BOUNDS ON A MIXED ČEBYŠEV FUNCTIONAL FOR THE RIEMANN-STIELTJES INTEGRAL

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Abstract. In this paper we provide some bounds for the error in approximating the Riemann-Stieltjes integral $\int_a^b f(t)\,g(t)\,du(t)$ by the simpler quantity

$$\frac{1}{x-a}\int_a^x f(s)\,ds\int_a^x g(t)\,du(t)+\frac{1}{b-x}\int_x^b f(s)\,ds\int_x^b g(t)\,du(t),$$

where $x\in(a,b)$, under various assumptions for the integrands f and g, and the integrator u for which the above integral exists. Applications for continuous functions of selfadjoint operators in Hilbert spaces are also given.

1. Introduction

In 1998, S. S. Dragomir and I. Fedotov [17], in order to approximate the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ with the simpler expression

$$\frac{1}{b-a}[u(b)-u(a)]\int_a^b f(t)\,dt$$

introduced the following error functional

$$D(f,u;a,b):=\int_a^b f(t)\,du(t)-\frac{1}{b-a}[u(b)-u(a)]\int_a^b f(t)\,dt \tag{1.1}$$

provided that both the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ and the Riemann integral $\int_a^b f(t)\,dt$ exist.

Assume that in the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$, the integrator u is *L-Lipschitzian*, i.e.,

$$|u(t)-u(s)|\le L|t-s| \qquad \text{for each } t,\ s\in[a,b], \tag{1.2}$$

where $L>0$ is given. It is well known that, in this case, the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists provided the integrand $f:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$.

Theorem 1 (Dragomir-Fedotov 1998, [17]). *If u is L-Lipschitzian on $[a,b]$ and f is Riemann integrable on $[a,b]$, then*

$$|D(f,u;a,b)|\le L\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt. \tag{1.3}$$

The inequality (1.3) is sharp.

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Moreover, if there exist the constants $m,\ M \in \mathbb{R}$ *such that*

$$m \leq f(t) \leq M \qquad \text{for any} \quad t \in [a,b], \tag{1.4}$$

then

$$|D(f,u;a,b)| \leq \frac{1}{2} L (M-m)(b-a). \tag{1.5}$$

The constant $\frac{1}{2}$ *is sharp in (1.5).*

Theorem 2 (Dragomir-Fedotov 2001, [18]). *If* u *is of bounded variation on* $[a,b]$ *and* f *is continuous on* $[a,b]$, *then*

$$|D(f,u;a,b)| \leq \max_{t\in[a,b]} \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\,ds \right| \bigvee_a^b (u). \tag{1.6}$$

The inequality (1.6) is sharp.

Moreover, if f *is* K*-Lipschitzian, then*

$$|D(f,u;a,b)| \leq \frac{1}{2} K (b-a) \bigvee_a^b (u). \tag{1.7}$$

The constant $\frac{1}{2}$ *is best possible in (1.7).*

For some related results for the functional $D(f,u;a,b)$, see [2]-[5], [9]-[12] and [17]-[20].

We can introduce the *mixed Čebyšev's functional*

$$\begin{aligned} D(f,g,u;x,a,b) &:= \int_a^b f(t)\,g(t)\,du(t) \\ &- \frac{1}{x-a}\int_a^x f(s)\,ds \int_a^x g(t)\,du(t) - \frac{1}{b-x}\int_x^b f(s)\,ds \int_x^b g(t)\,du(t), \end{aligned} \tag{1.8}$$

where $x \in (a,b)$, provided that the involved Riemann-Stieltjes integrals exist.

For $g(t) = 1,\ t \in [a,b]$, we also consider

$$\begin{aligned} D(f,u;x,a,b) &:= D(f,1,u;x,a,b) = \int_a^b f(t)\,du(t) \\ &- \frac{u(x)-u(a)}{x-a}\int_a^x f(s)\,ds - \frac{u(b)-u(x)}{b-x}\int_x^b f(s)\,ds, \end{aligned} \tag{1.9}$$

where $x \in (a,b)$.

Motivated by the above results, in this paper we establish some inequalities for the functional $D(f,g,u;x,a,b)$ under various assumptions of the integrands f and g and integrator u. Applications for continuous functions of selfadjoint operators in Hilbert spaces are also given.

2. Inequalities for Bounded Variation Integrators

Assume that $u,\ f : [a,b] \to \mathbb{C}$. If the Riemann-Stieltjes integral $\int_a^b f(u)\,du(t)$ exists, we write for simplicity, like in [1, p. 142] that $f \in \mathcal{R}_{\mathbb{C}}(u,[a,b])$, or $\mathcal{R}_{\mathbb{C}}(u)$ when the interval is implicitly known. If the functions $u,\ f$ are real valued, then we write $f \in \mathcal{R}(u,[a,b])$, or $\mathcal{R}(u)$.

We start with the following simple fact:

Lemma 1. *Let* $f, g, v : [a,b] \to \mathbb{C}$, $\lambda, \mu \in \mathbb{C}$ *and* $x \in [a,b]$. *If* $fg, g \in \mathcal{R}_{\mathbb{C}}(v, [a,x]) \cap \mathcal{R}_{\mathbb{C}}(v, [x,b])$, *then* $fg, g \in \mathcal{R}_{\mathbb{C}}(v, [a,b])$ *and*

$$\begin{aligned}
\int_a^b f(t) g(t) dv(t) &= \lambda \int_a^x g(t) dv(t) + \mu \int_x^b g(t) dv(t) \\
&\quad + \int_a^x [f(t) - \lambda] g(t) dv(t) + \int_x^b [f(t) - \mu] g(t) dv(t) \\
&= \mu \int_a^b g(t) dv(t) + (\lambda - \mu) \int_a^x g(t) dv(t) \\
&\quad + \int_a^x [f(t) - \lambda] g(t) dv(t) + \int_x^b [f(t) - \mu] g(t) dv(t).
\end{aligned} \tag{2.1}$$

In particular, for $\mu = \lambda$, *we have*

$$\begin{aligned}
\int_a^b f(t) g(t) dv(t) &= \lambda \int_a^b g(t) dv(t) \\
&\quad + \int_a^x [f(t) - \lambda] g(t) dv(t) + \int_x^b [f(t) - \lambda] g(t) dv(t) \\
&= \lambda \int_a^b g(t) dv(t) + \int_a^b [f(t) - \lambda] g(t) dv(t).
\end{aligned} \tag{2.2}$$

Proof. The integrability follows by Theorem 7. 4 from [1] which says that if a function is Riemann-Stieltjes integrable on the intervals $[a,x]$, $[x,b]$ with $x \in [a,b]$, then it is integrable on the whole interval $[a,b]$.

Using the properties of the Riemann-Stieltjes integral, we have

$$\begin{aligned}
&\int_a^x [f(t) - \lambda] g(t) dv(t) + \int_x^b [f(t) - \mu] g(t) dv(t) \\
&= \int_a^x f(t) g(t) dv(t) - \lambda \int_a^x g(t) dv(t) + \int_x^b f(t) g(t) dv(t) - \mu \int_x^b g(t) dv(t) \\
&= \int_a^b f(t) g(t) dv(t) - \lambda \int_a^x g(t) dv(t) - \mu \int_x^b g(t) dv(t),
\end{aligned}$$

which is equivalent to the first equality in (2.1).

The rest is obvious. □

Corollary 1. *Assume that* $f, v : [a,b] \to \mathbb{C}$ *and* $x \in [a,b]$ *are such that* $f \in \mathcal{R}_{\mathbb{C}}(v, [a,x]) \cap \mathcal{R}_{\mathbb{C}}(v, [x,b])$. *Then for any* $\lambda, \mu \in \mathbb{C}$ *we have the equality*

$$\begin{aligned}
\int_a^b f(t) dv(t) &= \lambda [v(x) - v(a)] + \mu [v(b) - v(x)] \\
&\quad + \int_a^x [f(t) - \lambda] dv(t) + \int_x^b [f(t) - \mu] dv(t).
\end{aligned} \tag{2.3}$$

In particular, for $\mu=\lambda$, we have

$$\begin{aligned}\int_a^b f(t)\,dv(t) &= \lambda\left[v(b)-v(a)\right] \\ &\quad+\int_a^x \left[f(t)-\lambda\right]dv(t)+\int_x^b \left[f(t)-\lambda\right]dv(t) \\ &= \lambda\left[v(b)-v(a)\right]+\int_a^b \left[f(t)-\lambda\right]dv(t).\end{aligned} \tag{2.4}$$

The proof follows by Lemma 1 for $g(t)=1$, $t\in[a,b]$.

Remark 1. *We observe that, see* [1, Theorem 7.27]*, if $f,\ g\in\mathcal{C}_{\mathbb{C}}[a,b]$, namely, are continuous on $[a,b]$ and $v\in\mathcal{BV}_{\mathbb{C}}[a,b]$, namely of bounded variation on $[a,b]$, then for any $x\in[a,b]$ the Riemann-Stieltjes integrals in Lemma 1 exist and the equalities (2.1) and (2.2) hold.*

If we take $\lambda=\frac{1}{x-a}\int_a^x f(s)\,ds$ and $\mu=\frac{1}{b-x}\int_x^b f(s)\,ds$ in (2.1), then we get

$$\begin{aligned}&\int_a^b f(t)\,g(t)\,du(t) \\ &=\frac{1}{x-a}\int_a^x f(s)\,ds\int_a^x g(t)\,du(t)+\frac{1}{b-x}\int_x^b f(s)\,ds\int_x^b g(t)\,du(t) \\ &\quad+\int_a^x\left[f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right]g(t)\,du(t) \\ &\quad+\int_x^b\left[f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right]g(t)\,du(t)\end{aligned} \tag{2.5}$$

for any $x\in(a,b)$.

In particular, for $g(t)=1$, $t\in[a,b]$, we have for $x\in[a,b]$ that

$$\begin{aligned}\int_a^b f(t)\,du(t) &= \frac{u(x)-u(a)}{x-a}\int_a^x f(s)\,ds+\frac{u(b)-u(x)}{b-x}\int_x^b f(s)\,ds \\ &\quad+\int_a^x\left[f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right]du(t) \\ &\quad+\int_x^b\left[f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right]du(t).\end{aligned} \tag{2.6}$$

We have:

Theorem 3. *Assume that* $f,\ g\in\mathcal{C}_{\mathbb{C}}[a,b]$ *and* $u\in\mathcal{BV}_{\mathbb{C}}[a,b]$, *then for* $x\in(a,b)$,

$$|D(f,g,u;x,a,b)| \tag{2.7}$$
$$\leq\begin{cases}\max_{t\in[a,x]}\left|f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right|\int_a^x|g(t)|\,d\left(\bigvee_a^t(u)\right),\\ \left(\int_a^x\left|f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right|^p d\left(\bigvee_a^t(u)\right)\right)^{1/p}\\ \times\left(\int_a^x|g(t)|^q\,d\left(\bigvee_a^t(u)\right)\right)^{1/q},\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \max_{t\in[a,x]}|g(t)|\int_a^x\left|f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right|d\left(\bigvee_a^t(u)\right)\end{cases}$$
$$+\begin{cases}\max_{t\in[x,b]}\left|f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right|\int_x^b|g(t)|\,d\left(\bigvee_a^t(u)\right),\\ \left(\int_x^b\left|f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right|^p d\left(\bigvee_a^t(u)\right)\right)^{1/p}\\ \times\left(\int_x^b|g(t)|^q\,d\left(\bigvee_a^t(u)\right)\right)^{1/q},\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \max_{t\in[x,b]}|g(t)|\int_x^b\left|f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right|d\left(\bigvee_a^t(u)\right).\end{cases}$$

Proof. It is well known that if $p\in\mathcal{R}(u,[a,b])$ where $u\in\mathcal{BV}_{\mathbb{C}}[a,b]$ then we have [1, p. 177]

$$\left|\int_a^b p(t)\,du(t)\right|\leq\int_a^b|p(t)|\,d\left(\bigvee_a^t(u)\right)\leq\sup_{t\in[a,b]}|p(t)|\bigvee_a^b(u). \tag{2.8}$$

Using the equality (2.5), we have

$$\begin{aligned}|D(f,g,u;x,a,b)| &\leq\left|\int_a^x\left[f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right]g(t)\,du(t)\right|\\ &\quad+\left|\int_x^b\left[f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right]g(t)\,du(t)\right|\\ &\leq\int_a^x\left|f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right||g(t)|\,du(t)\\ &\qquad+\int_x^b\left|f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right||g(t)|\,du(t)\end{aligned} \tag{2.9}$$

$$
\begin{aligned}
&\leq \int_a^x \left| f(t) - \frac{1}{x-a}\int_a^x f(s)\,ds \right| |g(t)|\, d\left(\bigvee_a^t (u)\right) \\
&\qquad + \int_x^b \left| f(t) - \frac{1}{b-x}\int_x^b f(s)\,ds \right| |g(t)|\, d\left(\bigvee_x^t (u)\right) \\
&\qquad = \int_a^x \left| f(t) - \frac{1}{x-a}\int_a^x f(s)\,ds \right| |g(t)|\, d\left(\bigvee_a^t (u)\right) \\
&\qquad + \int_x^b \left| f(t) - \frac{1}{b-x}\int_x^b f(s)\,ds \right| |g(t)|\, d\left(\bigvee_a^t (u)\right) =: B(f,g,u,x).
\end{aligned}
$$

Using Hölder's inequality for monotonic nondecreasing integrators, we have

$$
\begin{aligned}
&\int_a^x \left| f(t) - \frac{1}{x-a}\int_a^x f(s)\,ds \right| |g(t)|\, d\left(\bigvee_a^t (u)\right) \\
&\leq \begin{cases}
\max_{t\in[a,x]} \left| f(t) - \frac{1}{x-a}\int_a^x f(s)\,ds \right| \int_a^x |g(t)|\, d\left(\bigvee_a^t (u)\right), \\[2ex]
\left(\int_a^x \left| f(t) - \frac{1}{x-a}\int_a^x f(s)\,ds \right|^p d\left(\bigvee_a^t (u)\right)\right)^{1/p} \\
\times \left(\int_a^x |g(t)|^q\, d\left(\bigvee_a^t (u)\right)\right)^{1/q}, \ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\[2ex]
\max_{t\in[a,x]} |g(t)| \int_a^x \left| f(t) - \frac{1}{x-a}\int_a^x f(s)\,ds \right| d\left(\bigvee_a^t (u)\right)
\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_x^b \left| f(t) - \frac{1}{b-x}\int_x^b f(s)\,ds \right| |g(t)|\, d\left(\bigvee_a^t (u)\right) \\
&\leq \begin{cases}
\max_{t\in[x,b]} \left| f(t) - \frac{1}{b-x}\int_x^b f(s)\,ds \right| \int_x^b |g(t)|\, d\left(\bigvee_a^t (u)\right), \\[2ex]
\left(\int_x^b \left| f(t) - \frac{1}{b-x}\int_x^b f(s)\,ds \right|^p d\left(\bigvee_a^t (u)\right)\right)^{1/p} \\
\times \left(\int_x^b |g(t)|^q\, d\left(\bigvee_a^t (u)\right)\right)^{1/q}, \ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\[2ex]
\max_{t\in[x,b]} |g(t)| \int_x^b \left| f(t) - \frac{1}{b-x}\int_x^b f(s)\,ds \right| d\left(\bigvee_a^t (u)\right),
\end{cases}
\end{aligned}
$$

which by (2.9) we get (2.7). □

Corollary 2. *Assume that $f \in \mathcal{C}_{\mathbb{C}}[a,b]$ and $u \in \mathcal{BV}_{\mathbb{C}}[a,b]$, then for $x \in (a,b)$,*

$$
\begin{aligned}
(2.10)\quad & |D(f,u;x,a,b)| \\
& \leq \begin{cases}
\max_{t\in[a,x]} \left| f(t) - \frac{1}{x-a}\int_a^x f(s)\,ds \right| \bigvee_a^x (u), \\
\left(\int_a^x \left| f(t) - \frac{1}{x-a}\int_a^x f(s)\,ds \right|^p d\left(\bigvee_a^t (u) \right) \right)^{1/p} \\
\times [u(x)-u(a)]^{1/q}, \ p,q>1, \ \frac{1}{p}+\frac{1}{q}=1, \\
\int_a^x \left| f(t) - \frac{1}{x-a}\int_a^x f(s)\,ds \right| d\left(\bigvee_a^t (u) \right)
\end{cases} \\
& \quad + \begin{cases}
\max_{t\in[x,b]} \left| f(t) - \frac{1}{b-x}\int_x^b f(s)\,ds \right| \bigvee_x^b (u), \\
\left(\int_x^b \left| f(t) - \frac{1}{b-x}\int_x^b f(s)\,ds \right|^p d\left(\bigvee_a^t (u) \right) \right)^{1/p} \\
\times [u(b)-u(x)]^{1/q}, \ p,q>1, \ \frac{1}{p}+\frac{1}{q}=1, \\
\int_x^b \left| f(t) - \frac{1}{b-x}\int_x^b f(s)\,ds \right| d\left(\bigvee_a^t (u) \right).
\end{cases}
\end{aligned}
$$

Remark 2. *Using the Ostrowski type inequality for functions of bounded variation $h:[c,d]\to\mathbb{C}$,* [6], [8] *(or the survey* [16]*)*

$$
(2.11)\qquad \left| h(t) - \frac{1}{d-c}\int_c^d h(s)\,ds \right| \leq \left[\frac{1}{2} + \left| \frac{t-\frac{c+d}{2}}{d-c} \right| \right] \bigvee_c^d (h), \ t\in[c,d].
$$

This implies that

$$
\max_{t\in[c,d]} \left| h(t) - \frac{1}{d-c}\int_c^d h(s)\,ds \right| \leq \bigvee_c^d (h).
$$

Assume that $f,\ g \in \mathcal{C}_{\mathbb{C}}[a,b]$ and $f,\ u \in \mathcal{BV}_{\mathbb{C}}[a,b]$, then for $x\in(a,b)$, we have from the first branch of (2.7) that

$$
\begin{aligned}
(2.12)\quad & |D(f,g,u;x,a,b)| \\
& \leq \bigvee_a^x (f) \int_a^x |g(t)|\, d\left(\bigvee_a^t (u) \right) + \bigvee_x^b (f) \int_x^b |g(t)|\, d\left(\bigvee_a^t (u) \right)
\end{aligned}
$$

$$\leq \frac{1}{2}\begin{cases} \left(\bigvee_a^b(f)+\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right)\int_a^b|g(t)|\,d\left(\bigvee_a^t(u)\right), \\ \left[\int_a^b|g(t)|\,d\left(\bigvee_a^t(u)\right)+\left|\int_a^x|g(t)|\,d\left(\bigvee_a^t(u)\right)-\int_x^b|g(t)|\,d\left(\bigvee_a^t(u)\right)\right|\right] \\ \times\bigvee_a^b(f) \end{cases}$$

for any $x\in(a,b)$.

In particular, for $g(t)=1$, $t\in[a,b]$, *we have*

$$|D(f,u;x,a,b)|\leq\bigvee_a^x(u)\bigvee_a^x(f)+\bigvee_x^b(u)\bigvee_x^b(f) \tag{2.13}$$

$$\leq\begin{cases} \frac{1}{2}\left(\bigvee_a^b(f)+\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right)\bigvee_a^b(u), \\ \frac{1}{2}\left[\bigvee_a^b(u)+\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right]\bigvee_a^b(f) \end{cases}$$

for any $x\in(a,b)$.

3. Inequalities for Lipschitzian Integrators

We have:

Theorem 4. *Assume that* f *is Riemann-integrable on* $[a,b]$, $g\in\mathcal{C}_{\mathbb{C}}[a,b]$ *and* u *is Lipschitzian with the constant* $L>0$, *namely* $|u(t)-u(s)|\leq L|t-s|$ *for any* t, $s\in[a,b]$. *Then*

$$|D(f,g,u;x,a,b)| \tag{3.1}$$

$$\leq L\times\begin{cases} \sup_{t\in[a,x]}\left|f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right|\int_a^x|g(t)|\,dt, \\ \left(\int_a^x\left|f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right|^p dt\right)^{1/p}\left(\int_a^x|g(t)|^q\,dt\right)^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \max_{t\in[a,x]}|g(t)|\int_a^x\left|f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right|dt \end{cases}$$

$$+L\times\begin{cases} \sup_{t\in[x,b]}\left|f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right|\int_x^b|g(t)|\,dt, \\ \left(\int_x^b\left|f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right|^p dt\right)^{1/p}\left(\int_x^b|g(t)|^q\,dt\right)^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \max_{t\in[x,b]}|g(t)|\int_x^b\left|f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right|dt \end{cases}$$

for $x\in(a,b)$.

Proof. It is known that if $p:[c,d]\to\mathbb{C}$ is Riemann integrable and $u:[c,d]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, then the Riemann-Stieltjes integral $\int_a^b p(t)\,du(t)$ exists and

$$\left|\int_a^b p(t)\,du(t)\right|\le L\int_a^b |p(t)|\,dt.$$

Using the equality (2.5), we have

$$\begin{aligned}
|D(f,g,u;x,a,b)| &\le \left|\int_a^x \left[f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right] g(t)\,du(t)\right| \\
&\quad + \left|\int_x^b \left[f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right] g(t)\,du(t)\right| \\
&\le L\int_a^x \left|f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right| |g(t)|\,dt \\
&\quad + L\int_x^b \left|f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right| |g(t)|\,dt =: B(f,g,x)
\end{aligned} \tag{3.2}$$

for $x\in(a,b)$.

Utilising Hölder's integral inequality, we have

$$\int_a^x \left|f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right| |g(t)|\,dt \tag{3.3}$$

$$\le\begin{cases}
\sup_{t\in[a,x]}\left|f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right|\int_a^x |g(t)|\,dt, \\[2mm]
\left(\int_a^x \left|f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right|^p dt\right)^{1/p}\left(\int_a^x |g(t)|^q\,dt\right)^{1/q}, \\
p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\[2mm]
\max_{t\in[a,x]}|g(t)|\int_a^x \left|f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right| dt
\end{cases}$$

and

$$\int_x^b \left|f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right| |g(t)|\,dt \tag{3.4}$$

$$\le\begin{cases}
\sup_{t\in[x,b]}\left|f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right|\int_x^b |g(t)|\,dt, \\[2mm]
\left(\int_x^b \left|f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right|^p dt\right)^{1/p}\left(\int_x^b |g(t)|^q\,dt\right)^{1/q}, \\
p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\[2mm]
\max_{t\in[x,b]}|g(t)|\int_x^b \left|f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right| dt.
\end{cases}$$

By making use of (3.2), (3.3) and (3.4) we get (3.1). □

Corollary 3. *Assume that f is Riemann-integrable on $[a,b]$ and u is Lipschitzian with the constant $L>0$, then*

$$|D(f,u;x,a,b)| \tag{3.5}$$

$$\leq L\times\begin{cases}\sup_{t\in[a,x]}\left|f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right|(x-a), \\ \\ \left(\int_a^x\left|f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right|^p dt\right)^{1/p}(x-a)^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \int_a^x\left|f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right|dt\end{cases}$$

$$+L\times\begin{cases}\sup_{t\in[x,b]}\left|f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right|(b-x), \\ \\ \left(\int_x^b\left|f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right|^p dt\right)^{1/p}(b-x)^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \int_x^b\left|f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right|dt,\end{cases}$$

for $x\in(a,b)$.

4. Some Related Facts

If we take $p=q=2$ in (3.1), then we get

$$\begin{aligned}&|D(f,g,u;x,a,b)| \\ &\leq L\left(\int_a^x\left|f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right|^2 dt\right)^{1/2}\left(\int_a^x|g(t)|^2\,dt\right)^{1/2} \\ &+L\left(\int_x^b\left|f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right|^2 dt\right)^{1/2}\left(\int_x^b|g(t)|^2\,dt\right)^{1/2} \\ &=L(x-a)\left(\frac{1}{x-a}\int_a^x\left|f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right|^2 dt\right)^{1/2} \\ &\qquad\times\left(\frac{1}{x-a}\int_a^x|g(t)|^2\,dt\right)^{1/2} \\ &+L(b-x)\left(\frac{1}{b-x}\int_x^b\left|f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right|^2 dt\right)^{1/2} \\ &\qquad\qquad\times\left(\frac{1}{b-x}\int_x^b|g(t)|^2\,dt\right)^{1/2}\end{aligned}$$

for $x\in(a,b)$.

In [14] we proved amongst other that, if $h:[c,d]\to\mathbb{C}$ is of bounded variation, then

$$\left(\frac{1}{d-c}\int_c^d\left|h(t)-\frac{1}{d-c}\int_c^d h(s)\,ds\right|^2 dt\right)^{1/2} = \left[\frac{1}{d-c}\int_c^d|h(t)|^2\,dt-\left|\frac{1}{d-c}\int_c^d h(s)\,ds\right|^2\right]^{1/2}\le\frac{1}{2}\bigvee_c^d(h) \tag{4.1}$$

with the constant $\frac{1}{2}$ as best possible.

So, if $f:[a,b]\to\mathbb{C}$ is of bounded variation, then

$$\left(\frac{1}{x-a}\int_a^x\left|f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right|^2 dt\right)^{1/2}\le\frac{1}{2}\bigvee_a^x(f)$$

and

$$\left(\frac{1}{b-x}\int_x^b\left|f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right|^2 dt\right)^{1/2}\le\frac{1}{2}\bigvee_x^b(f),$$

where $x\in(a,b)$.

We can state the following fact:

Proposition 1. *Assume that f is of bounded variation on $[a,b]$, $g\in\mathcal{C}_{\mathbb{C}}[a,b]$ and u is Lipschitzian with the constant $L>0$, then*

$$|D(f,g,u;x,a,b)|\le\frac{1}{2}L\left[(x-a)\bigvee_a^x(f)\left(\frac{1}{x-a}\int_a^x|g(t)|^2\,dt\right)^{1/2} + (b-x)\bigvee_x^b(f)\left(\frac{1}{b-x}\int_x^b|g(t)|^2\,dt\right)^{1/2}\right] \tag{4.2}$$

for $x\in(a,b)$.

In particular, we have

$$|D(f,u;x,a,b)|\le\frac{1}{2}L\left[(x-a)\bigvee_a^x(f)+(b-x)\bigvee_x^b(f)\right] \tag{4.3}$$

for $x\in(a,b)$.

From (4.3) we get the simpler bounds

$$|D(f,u;x,a,b)|\le\frac{1}{2}L\times\begin{cases}\left(\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right)(b-a),\\[2ex] \left(\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right)\bigvee_a^b(f)\end{cases} \tag{4.4}$$

for $x\in(a,b)$.

In particular, we have

$$\left|D\left(f,u;\frac{a+b}{2},a,b\right)\right|\leq\frac{1}{4}(b-a)L\bigvee_{a}^{b}(f). \tag{4.5}$$

If $m\in(a,b)$ is such that $\bigvee_{a}^{m}(f)=\bigvee_{m}^{b}(f)$, then by (4.4) we get

$$|D(f,u;m,a,b)|\leq\frac{1}{4}(b-a)L\bigvee_{a}^{b}(f). \tag{4.6}$$

Now, for $\gamma,\ \Gamma\in\mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{[a,b]}(\gamma,\Gamma):=\left\{f:[a,b]\to\mathbb{C}|\operatorname{Re}\left[(\Gamma-f(t))\left(\overline{f(t)}-\overline{\gamma}\right)\right]\geq0\ \text{ for each }\ t\in[a,b]\right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma):=\left\{f:[a,b]\to\mathbb{C}|\ \left|f(t)-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma|\ \text{ for each }\ t\in[a,b]\right\}.$$

This family of functions is a particular case of the class introduced in [13]

$$\begin{aligned}&\bar{\Delta}_{[a,b],g}(\gamma,\Gamma)\\&:=\left\{\ f:[a,b]\to\mathbb{C}|\ \left|f(t)-\frac{\gamma+\Gamma}{2}g(t)\right|\leq\frac{1}{2}|\Gamma-\gamma||g(t)|\ \text{ for each }\ t\in[a,b]\right\},\end{aligned}$$

where $g:[a,b]\to\mathbb{C}$.

The following representation result may be stated.

Proposition 2. *For any* $\gamma,\ \Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, *we have that* $\bar{U}_{[a,b]}(\gamma,\Gamma)$ *and* $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ *are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma)=\bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{4.7}$$

Proof. We observe that for any $z\in\mathbb{C}$ we have the equivalence

$$\left|z-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma|$$

if and only if

$$\operatorname{Re}[(\Gamma-z)(\bar{z}-\bar{\gamma})]\geq0.$$

This follows by the equality

$$\frac{1}{4}|\Gamma-\gamma|^{2}-\left|z-\frac{\gamma+\Gamma}{2}\right|^{2}=\operatorname{Re}[(\Gamma-z)(\bar{z}-\bar{\gamma})]$$

that holds for any $z\in\mathbb{C}$.

The equality (4.7) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 4. *For any* $\gamma,\ \Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$,*we have that*

$$\begin{aligned}\bar{U}_{[a,b]}(\gamma,\Gamma)=\{f:[a,b]\to\mathbb{C}\ |\ &(\operatorname{Re}\Gamma-\operatorname{Re}f(t))(\operatorname{Re}f(t)-\operatorname{Re}\gamma)\\&+(\operatorname{Im}\Gamma-\operatorname{Im}f(t))(\operatorname{Im}f(t)-\operatorname{Im}\gamma)\geq0\ \textit{for each}\ t\in[a,b]\}.\end{aligned} \tag{4.8}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\bar{S}_{[a,b]}(\gamma, \Gamma) := \{f : [a,b] \to \mathbb{C} \mid \operatorname{Re}(\Gamma) \geq \operatorname{Re} f(t) \geq \operatorname{Re}(\gamma) \text{ and } \operatorname{Im}(\Gamma) \geq \operatorname{Im} f(t) \geq \operatorname{Im}(\gamma) \text{ for each } t \in [a,b]\}. \tag{4.9}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma, \Gamma)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\gamma, \Gamma) \subseteq \bar{U}_{[a,b]}(\gamma, \Gamma). \tag{4.10}$$

If $f \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$ is square integrable on $[a,b]$, then we have the following Grüss type inequality (see for instance [7])

$$\left[\frac{1}{b-a}\int_a^b |f(t)|^2 dt - \left|\frac{1}{b-a}\int_a^b f(s)\, ds\right|^2\right]^{1/2} \leq \frac{1}{2}|\Gamma - \gamma|. \tag{4.11}$$

Proposition 3. *Assume that f is Riemann-integrable on $[a,b]$, $g \in \mathcal{C}_{\mathbb{C}}[a,b]$ and u is Lipschitzian with the constant $L > 0$. Let $x \in (a,b)$ and suppose that $f \in \bar{\Delta}_{[a,x]}(\gamma_x, \Gamma_x)$ and $f \in \bar{\Delta}_{[x,b]}(\delta_x, \Delta_x)$, then*

$$\begin{aligned} |D(f,g,u;x,a,b)| &\leq \frac{1}{2}|\Gamma_x - \gamma_x| L(x-a)\left(\frac{1}{x-a}\int_a^x |g(t)|^2 dt\right)^{1/2} \\ &\quad + \frac{1}{2}|\Delta_x - \delta_x| L(b-x)\left(\frac{1}{b-x}\int_x^b |g(t)|^2 dt\right)^{1/2}. \end{aligned}$$

In particular,

$$|D(f,u;x,a,b)| \leq \frac{1}{2}L[|\Gamma_x - \gamma_x|(x-a) + |\Delta_x - \delta_x|(b-x)]. \tag{4.12}$$

We observe that if $f \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$, then by (4.12) we get

$$|D(f,u;x,a,b)| \leq \frac{1}{2}(b-a)L(\Gamma - \gamma), \tag{4.13}$$

for any $x \in (a,b)$.

Remark 3. *With the assumptions in Proposition 3 and if $f \in \bar{\Delta}_{\left[a,\frac{a+b}{2}\right]}(\gamma, \Gamma)$ and $f \in \bar{\Delta}_{\left[\frac{a+b}{2},b\right]}(\delta, \Delta)$, then*

$$\left|D\left(f,u;\frac{a+b}{2},a,b\right)\right| \leq \frac{1}{4}(b-a)L[|\Gamma - \gamma| + |\Delta - \delta|]. \tag{4.14}$$

5. Applications for Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H; \langle\cdot,\cdot\rangle)$. Let $A \in \mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda \in \mathbb{R}$ as follows

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \leq \lambda, \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$E_\lambda := \varphi_\lambda (A) \tag{5.1}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [19, p. 256]:

Theorem 5 (Spectral Representation Theorem)**.** *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a = \min \{\lambda \,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $b = \max \{\lambda \,|\lambda \in Sp(A)\} =: \max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_\lambda \le E_{\lambda'}$ *for* $\lambda \le \lambda'$;
b) $E_{a-0} = 0, E_b = 1_H$ *and* $E_{\lambda+0} = E_\lambda$ *for all* $\lambda \in \mathbb{R}$*;*
c) *We have the representation*

$$A = \int_{a-0}^{b} \lambda dE_\lambda.$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ there exists a unique operator $\varphi(A) \in \mathcal{B}(H)$ such that for every $\varepsilon > 0$ there exists a $\delta > 0$ satisfying the inequality

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi(\lambda_k') \left[E_{\lambda_k} - E_{\lambda_{k-1}}\right] \right\| \le \varepsilon$$

whenever

$$\begin{cases} \lambda_0 < a = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = b, \\ \lambda_k - \lambda_{k-1} \le \delta \text{ for } 1 \le k \le n, \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \le k \le n \end{cases}$$

this means that

$$\varphi(A) = \int_{a-0}^{b} \varphi(\lambda)\, dE_\lambda, \tag{5.2}$$

where the integral is of Riemann-Stieltjes type.

Corollary 5. *With the assumptions of Theorem 5 for A, E_λ and φ we have the representations*

$$\varphi(A)\, x = \int_{a-0}^{b} \varphi(\lambda)\, dE_\lambda x \quad \text{for all } x \in H$$

and

$$\langle \varphi(A)\, x, y\rangle = \int_{a-0}^{b} \varphi(\lambda)\, d\langle E_\lambda x, y\rangle \quad \text{for all } x,\; y \in H. \tag{5.3}$$

In particular,

$$\langle \varphi(A)\, x, x\rangle = \int_{a-0}^{b} \varphi(\lambda)\, d\langle E_\lambda x, x\rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(A)x\|^2 = \int_{a-0}^{b} |\varphi(\lambda)|^2 \, d\|E_\lambda x\|^2 \quad \textit{for all } x \in H.$$

We need the following result that provides an upper bound for the total variation of the function $\mathbb{R} \ni \lambda \mapsto \langle E_\lambda x, y\rangle \in \mathbb{C}$ on an interval $[\alpha, \beta]$, see [15].

Lemma 2. *Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A. Then for any x, $y \in H$ and $\alpha < \beta$ we have the inequality*

$$\left[\bigvee_{\alpha}^{\beta}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right]^2 \le \langle (E_\beta - E_\alpha)x, x\rangle \langle (E_\beta - E_\alpha)y, y\rangle, \tag{5.4}$$

where $\bigvee_{\alpha}^{\beta}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$ denotes the total variation of the function $\left\langle E_{(\cdot)}x, y\right\rangle$ on $[\alpha, \beta]$.

Remark 4. *For $\alpha = a - \varepsilon$ with $\varepsilon > 0$ and $\beta = b$ we get from (5.4) the inequality*

$$\bigvee_{a-\varepsilon}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \le \langle (1_H - E_{a-\varepsilon})x, x\rangle^{1/2} \langle (1_H - E_{a-\varepsilon})y, y\rangle^{1/2} \tag{5.5}$$

for any x, $y \in H$.

This implies, for any x, $y \in H$, that

$$\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \le \|x\| \, \|y\|, \tag{5.6}$$

where $\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$ denotes the limit $\lim_{\varepsilon\to 0+}\left[\bigvee_{a-\varepsilon}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right]$.

We also have:

Theorem 6. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a = \min\{\lambda \,|\, \lambda \in Sp(A)\} =: \min Sp(A)$ and $b = \max\{\lambda \,|\, \lambda \in Sp(A)\} =: \max Sp(A)$. Also, assume that $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A and f of locally bounded variation on I, with $[a, b] \subset \mathring{I}$. Then for all $s \in [a, b]$,*

$$\begin{aligned} &\Bigg|\langle f(A)x, y\rangle \\ &\quad - \left(\frac{1}{s-a}\int_a^s f(t)\,dt\right)\langle E_s x, y\rangle - \left(\frac{1}{b-s}\int_s^b f(t)\,dt\right)\langle (1_H - E_s)x, y\rangle\Bigg| \\ &\qquad \le \frac{1}{2}\left(\bigvee_a^b(f) + \left|\bigvee_a^s(f) - \bigvee_s^b(f)\right|\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \\ &\qquad\qquad \le \frac{1}{2}\left(\bigvee_a^b(f) + \left|\bigvee_a^s(f) - \bigvee_s^b(f)\right|\right)\|x\| \, \|y\| \end{aligned} \tag{5.7}$$

for any x, $y \in H$.

In particular, if $m \in (a,b)$ *is such that* $\bigvee_a^m (f) = \bigvee_m^b (f)$, *then*

$$\begin{aligned}(5.8)\quad &\Big|\langle f(A)x,y\rangle \\ &-\left(\frac{1}{m-a}\int_a^m f(t)\,dt\right)\langle E_m x,y\rangle - \left(\frac{1}{b-m}\int_m^b f(t)\,dt\right)\langle (1_H - E_m)x,y\rangle\Big| \\ &\leq \frac{1}{2}\bigvee_a^b (f)\bigvee_{a-0}^b \left(\langle E_{(\cdot)}x,y\rangle\right) \leq \frac{1}{2}\bigvee_a^b (f)\|x\|\|y\|\end{aligned}$$

for any $x,\ y \in H$.

Proof. Using the inequality (2.13) we have for $s \in [a,b]$, for small $\varepsilon > 0$ and for any $x,\ y \in H$ that

$$\begin{aligned}&\Big|\int_{a-\varepsilon}^b f(t)\,d\langle E_t x,y\rangle - \left(\frac{1}{s-a+\varepsilon}\int_{a-\varepsilon}^x f(t)\,dt\right)[\langle E_s x,y\rangle - \langle E_{a-\varepsilon}x,y\rangle] \\ &- \left(\frac{1}{b-s}\int_s^b f(t)\,dt\right)[\langle E_b x,y\rangle - \langle E_s x,y\rangle]\Big| \\ &\leq \frac{1}{2}\left(\bigvee_{a-\varepsilon}^b (f) + \left|\bigvee_{a-\varepsilon}^s (f) - \bigvee_s^b (f)\right|\right)\bigvee_{a-\varepsilon}^b \left(\langle E_{(\cdot)}x,y\rangle\right).\end{aligned}$$

Taking the limit over $\varepsilon \to 0+$ and using the continuity of f and the Spectral Representation Theorem, we deduce the desired result (5.7). □

Remark 5. *The above inequalities (5.7)-(5.8) can produce several particular examples of interest. For example if* $[a,b] \subset (0,\infty)$ *and we take* $f(t) = t^{-1}$, *then we get for any* $x,\ y \in H$ *that*

$$\begin{aligned}(5.9)\quad &\left|\left\langle A^{-1}x,y\right\rangle - \frac{\ln s - \ln a}{s-a}\langle E_s x,y\rangle - \frac{\ln b - \ln s}{b-s}\langle (1_H - E_s)x,y\rangle\right| \\ &\leq \frac{1}{2}\left(\frac{b-a}{ba} + \left|\frac{1}{a}+\frac{1}{b}-\frac{2}{s}\right|\right)\bigvee_{a-0}^b \left(\langle E_{(\cdot)}x,y\rangle\right) \\ &\leq \frac{1}{2}\left(\frac{b-a}{ba} + \left|\frac{1}{a}+\frac{1}{b}-\frac{2}{s}\right|\right)\|x\|\|y\|.\end{aligned}$$

In particular, if we put $H(a,b) := \frac{2ab}{a+b}$, *the harmonic mean, then for any* $x,\ y \in H$,

$$\begin{aligned}(5.10)\quad &\Big|\left\langle A^{-1}x,y\right\rangle - \frac{\ln H(a,b) - \ln a}{H(a,b)-a}\left\langle E_{H(a,b)}x,y\right\rangle \\ &-\frac{\ln b - \ln H(a,b)}{b-H(a,b)}\left\langle \left(1_H - E_{H(a,b)}\right)x,y\right\rangle\Big| \\ &\leq \frac{1}{2}\frac{b-a}{ba}\bigvee_{a-0}^b \left(\langle E_{(\cdot)}x,y\rangle\right) \leq \frac{1}{2}\frac{b-a}{ba}\|x\|\|y\|.\end{aligned}$$

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