Main Results

Assume that u, $f : [a,b] \to \mathbb{C}$. If the Riemann-Stieltjes integral $\int_a^b f(u)\,du(t)$ exists, we write for simplicity, like in [1, p. 142] that $f \in \mathcal{R}_{\mathbb{C}}(u,[a,b])$, or $\mathcal{R}_{\mathbb{C}}(u)$ when the interval is implicitly known. If the functions u, f are real valued, then we write $f \in \mathcal{R}(u,[a,b])$, or $\mathcal{R}(u)$.

We start with the following simple fact:

Lemma 1. *Let f, g, $v : [a,b] \to \mathbb{C}$, λ, $\mu \in \mathbb{C}$ and $x \in [a,b]$. If fg, $g \in \mathcal{R}_{\mathbb{C}}(v,[a,x]) \cap \mathcal{R}_{\mathbb{C}}(v,[x,b])$, then fg, $g \in \mathcal{R}_{\mathbb{C}}(v,[a,b])$ and*

$$\begin{aligned}\int_a^b f(t)\,g(t)\,dv(t) &= \lambda\int_a^x g(t)\,dv(t) + \mu\int_x^b g(t)\,dv(t) \\ &\quad + \int_a^x [f(t)-\lambda]\,g(t)\,dv(t) + \int_x^b [f(t)-\mu]\,g(t)\,dv(t) \\ &= \mu\int_a^b g(t)\,dv(t) + (\lambda-\mu)\int_a^x g(t)\,dv(t) \\ &\quad + \int_a^x [f(t)-\lambda]\,g(t)\,dv(t) + \int_x^b [f(t)-\mu]\,g(t)\,dv(t).\end{aligned} \tag{2.1}$$

In particular, for $\mu = \lambda$, we have

$$\begin{aligned}\int_a^b f(t)\,g(t)\,dv(t) &= \lambda\int_a^b g(t)\,dv(t) \\ &\quad + \int_a^x [f(t)-\lambda]\,g(t)\,dv(t) + \int_x^b [f(t)-\lambda]\,g(t)\,dv(t) \\ &= \lambda\int_a^b g(t)\,dv(t) + \int_a^b [f(t)-\lambda]\,g(t)\,dv(t).\end{aligned} \tag{2.2}$$

Proof. The integrability follows by Theorem 7. 4 from [1] which says that if a function is Riemann-Stieltjes integrable on the intervals $[a,x]$, $[x,b]$ with $x \in [a,b]$, then it is integrable on the whole interval $[a,b]$.

Using the properties of the Riemann-Stieltjes integral, we have

$$\begin{aligned}
&\int_a^x [f(t)-\lambda]\, g(t)\, dv(t) + \int_x^b [f(t)-\mu]\, g(t)\, dv(t) \\
&= \int_a^x f(t)\, g(t)\, dv(t) - \lambda \int_a^x g(t)\, dv(t) + \int_x^b f(t)\, g(t)\, dv(t) - \mu \int_x^b g(t)\, dv(t) \\
&= \int_a^b f(t)\, g(t)\, dv(t) - \lambda \int_a^x g(t)\, dv(t) - \mu \int_x^b g(t)\, dv(t),
\end{aligned}$$

which is equivalent to the first equality in (2.1).

The rest is obvious. □

Corollary 1. *Assume that $f,\ v : [a,b] \to \mathbb{C}$ and $x \in [a,b]$ are such that $f \in \mathcal{R}_{\mathbb{C}}(v,[a,x]) \cap \mathcal{R}_{\mathbb{C}}(v,[x,b])$. Then for any $\lambda,\ \mu \in \mathbb{C}$ we have the equality*

$$\begin{aligned}
(2.3)\qquad \int_a^b f(t)\, dv(t) &= \lambda [v(x)-v(a)] + \mu [v(b)-v(x)] \\
&\quad + \int_a^x [f(t)-\lambda]\, dv(t) + \int_x^b [f(t)-\mu]\, dv(t).
\end{aligned}$$

In particular, for $\mu = \lambda$, we have

$$\begin{aligned}
(2.4)\qquad \int_a^b f(t)\, dv(t) &= \lambda [v(b)-v(a)] \\
&\quad + \int_a^x [f(t)-\lambda]\, dv(t) + \int_x^b [f(t)-\lambda]\, dv(t) \\
&= \lambda [v(b)-v(a)] + \int_a^b [f(t)-\lambda]\, dv(t).
\end{aligned}$$

The proof follows by Lemma 1 for $g(t) = 1$, $t \in [a,b]$.

Remark 1. *We observe that, see* [1, Theorem 7.27], *if $f,\ g \in \mathcal{C}_{\mathbb{C}}[a,b]$, namely, are continuous on $[a,b]$ and $v \in \mathcal{BV}_{\mathbb{C}}[a,b]$, namely of bounded variation on $[a,b]$, then for any $x \in [a,b]$ the Riemann-Stieltjes integrals in Lemma 1 exist and the equalities (2.1) and (2.2) hold.*

If we use the equality (2.2) for $\lambda = f\left(\frac{a+x}{2}\right)$ and $\mu = f\left(\frac{x+b}{2}\right)$ in (2.1), then we get for $x \in [a,b]$ that

$$\begin{aligned}
(2.5)\qquad &\int_a^b f(t)\, g(t)\, du(t) \\
&= f\left(\frac{a+x}{2}\right) \int_a^x g(t)\, du(t) + f\left(\frac{x+b}{2}\right) \int_x^b g(t)\, du(t) \\
&\quad + \int_a^x \left[f(t) - f\left(\frac{a+x}{2}\right)\right] g(t)\, du(t) \\
&\quad + \int_x^b \left[f(t) - f\left(\frac{x+b}{2}\right)\right] g(t)\, du(t).
\end{aligned}$$

Also, if we take $\lambda = \frac{f(a)+f(x)}{2}$ and $\mu = \frac{f(x)+f(b)}{2}$ in (2.1), then we get for $x \in [a,b]$ that

$$\begin{aligned}(2.6)\quad & \int_a^b f(t)\, g(t)\, du(t) \\ &= \frac{f(a)+f(x)}{2} \int_a^x g(t)\, du(t) + \frac{f(x)+f(b)}{2} \int_x^b g(t)\, du(t) \\ &\quad + \int_a^x \left[f(t) - \frac{f(a)+f(x)}{2} \right] g(t)\, du(t) \\ &\quad + \int_x^b \left[f(t) - \frac{f(x)+f(b)}{2} \right] g(t)\, du(t).\end{aligned}$$

In particular, for $g(t) = 1$, $t \in [a,b]$, we have for $x \in [a,b]$ that

$$\begin{aligned}(2.7)\quad \int_a^b f(t)\, du(t) &= [u(x) - u(a)]\, f\left(\frac{a+x}{2}\right) + [u(b) - u(x)]\, f\left(\frac{x+b}{2}\right) \\ &\quad + \int_a^x \left[f(t) - f\left(\frac{a+x}{2}\right) \right] du(t) + \int_x^b \left[f(t) - f\left(\frac{x+b}{2}\right) \right] du(t),\end{aligned}$$

and

$$\begin{aligned}(2.8)\quad \int_a^b f(t)\, du(t) &= [u(x) - u(a)]\, \frac{f(a)+f(x)}{2} + [u(b) - u(x)]\, \frac{f(x)+f(b)}{2} \\ &\quad + \int_a^x \left[f(t) - \frac{f(a)+f(x)}{2} \right] du(t) + \int_x^b \left[f(t) - \frac{f(x)+f(b)}{2} \right] du(t).\end{aligned}$$

We have:

Theorem 1. *Assume that* $g \in \mathcal{C}_{\mathbb{C}}[a,b]$ *and* $u \in \mathcal{BV}_{\mathbb{C}}[a,b]$. *If* f *is Lipschitzian with the constant* $L > 0$, *namely*

$$|f(t) - f(s)| \le L\,|t - s| \text{ for any } t,\ s \in [a,b],$$

then for $x \in [a,b]$

$$\begin{aligned}(2.9)\quad & |F(f, g, u; a, x, b)| \\ &\le L \max_{t \in [a,x]} |g(t)| \left[\int_a^{\frac{a+x}{2}} \left(\bigvee_a^t (u) \right) dt + \int_{\frac{a+x}{2}}^x \left(\bigvee_t^x (u) \right) dt \right] \\ &\quad + L \max_{t \in [x,b]} |g(t)| \left[\int_x^{\frac{x+b}{2}} \left(\bigvee_x^t (u) \right) dt + \int_{\frac{x+b}{2}}^b \left(\bigvee_t^b (u) \right) dt \right]\end{aligned}$$

$$\le \frac{1}{2}L\left[(x-a)\max_{t\in[a,x]}|g(t)|\bigvee_a^x(u)+(b-x)\max_{t\in[x,b]}|g(t)|\bigvee_x^b(u)\right]$$

$$\le \frac{1}{2}L\max_{t\in[a,b]}|g(t)|\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]$$

$$\le \frac{1}{2}L\max_{t\in[a,b]}|g(t)|\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(u),\\ \\ \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right](b-a).\end{cases}$$

In particular,

$$(2.10)\qquad |F(f,u;a,x,b)|$$
$$\le L\left[\int_a^{\frac{a+x}{2}}\left(\bigvee_a^t(u)\right)dt+\int_{\frac{a+x}{2}}^x\left(\bigvee_t^x(u)\right)dt\right]$$
$$+L\left[\int_x^{\frac{x+b}{2}}\left(\bigvee_x^t(u)\right)dt+\int_{\frac{x+b}{2}}^b\left(\bigvee_t^b(u)\right)dt\right]$$
$$\le\frac{1}{2}L\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]$$
$$\le\frac{1}{2}L\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(u),\\ \\ \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right](b-a)\end{cases}$$

for $x\in[a,b]$.

Proof. It is well known that if $p\in\mathcal{R}(u,[a,b])$ where $u\in\mathcal{BV}_{\mathbb{C}}[a,b]$ then we have [1, p. 177]

$$(2.11)\qquad\left|\int_a^b p(t)\,du(t)\right|\le\int_a^b|p(t)|\,d\left(\bigvee_a^t(u)\right)\le\sup_{t\in[a,b]}|p(t)|\bigvee_a^b(u).$$

Using the representation (2.5) and the property (2.11), we have

$$(2.12)\qquad |F(f,g,u;a,x,b)|\le\left|\int_a^x\left[f(t)-f\left(\frac{a+x}{2}\right)\right]g(t)\,du(t)\right|$$
$$+\left|\int_x^b\left[f(t)-f\left(\frac{x+b}{2}\right)\right]g(t)\,du(t)\right|$$

$$\le \int_a^x \left| f(t) - f\left(\frac{a+x}{2}\right)\right| |g(t)|\, d\left(\bigvee_a^t (u)\right) + \int_x^b \left| f(t) - f\left(\frac{x+b}{2}\right)\right| |g(t)|\, d\left(\bigvee_x^t (u)\right)$$

$$\le L \int_a^x \left| t - \frac{a+x}{2}\right| |g(t)|\, d\left(\bigvee_a^t (u)\right) + L \int_x^b \left| t - \frac{x+b}{2}\right| |g(t)|\, d\left(\bigvee_x^t (u)\right)$$
$$\le L \max_{t\in[a,x]} |g(t)| \int_a^x \left| t - \frac{a+x}{2}\right| d\left(\bigvee_a^t (u)\right) + L \max_{t\in[x,b]} |g(t)| \int_x^b \left| t - \frac{x+b}{2}\right| d\left(\bigvee_x^t (u)\right) =: B(g,u,x)$$

for $x \in [a,b]$.

Using integration by parts, we have

$$\int_a^x \left| t - \frac{a+x}{2}\right| d\left(\bigvee_a^t (u)\right)$$
$$= \int_a^{\frac{a+x}{2}} \left(\frac{a+x}{2} - t\right) d\left(\bigvee_a^t (u)\right) + \int_{\frac{a+x}{2}}^x \left(t - \frac{a+x}{2}\right) d\left(\bigvee_a^t (u)\right)$$
$$= \left(\frac{a+x}{2} - t\right) \bigvee_a^t (u)\Big|_a^{\frac{a+x}{2}} + \int_a^{\frac{a+x}{2}} \left(\bigvee_a^t (u)\right) dt + \left(t - \frac{a+x}{2}\right) \bigvee_a^t (u)\Big|_{\frac{a+x}{2}}^x - \int_{\frac{a+x}{2}}^x \left(\bigvee_a^t (u)\right) dt$$
$$= \int_a^{\frac{a+x}{2}} \left(\bigvee_a^t (u)\right) dt + \frac{x-a}{2} \bigvee_a^x (u) - \int_{\frac{a+x}{2}}^x \left(\bigvee_a^t (u)\right) dt$$
$$= \int_a^{\frac{a+x}{2}} \left(\bigvee_a^t (u)\right) dt + \int_{\frac{a+x}{2}}^x \left(\bigvee_a^x (u) - \bigvee_a^t (u)\right) dt$$
$$= \int_a^{\frac{a+x}{2}} \left(\bigvee_a^t (u)\right) dt + \int_{\frac{a+x}{2}}^x \left(\bigvee_t^x (u)\right) dt$$

and

$$\begin{aligned}\int_x^b \left|t-\frac{x+b}{2}\right| d\left(\bigvee_x^t (u)\right) &= \int_x^{\frac{x+b}{2}} \left(\frac{x+b}{2}-t\right) d\left(\bigvee_x^t (u)\right) + \int_{\frac{x+b}{2}}^b \left(t-\frac{x+b}{2}\right) d\left(\bigvee_x^t (u)\right)\\ &= \left(\frac{x+b}{2}-t\right)\bigvee_x^t (u)\Big|_x^{\frac{x+b}{2}} + \int_x^{\frac{x+b}{2}} \left(\bigvee_x^t (u)\right) dt\\ &\quad + \left(t-\frac{x+b}{2}\right)\bigvee_x^t (u)\Big|_{\frac{x+b}{2}}^b - \int_{\frac{x+b}{2}}^b \left(\bigvee_x^t (u)\right) dt\end{aligned}$$

$$\begin{aligned}&= \int_x^{\frac{x+b}{2}} \left(\bigvee_x^t (u)\right) dt + \left(\frac{b-x}{2}\right)\bigvee_x^b (u) - \int_{\frac{x+b}{2}}^b \left(\bigvee_x^t (u)\right) dt\\ &= \int_x^{\frac{x+b}{2}} \left(\bigvee_x^t (u)\right) dt + \int_{\frac{x+b}{2}}^b \left(\bigvee_x^b (u) - \bigvee_x^t (u)\right) dt\\ &= \int_x^{\frac{x+b}{2}} \left(\bigvee_x^t (u)\right) dt + \int_{\frac{x+b}{2}}^b \left(\bigvee_t^b (u)\right) dt,\end{aligned}$$

where $x \in [a,b]$.

Therefore

$$\begin{aligned}B(g,u,x) &\le L \max_{t\in[a,x]} |g(t)| \left[\int_a^{\frac{a+x}{2}} \left(\bigvee_a^t (u)\right) dt + \int_{\frac{a+x}{2}}^x \left(\bigvee_t^x (u)\right) dt\right]\\ &\quad + L \max_{t\in[x,b]} |g(t)| \left[\int_x^{\frac{x+b}{2}} \left(\bigvee_x^t (u)\right) dt + \int_{\frac{x+b}{2}}^b \left(\bigvee_t^b (u)\right) dt\right],\end{aligned}$$

for $x \in [a,b]$, which proves the first inequality in (2.9).

We have

$$\begin{aligned}&\int_a^{\frac{a+x}{2}} \left(\bigvee_a^t (u)\right) dt + \int_{\frac{a+x}{2}}^x \left(\bigvee_t^x (u)\right) dt\\ &\qquad \le \frac{x-a}{2}\bigvee_a^{\frac{a+x}{2}} (u) + \frac{x-a}{2}\bigvee_{\frac{a+x}{2}}^x (u) = \frac{x-a}{2}\bigvee_a^x (u)\end{aligned}$$

and

$$\int_{x}^{\frac{x+b}{2}}\left(\bigvee_{x}^{t}(u)\right)dt+\int_{\frac{x+b}{2}}^{b}\left(\bigvee_{t}^{b}(u)\right)dt \le \frac{b-x}{2}\bigvee_{x}^{\frac{x+b}{2}}(u)+\frac{b-x}{2}\bigvee_{\frac{x+b}{2}}^{b}(u)=\frac{b-x}{2}\bigvee_{x}^{b}(u),$$

which proves the second inequality in (2.9).

The rest is obvious. □

Remark 2. *If we take in (2.9) and (2.10) $x=\frac{a+b}{2}$, then we get*

$$\begin{aligned}(2.13)\quad &\left|F\left(f,g,u;a,\frac{a+b}{2},b\right)\right| \\ &\le L\max_{t\in\left[a,\frac{a+b}{2}\right]}|g(t)|\left[\int_{a}^{\frac{3a+b}{2}}\left(\bigvee_{a}^{t}(u)\right)dt+\int_{\frac{3a+b}{2}}^{\frac{a+b}{2}}\left(\bigvee_{t}^{\frac{a+b}{2}}(u)\right)dt\right] \\ &+L\max_{t\in\left[\frac{a+b}{2},b\right]}|g(t)|\left[\int_{\frac{a+b}{2}}^{\frac{a+3b}{2}}\left(\bigvee_{\frac{a+b}{2}}^{t}(u)\right)dt+\int_{\frac{a+3b}{2}}^{b}\left(\bigvee_{t}^{b}(u)\right)dt\right] \\ &\le\frac{1}{4}L(b-a)\left[\max_{t\in\left[a,\frac{a+b}{2}\right]}|g(t)|\bigvee_{a}^{\frac{a+b}{2}}(u)+\max_{t\in\left[\frac{a+b}{2},b\right]}|g(t)|\bigvee_{\frac{a+b}{2}}^{b}(u)\right] \\ &\le\frac{1}{4}L\max_{t\in[a,b]}|g(t)|(b-a)\bigvee_{a}^{b}(u)\end{aligned}$$

and

$$\begin{aligned}(2.14)\quad &\left|F\left(f,u;a,\frac{a+b}{2},b\right)\right| \\ &\le L\left[\int_{a}^{\frac{3a+b}{2}}\left(\bigvee_{a}^{t}(u)\right)dt+\int_{\frac{3a+b}{2}}^{\frac{a+b}{2}}\left(\bigvee_{t}^{\frac{a+b}{2}}(u)\right)dt\right] \\ &+L\left[\int_{\frac{a+b}{2}}^{\frac{a+3b}{2}}\left(\bigvee_{\frac{a+b}{2}}^{t}(u)\right)dt+\int_{\frac{a+3b}{2}}^{b}\left(\bigvee_{t}^{b}(u)\right)dt\right]\le\frac{1}{4}L(b-a)\bigvee_{a}^{b}(u).\end{aligned}$$

Also, if $p\in[a,b]$ is such that $\bigvee_{a}^{p}(u)=\bigvee_{p}^{b}(u)$, then by (2.9) and (2.10) we get

$$(2.15)\quad \left|F\left(f,g,u;a,p,b\right)\right| \leq L\max_{t\in[a,p]}\left|g\left(t\right)\right|\left[\int_{a}^{\frac{a+p}{2}}\left(\bigvee_{a}^{t}\left(u\right)\right)dt+\int_{\frac{a+p}{2}}^{p}\left(\bigvee_{t}^{p}\left(u\right)\right)dt\right] + L\max_{t\in[p,b]}\left|g\left(t\right)\right|\left[\int_{p}^{\frac{p+b}{2}}\left(\bigvee_{p}^{t}\left(u\right)\right)dt+\int_{\frac{p+b}{2}}^{b}\left(\bigvee_{t}^{b}\left(u\right)\right)dt\right]$$

$$\leq\frac{1}{4}L\left[\left(p-a\right)\max_{t\in[a,p]}\left|g\left(t\right)\right|+\left(b-p\right)\max_{t\in[p,b]}\left|g\left(t\right)\right|\right]\bigvee_{a}^{b}\left(u\right) \leq\frac{1}{4}L\max_{t\in[a,b]}\left|g\left(t\right)\right|\left(b-a\right)\bigvee_{a}^{b}\left(u\right)$$

and

$$(2.16)\quad \left|F\left(f,u;a,p,b\right)\right| \leq L\left[\int_{a}^{\frac{a+p}{2}}\left(\bigvee_{a}^{t}\left(u\right)\right)dt+\int_{\frac{a+p}{2}}^{p}\left(\bigvee_{t}^{p}\left(u\right)\right)dt\right] + L\left[\int_{p}^{\frac{p+b}{2}}\left(\bigvee_{p}^{t}\left(u\right)\right)dt+\int_{\frac{p+b}{2}}^{b}\left(\bigvee_{t}^{b}\left(u\right)\right)dt\right]\leq\frac{1}{4}L\left(b-a\right)\bigvee_{a}^{b}\left(u\right).$$

We also have:

Theorem 2. *Assume that* $f,\ g\in\mathcal{C}_{\mathbb{C}}\left[a,b\right]$ *and* $u\in\mathcal{BV}_{\mathbb{C}}\left[a,b\right]$. *If* $f\in\mathcal{BV}_{\mathbb{C}}\left[a,b\right]$, *then we have for all* $x\in\left[a,b\right]$

$$(2.17)\quad \left|\Upsilon\left(f,g,u;a,x,b\right)\right| \leq\frac{1}{2}\left[\bigvee_{a}^{x}\left(f\right)\int_{a}^{x}\left|g\left(t\right)\right|d\left(\bigvee_{a}^{t}\left(u\right)\right)+\bigvee_{x}^{b}\left(f\right)\int_{x}^{b}\left|g\left(t\right)\right|d\left(\bigvee_{x}^{t}\left(u\right)\right)\right]$$

$$\leq\frac{1}{4}\times\begin{cases}\left[\bigvee_{a}^{b}\left(f\right)+\left|\bigvee_{a}^{x}\left(f\right)-\bigvee_{x}^{b}\left(f\right)\right|\right]\int_{a}^{b}\left|g\left(t\right)\right|d\left(\bigvee_{a}^{t}\left(u\right)\right),\\[2ex] \left[\int_{a}^{b}\left|g\left(t\right)\right|d\left(\bigvee_{a}^{t}\left(u\right)\right)+\left|\int_{a}^{x}\left|g\left(t\right)\right|d\left(\bigvee_{a}^{t}\left(u\right)\right)-\int_{x}^{b}\left|g\left(t\right)\right|d\left(\bigvee_{a}^{t}\left(u\right)\right)\right|\right]\bigvee_{a}^{b}\left(f\right).\end{cases}$$

In particular,

$$|\Upsilon(f,u;a,x,b)| \leq \frac{1}{2}\left[\bigvee_a^x (f) \bigvee_a^x (u) + \bigvee_x^b (f) \bigvee_x^b (u)\right] \leq \frac{1}{4} \times \begin{cases} \left[\bigvee_a^b (f) + \left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right] \bigvee_a^b (u), \\ \left[\bigvee_a^b (u) + \left|\bigvee_a^x (u) - \bigvee_x^b (u)\right|\right] \bigvee_a^b (f) \end{cases} \tag{2.18}$$

for all $x \in [a,b]$.

Proof. Let $x \in (a,b)$. Since $f : [a,b] \to \mathbb{C}$ is of bounded variation, then for any $t \in [a,x]$ we have

$$\left|f(t) - \frac{f(a)+f(x)}{2}\right| = \frac{1}{2}\left[|f(t) - f(a) + f(t) - f(x)|\right] \leq \frac{1}{2}\left[|f(t) - f(a)| + |f(x) - f(t)|\right] \leq \frac{1}{2}\bigvee_a^x (f)$$

and, similarly

$$\left|f(t) - \frac{f(x)+f(b)}{2}\right| \leq \frac{1}{2}\bigvee_x^b (f)$$

for $t \in [x,b]$.

Using the equality (2.6) and the property (2.11), we have

$$\begin{aligned} |\Upsilon(f,g,u;a,x,b)| &\leq \left|\int_a^x \left[f(t) - \frac{f(a)+f(x)}{2}\right] g(t)\,du(t)\right| \\ &\quad + \left|\int_x^b \left[f(t) - \frac{f(x)+f(b)}{2}\right] g(t)\,du(t)\right| \\ &\leq \int_a^x \left|f(t) - \frac{f(a)+f(x)}{2}\right| |g(t)|\, d\left(\bigvee_a^t (u)\right) \\ &\quad + \int_x^b \left[\left|f(t) - \frac{f(x)+f(b)}{2}\right|\right] |g(t)|\, d\left(\bigvee_x^t (u)\right) \\ &\leq \frac{1}{2}\bigvee_a^x (f) \int_a^x |g(t)|\, d\left(\bigvee_a^t (u)\right) + \frac{1}{2}\bigvee_x^b (f) \int_x^b |g(t)|\, d\left(\bigvee_x^t (u)\right), \end{aligned} \tag{2.19}$$

which proves the first inequality in (2.17).

Observe that

$$\begin{aligned}
&\bigvee_a^x(f)\int_a^x|g(t)|\,d\left(\bigvee_a^t(u)\right)+\bigvee_x^b(f)\int_x^b|g(t)|\,d\left(\bigvee_x^t(u)\right)\\
&\qquad=\bigvee_a^x(f)\int_a^x|g(t)|\,d\left(\bigvee_a^t(u)\right)+\bigvee_x^b(f)\int_x^b|g(t)|\,d\left(\bigvee_a^t(u)\right)\\
&\le\begin{cases}\max\left\{\bigvee_a^x(f),\bigvee_x^b(f)\right\}\left[\int_a^x|g(t)|\,d\left(\bigvee_a^t(u)\right)+\int_x^b|g(t)|\,d\left(\bigvee_a^t(u)\right)\right]\\[2ex]
\max\left\{\int_a^x|g(t)|\,d\left(\bigvee_a^t(u)\right),\int_x^b|g(t)|\,d\left(\bigvee_a^t(u)\right)\right\}\left[\bigvee_a^x(f)+\bigvee_x^b(f)\right]\end{cases}\\
&\qquad=\begin{cases}\max\left\{\bigvee_a^x(f),\bigvee_x^b(f)\right\}\int_a^b|g(t)|\,d\left(\bigvee_a^t(u)\right)\\[2ex]
\max\left\{\int_a^x|g(t)|\,d\left(\bigvee_a^t(u)\right),\int_x^b|g(t)|\,d\left(\bigvee_a^t(u)\right)\right\}\bigvee_a^b(f),\end{cases}
\end{aligned}$$

which proves the last part of (2.17). □

Remark 3. *If $m\in[a,b]$ is such that $\bigvee_a^m(f)=\bigvee_m^b(f)$, then by (2.17) we have*

$$|\Upsilon(f,g,u;a,m,b)|\le\frac{1}{4}\bigvee_a^b(f)\int_a^b|g(t)|\,d\left(\bigvee_a^t(u)\right),\tag{2.20}$$

while by (2.18) we get

$$|\Upsilon(f,u;a,m,b)|\le\frac{1}{4}\bigvee_a^b(f)\bigvee_a^b(u).\tag{2.21}$$

If $q\in[a,b]$ is such that

$$\int_a^q|g(t)|\,d\left(\bigvee_a^t(u)\right)=\int_q^b|g(t)|\,d\left(\bigvee_a^t(u)\right),$$

then by (2.17) we get

$$|\Upsilon(f,g,u;a,q,b)|\le\frac{1}{4}\bigvee_a^b(f)\int_a^b|g(t)|\,d\left(\bigvee_a^t(u)\right).\tag{2.22}$$

If $p\in[a,b]$ is such that $\bigvee_a^p(u)=\bigvee_p^b(u)$, then by (2.18) we get

$$|\Upsilon(f,u;a,p,b)|\le\frac{1}{4}\bigvee_a^b(f)\bigvee_a^b(u).\tag{2.23}$$

3. Applications for Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Let $A\in\mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda\in\mathbb{R}$ as follows

$$\varphi_\lambda(s):=\begin{cases}1, & \text{for } -\infty<s\le\lambda,\\ \\ 0, & \text{for } \lambda<s<+\infty.\end{cases}$$

Then for every $\lambda\in\mathbb{R}$ the operator

$$E_\lambda:=\varphi_\lambda(A) \tag{3.1}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [27, p. 256]:

Theorem 3 (Spectral Representation Theorem)**.** *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a=\min\{\lambda\,|\lambda\in Sp(A)\}=:\min Sp(A)$ and $b=\max\{\lambda\,|\lambda\in Sp(A)\}=:\max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_\lambda\le E_{\lambda'}$ *for* $\lambda\le\lambda'$;
b) $E_{a-0}=0, E_b=1_H$ *and* $E_{\lambda+0}=E_\lambda$ *for all* $\lambda\in\mathbb{R}$;
c) *We have the representation*

$$A=\int_{a-0}^{b}\lambda dE_\lambda.$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ there exists a unique operator $\varphi(A)\in\mathcal{B}(H)$ such that for every $\varepsilon>0$ there exists a $\delta>0$ satisfying the inequality

$$\left\|\varphi(A)-\sum_{k=1}^{n}\varphi\left(\lambda_k'\right)\left[E_{\lambda_k}-E_{\lambda_{k-1}}\right]\right\|\le\varepsilon$$

whenever

$$\begin{cases}\lambda_0<a=\lambda_1<...<\lambda_{n-1}<\lambda_n=b,\\ \\ \lambda_k-\lambda_{k-1}\le\delta \text{ for } 1\le k\le n,\\ \\ \lambda_k'\in[\lambda_{k-1},\lambda_k] \text{ for } 1\le k\le n\end{cases}$$

this means that

$$\varphi(A)=\int_{a-0}^{b}\varphi(\lambda)\,dE_\lambda, \tag{3.2}$$

where the integral is of Riemann-Stieltjes type.

Corollary 2. *With the assumptions of Theorem 3 for A, E_λ and φ we have the representations*

$$\varphi(A)x=\int_{a-0}^{b}\varphi(\lambda)\,dE_\lambda x \quad \text{for all } x\in H$$

and

$$\left\langle \varphi\left(A\right)x,y\right\rangle =\int_{a-0}^{b}\varphi\left(\lambda\right)d\left\langle E_{\lambda}x,y\right\rangle \quad \text{for all } x,\ y\in H. \tag{3.3}$$

In particular,

$$\left\langle \varphi\left(A\right)x,x\right\rangle =\int_{a-0}^{b}\varphi\left(\lambda\right)d\left\langle E_{\lambda}x,x\right\rangle \quad \text{for all } x\in H.$$

Moreover, we have the equality

$$\left\Vert \varphi\left(A\right)x\right\Vert ^{2}=\int_{a-0}^{b}\left\vert \varphi\left(\lambda\right)\right\vert ^{2}d\left\Vert E_{\lambda}x\right\Vert ^{2}\quad \text{for all } x\in H.$$

We need the following result that provides an upper bound for the total variation of the function $\mathbb{R}\ni\lambda\mapsto\left\langle E_{\lambda}x,y\right\rangle \in\mathbb{C}$ on an interval $\left[\alpha,\beta\right]$, see [23].

Lemma 2. *Let $\left\{E_{\lambda}\right\}_{\lambda\in\mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A. Then for any $x,\ y\in H$ and $\alpha<\beta$ we have the inequality*

$$\left[\bigvee_{\alpha}^{\beta}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right]^{2}\leq\left\langle \left(E_{\beta}-E_{\alpha}\right)x,x\right\rangle \left\langle \left(E_{\beta}-E_{\alpha}\right)y,y\right\rangle , \tag{3.4}$$

where $\bigvee_{\alpha}^{\beta}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$ denotes the total variation of the function $\left\langle E_{(\cdot)}x,y\right\rangle$ on $\left[\alpha,\beta\right]$.

Remark 4. *For $\alpha=a-\varepsilon$ with $\varepsilon>0$ and $\beta=b$ we get from (3.4) the inequality*

$$\bigvee_{a-\varepsilon}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\leq\left\langle \left(1_{H}-E_{a-\varepsilon}\right)x,x\right\rangle ^{1/2}\left\langle \left(1_{H}-E_{a-\varepsilon}\right)y,y\right\rangle ^{1/2} \tag{3.5}$$

for any $x,\ y\in H$.

This implies, for any $x,\ y\in H$, that

$$\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\leq\left\Vert x\right\Vert \left\Vert y\right\Vert , \tag{3.6}$$

where $\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$ denotes the limit $\lim_{\varepsilon\rightarrow0+}\left[\bigvee_{a-\varepsilon}^{b}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right]$.

We can state the following result for functions of selfadjoint operators:

Theorem 4. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a=\min\left\{\lambda\left\vert\lambda\in Sp\left(A\right)\right.\right\}=:\min Sp\left(A\right)$ and $b=\max\left\{\lambda\left\vert\lambda\in Sp\left(A\right)\right.\right\}=:$ $\max Sp\left(A\right)$. Also, assume that $\left\{E_{\lambda}\right\}_{\lambda\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A and $f:I\rightarrow\mathbb{C}$ is Lipschitzian with the constant $L>0$ continuous*

$[a,b]\subset \mathring{I}$ *(the interior of I). Then for all $s\in[a,b]$,*

$$\left|\langle f(A)x,y\rangle - f\left(\frac{a+s}{2}\right)\langle E_s x,y\rangle - f\left(\frac{s+b}{2}\right)\langle (1_H-E_s)x,y\rangle\right| \tag{3.7}$$
$$\le \frac{1}{2}L\left[\frac{1}{2}(b-a)+\left|s-\frac{a+b}{2}\right|\right]\bigvee_{a-0}^{b}\left(\langle E_{(\cdot)}x,y\rangle\right)$$
$$\le \frac{1}{2}L\left[\frac{1}{2}(b-a)+\left|s-\frac{a+b}{2}\right|\right]\|x\|\,\|y\|$$

for any $x,\ y\in H$.

In particular,

$$\left|\langle f(A)x,y\rangle - f\left(\frac{3a+b}{4}\right)\left\langle E_{\frac{a+b}{2}}x,y\right\rangle - f\left(\frac{a+3b}{4}\right)\left\langle\left(1_H-E_{\frac{a+b}{2}}\right)x,y\right\rangle\right| \tag{3.8}$$
$$\le \frac{1}{4}L(b-a)\bigvee_{a-0}^{b}\left(\langle E_{(\cdot)}x,y\rangle\right)\le \frac{1}{4}L(b-a)\|x\|\,\|y\|$$

for any $x,\ y\in H$.

Proof. Using the inequality (2.10) we have for $s\in[a,b]$, for small $\varepsilon>0$ and for any $x,\ y\in H$ that

$$\left|\int_{a-\varepsilon}^{b} f(t)\,d\langle E_t x,y\rangle \right.$$
$$\left. -f\left(\frac{a-\varepsilon+s}{2}\right)\left[\langle E_s x,y\rangle-\langle E_{a-\varepsilon}x,y\rangle\right]-f\left(\frac{s+b}{2}\right)\left[\langle E_b x,y\rangle-\langle E_s x,y\rangle\right]\right|$$
$$\le \frac{1}{2}L\left[\frac{1}{2}(b-a)+\left|s-\frac{a+b}{2}\right|\right]\bigvee_{a-\varepsilon}^{b}\left(\langle E_{(\cdot)}x,y\rangle\right).$$

Taking the limit over $\varepsilon\to 0+$ and using the continuity of f and the Spectral Representation Theorem, we deduce the desired result (3.7). □

We also have:

Theorem 5. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a=\min\{\lambda\,|\lambda\in Sp(A)\}=:\min Sp(A)$ and $b=\max\{\lambda\,|\lambda\in Sp(A)\}=:\max Sp(A)$. Also, assume that $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A and f of locally bounded variation on I, with $[a,b]\subset\mathring{I}$. Then for all $s\in[a,b]$,*

$$\left|\langle f(A)x,y\rangle-\frac{f(a)+f(s)}{2}\langle E_s x,y\rangle-\frac{f(s)+f(b)}{2}\langle(1_H-E_s)x,y\rangle\right| \tag{3.9}$$
$$\le\frac{1}{4}\left[\bigvee_a^b(f)+\left|\bigvee_a^s(f)-\bigvee_s^b(f)\right|\right]\bigvee_{a-0}^{b}\left(\langle E_{(\cdot)}x,y\rangle\right)$$
$$\le\frac{1}{4}\left[\bigvee_a^b(f)+\left|\bigvee_a^s(f)-\bigvee_s^b(f)\right|\right]\|x\|\,\|y\|$$

for any $x,\ y\in H$.

In particular, if $m \in [a,b]$ *is such that* $\bigvee_a^m (f) = \bigvee_m^b (f)$, *then*

$$\left|\langle f(A)x,y\rangle - \frac{f(a)+f(m)}{2}\langle E_m x,y\rangle - \frac{f(m)+f(b)}{2}\langle (1_H - E_m)x,y\rangle\right| \le \frac{1}{4}\bigvee_a^b (f) \bigvee_{a-0}^b \left(\langle E_{(\cdot)}x,y\rangle\right) \le \frac{1}{4}\bigvee_a^b (f)\|x\|\|y\| \tag{3.10}$$

for any $x,\ y \in H$.

Remark 5. *The above inequalities (3.7)-(3.10) can produce several particular examples of interest. For example if* $[a,b] \subset (0,\infty)$ *and we take* $f(t) = \ln t$, *then we get*

$$\left|\langle \ln Ax,y\rangle - \ln\left[\left(\frac{a+s}{2}\right)^{\langle E_s x,y\rangle}\left(\frac{s+b}{2}\right)^{\langle (1_H - E_s)x,y\rangle}\right]\right| \le \frac{1}{2a}\left[\frac{1}{2}(b-a) + \left|s - \frac{a+b}{2}\right|\right]\bigvee_{a-0}^b \left(\langle E_{(\cdot)}x,y\rangle\right) \le \frac{1}{2a}\left[\frac{1}{2}(b-a) + \left|s - \frac{a+b}{2}\right|\right]\|x\|\|y\| \tag{3.11}$$

for any $x,\ y \in H$.

In particular,

$$\left|\langle \ln Ax,y\rangle - \ln\left[\left(\frac{3a+b}{4}\right)^{\left\langle E_{\frac{a+b}{2}}x,y\right\rangle}\left(\frac{a+3b}{4}\right)^{\left\langle\left(1_H - E_{\frac{a+b}{2}}\right)x,y\right\rangle}\right]\right| \le \frac{1}{4a}(b-a)\bigvee_{a-0}^b \left(\langle E_{(\cdot)}x,y\rangle\right) \le \frac{1}{4a}(b-a)\|x\|\|y\| \tag{3.12}$$

for any $x,\ y \in H$.

We also have

$$\left|\langle \ln Ax,y\rangle - \ln\left(\sqrt{as}\right)\langle E_s x,y\rangle - \ln\left(\sqrt{sb}\right)\langle (1_H - E_s)x,y\rangle\right| \le \frac{1}{4}\left[\ln(ab) + \left|\ln\left(\frac{s^2}{ab}\right)\right|\right]\bigvee_{a-0}^b \left(\langle E_{(\cdot)}x,y\rangle\right) \le \frac{1}{4}\left[\ln(ab) + \left|\ln\left(\frac{s^2}{ab}\right)\right|\right]\|x\|\|y\| \tag{3.13}$$

for any $x,\ y \in H$.

In particular, we have

$$\left|\langle \ln Ax,y\rangle - \ln\left(a^{3/4}b^{1/4}\right)\left\langle E_{\sqrt{ab}}x,y\right\rangle - \ln\left(a^{1/4}b^{3/4}\right)\left\langle\left(1_H - E_{\sqrt{ab}}\right)x,y\right\rangle\right| \le \frac{1}{4}\ln\left(\frac{b}{a}\right)\bigvee_{a-0}^b \left(\langle E_{(\cdot)}x,y\rangle\right) \le \frac{1}{4}\ln\left(\frac{b}{a}\right)\|x\|\|y\| \tag{3.14}$$

for any $x,\ y \in H$.

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