SOME RIEMANN-STIELTJES INTEGRAL INEQUALITIES FOR α-TRAPEZOID RULE WITH APPLICATIONS

SILVESTRU SEVER DRAGOMIR

Abstract. In this paper we provide some bounds for the error in approximating the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by the α-trapezoid rule

$$[(1-\alpha)f(b)+\alpha f(a)][u(b)-u(a)]$$

under various assumptions for the integrand f and the integrator u for which the above integral exists. Applications for continuous functions of selfadjoint operators in Hilbert spaces are provided as well.

1. Introduction

The following theorem generalizing the classical trapezoid inequality to the Riemann-Stieltjes integral for integrators of bounded variation and Hölder-continuous integrands was obtained by the author in 2001, see [4]:

Theorem 1. *Let $f:[a,b]\to\mathbb{C}$ be a p-H-Hölder type function, that is, it satisfies the condition*

$$|f(x)-f(y)|\le H|x-y|^p \text{ for all } x,\ y\in[a,b], \tag{1.1}$$

where $H>0$ and $p\in(0,1]$ are given, and $u:[a,b]\to\mathbb{C}$ is a function of bounded variation on $[a,b]$. Then we have the inequality:

$$\left|\frac{f(a)+f(b)}{2}[u(b)-u(a)]-\int_a^b f(t)\,du(t)\right|\le\frac{1}{2^p}H(b-a)^p\bigvee_a^b(u). \tag{1.2}$$

The constant $C=1$ on the right hand side of (1.2) cannot be replaced by a smaller quantity.

The case when the integrator is Lipschitzian is as follows, [8]:

Theorem 2. *Let $f:[a,b]\to\mathbb{C}$ be a p-H-Hölder type mapping where $H>0$ and $p\in(0,1]$ are given, and $u:[a,b]\to\mathbb{C}$ is a Lipschitzian function on $[a,b]$, this means that*

$$|u(x)-u(y)|\le L|x-y| \text{ for all } x,\ y\in[a,b], \tag{1.3}$$

where $L>0$ is given. Then we have the inequality:

$$\left|\frac{f(a)+f(b)}{2}[u(b)-u(a)]-\int_a^b f(t)\,du(t)\right|\le\frac{1}{p+1}HL(b-a)^{p+1}. \tag{1.4}$$

1991 *Mathematics Subject Classification.* 26D15, 41A55, 47A63.

Key words and phrases. Riemann-Stieltjes integral, Trapezoidal Quadrature Rule, Selfadjoint operators, Functions of Selfadjoint operators, Spectral representation, Inequalities for selfadjoint operators.

RGMIA Res. Rep. Coll. 21 (2018), Art. 93, 11 pp. Received 10/08/12

In the case when u is monotonic nondecreasing, we have the following result as well, [8]:

Theorem 3. *Let $f:[a,b]\to\mathbb{C}$ be a p-H-Hölder type mapping where $H>0$ and $p\in(0,1]$ are given, and $u:[a,b]\to\mathbb{R}$ a monotonic nondecreasing function on $[a,b]$. Then we have the inequality:*

$$\begin{aligned}(1.5)\quad &\left|\frac{f(a)+f(b)}{2}[u(b)-u(a)]-\int_a^b f(t)\,du(t)\right|\\ &\le\frac{1}{2}H\left\{(b-a)^p[u(b)-u(a)]-p\int_a^b\left[\frac{(b-t)^{1-p}-(t-a)^{1-p}}{(b-t)^{1-p}(t-a)^{1-p}}\right]u(t)\,dt\right\}\\ &\le\frac{1}{2^p}H(b-a)^p[u(b)-u(a)].\end{aligned}$$

The inequalities in (1.5) are sharp.

For other similar results, see [2]-[8].

In this paper we provide some bounds for the error in approximating the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by the *α-trapezoid rule*

$$[(1-\alpha)f(b)+\alpha f(a)][u(b)-u(a)]$$

under various assumptions for the integrand f and the integrator u for which the above integral exists. Applications for continuous functions of selfadjoint operators in Hilbert spaces are provided as well.

2. Inequalities for Integrands of Bounded Variation

Assume that $u,\ f:[a,b]\to\mathbb{C}$. If the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists, we write for simplicity, like in [1, p. 142] that $f\in\mathcal{R}_{\mathbb{C}}(u,[a,b])$, or $\mathcal{R}_{\mathbb{C}}(u)$ when the interval is implicitly known. If the functions $u,\ f$ are real valued, then we write $f\in\mathcal{R}(u,[a,b])$, or $\mathcal{R}(u)$.

We start with the following identity of interest.

Lemma 1. *Let $f,\ u:[a,b]\to\mathbb{C}$ and $x\in[a,b]$ such that $f\in\mathcal{R}_{\mathbb{C}}(u,[a,b])$. Then for any $\gamma,\ \mu\in\mathbb{C}$,*

$$\begin{aligned}(2.1)\quad &[u(b)-\mu]f(b)+[\gamma-u(a)]f(a)+(\mu-\gamma)f(x)-\int_a^b f(t)\,du(t)\\ &=\int_a^x[u(t)-\gamma]\,df(t)+\int_x^b[u(t)-\mu]\,df(t).\end{aligned}$$

In particular, for $\mu=\gamma$ we have

$$(2.2)\quad [u(b)-\gamma]f(b)+[\gamma-u(a)]f(a)-\int_a^b f(t)\,du(t)=\int_a^b[u(t)-\gamma]\,df(t).$$

Proof. Using integration by parts rule for the Riemann-Stieltjes integral, we have

$$\int_a^x[u(t)-\gamma]\,df(t)=[u(x)-\gamma]f(x)-[u(a)-\gamma]f(a)-\int_a^x f(t)\,du(t)$$

and

$$\int_x^b[u(t)-\mu]\,df(t)=[u(b)-\mu]f(b)-[u(x)-\mu]f(x)-\int_x^b f(t)\,du(t)$$

for any $x \in [a,b]$.

If we add these two equalities, we get

$$\begin{aligned}\int_a^x [u(t)-\gamma]\,df(t) + \int_x^b [u(t)-\mu]\,df(t) \\ &= [u(b)-\mu]\,f(b) + [\gamma - u(a)]\,f(a) + [\mu - u(x)]\,f(x) \\ &\quad + [u(x)-\gamma]\,f(x) - \int_a^x f(t)\,du(t) - \int_x^b f(t)\,du(t) \\ &= [u(b)-\mu]\,f(b) + [\gamma - u(a)]\,f(a) + (\mu-\gamma)\,f(x) - \int_a^b f(t)\,du(t)\end{aligned}$$

for any $x \in [a,b]$, which proves the desired equality (2.1). ☐

Now, if we take $\gamma = (1-\alpha)\,u(a) + \alpha u(b)$, $\alpha \in [0,1]$ in (2.2), then we get

$$[u(b)-u(a)]\,[(1-\alpha)\,f(b) + \alpha f(a)] - \int_a^b f(t)\,du(t) = \int_a^b [u(t) - (1-\alpha)\,u(a) - \alpha u(b)]\,df(t) \tag{2.3}$$

and in particular

$$[u(b)-u(a)]\,\frac{f(b)+f(a)}{2} - \int_a^b f(t)\,du(t) = \int_a^b \left[u(t) - \frac{u(a)+u(b)}{2}\right] df(t). \tag{2.4}$$

Define the *α-trapezoid error functional*

$$T(f,u;a,b;\alpha) := [(1-\alpha)\,f(b) + \alpha f(a)]\,[u(b)-u(a)] - \int_a^b f(t)\,du(t)$$

where $\alpha \in [0,1]$ and for $\alpha = \frac{1}{2}$, the trapezoid error functional

$$T(f,u;a,b) := \frac{f(b)+f(a)}{2}\,[u(b)-u(a)] - \int_a^b f(t)\,du(t)$$

provided the Riemann-Stieltjes integral exists.

We have:

Theorem 4. *Assume that u, $f \in \mathcal{BV}_{\mathbb{C}}[a,b]$ (of bounded variations) and $f \in \mathcal{C}_{\mathbb{C}}[a,b]$. Then the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and*

$$\begin{aligned}|T(f,u;a,b;\alpha)| &\le (1-\alpha)\int_a^b \left(\bigvee_a^t (u)\right) d\left(\bigvee_a^t (f)\right) + \alpha \int_a^b \left(\bigvee_t^b (u)\right) d\left(\bigvee_a^t (f)\right) \\ &= (1-\alpha)\int_a^b \left(\bigvee_t^b (f)\right) d\left(\bigvee_a^t (u)\right) + \alpha \int_a^b \left(\bigvee_a^t (f)\right) d\left(\bigvee_a^t (u)\right) \\ &\le \left[\frac{1}{2} + \left|\alpha - \frac{1}{2}\right|\right] \bigvee_a^b (u) \bigvee_a^b (f),\end{aligned} \tag{2.5}$$

for all $\alpha \in [0,1]$.

In particular

$$|T(f,u;a,b)| \leq \frac{1}{2} \bigvee_a^b (u) \bigvee_a^b (f), \tag{2.6}$$

that was obtained in [8].

Proof. It is well known that, if $p : [a,b] \to \mathbb{C}$ is continuous and $v : [a,b] \to \mathbb{C}$ of bounded variation, then

$$\left| \int_a^b p(t)\, dv(t) \right| \leq \int_a^b |p(t)|\, d\left(\bigvee_a^t (u) \right) \leq \max_{t\in[a,b]} |p(t)| \bigvee_a^b (u). \tag{2.7}$$

By making use of the equality (2.3) we have

$$\begin{aligned} |T(f,u;a,b;\alpha)| &= \left| \int_a^b [u(t) - (1-\alpha) u(a) - \alpha u(b)]\, df(t) \right| \\ &= \left| (1-\alpha) \int_a^b [u(t) - u(a)]\, df(t) + \alpha \int_a^b [u(t) - u(b)]\, df(t) \right| \\ &\leq (1-\alpha) \left| \int_a^b [u(t) - u(a)]\, df(t) \right| + \alpha \left| \int_a^b [u(t) - u(b)]\, df(t) \right| \\ &\leq (1-\alpha) \int_a^b |u(t) - u(a)|\, d\left(\bigvee_a^t (f) \right) + \alpha \int_a^b |u(t) - u(b)|\, d\left(\bigvee_a^t (f) \right) \\ &=: B(f,u;\alpha). \end{aligned} \tag{2.8}$$

Since u is of bounded variation, we have

$$|u(t) - u(a)| \leq \bigvee_a^t (u) \text{ for } t \in [a,b]$$

and

$$|u(t) - u(b)| \leq \bigvee_t^b (u) \text{ for } t \in [a,b],$$

which implies that

$$B(f,u;\alpha) \leq (1-\alpha) \int_a^b \left(\bigvee_a^t (u) \right) d\left(\bigvee_a^t (f) \right) + \alpha \int_a^b \left(\bigvee_t^b (u) \right) d\left(\bigvee_a^t (f) \right),$$

where $\alpha \in [0,1]$.

Using integration by parts formula for Riemann-Stieltjes integral, we have

$$
\begin{aligned}
&\int_a^b \left(\bigvee_a^t (u)\right) d\left(\bigvee_a^t (f)\right) \\
&= \left(\bigvee_a^t (u)\right)\left(\bigvee_a^t (f)\right)\Bigg|_a^b - \int_a^b \left(\bigvee_a^t (f)\right) d\left(\bigvee_a^t (u)\right) \\
&= \left(\bigvee_a^b (u)\right)\left(\bigvee_a^b (f)\right) - \int_a^b \left(\bigvee_a^t (f)\right) d\left(\bigvee_a^t (u)\right) \\
&= \int_a^b \left(\bigvee_a^b (f) - \bigvee_a^t (f)\right) d\left(\bigvee_a^t (u)\right) = \int_a^b \left(\bigvee_t^b (f)\right) d\left(\bigvee_a^t (u)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_a^b \left(\bigvee_t^b (u)\right) d\left(\bigvee_a^t (f)\right) \\
&= \left(\bigvee_t^b (u)\right)\left(\bigvee_a^t (f)\right)\Bigg|_a^b - \int_a^b \left(\bigvee_a^t (f)\right) d\left(\bigvee_t^b (u)\right) \\
&= -\int_a^b \left(\bigvee_a^t (f)\right) d\left(\bigvee_t^b (u)\right) = -\int_a^b \left(\bigvee_a^t (f)\right) d\left(\bigvee_a^b (u) - \bigvee_a^t (u)\right) \\
&= \int_a^b \left(\bigvee_a^t (f)\right) d\left(\bigvee_a^t (u)\right),
\end{aligned}
$$

which prove the equality in (2.5).

Now, observe that

$$
\begin{aligned}
(1-\alpha)&\int_a^b \left(\bigvee_t^b (f)\right) d\left(\bigvee_a^t (u)\right) + \alpha \int_a^b \left(\bigvee_a^t (f)\right) d\left(\bigvee_a^t (u)\right) \\
&\leq \max\{1-\alpha, \alpha\} \left[\int_a^b \left(\bigvee_t^b (f)\right) d\left(\bigvee_a^t (u)\right) + \int_a^b \left(\bigvee_a^t (f)\right) d\left(\bigvee_a^t (u)\right)\right] \\
&= \left[\frac{1}{2} + \left|\alpha - \frac{1}{2}\right|\right] \int_a^b \left(\bigvee_t^b (f) + \bigvee_a^t (f)\right) d\left(\bigvee_a^t (u)\right) \\
&= \left[\frac{1}{2} + \left|\alpha - \frac{1}{2}\right|\right] \bigvee_a^b (f) \int_a^b d\left(\bigvee_a^t (u)\right) = \left[\frac{1}{2} + \left|\alpha - \frac{1}{2}\right|\right] \bigvee_a^b (f) \bigvee_a^b (u),
\end{aligned}
$$

which proves the last part of (2.5). □

Corollary 1. *Assume that* $u \in \mathcal{BV}_{\mathbb{C}}[a,b]$, $f \in \mathcal{M}^{\nearrow}[a,b]$ *(monotonic nondecreasing) and* $f \in \mathcal{C}_{\mathbb{C}}[a,b]$. *Then the Riemann-Stieltjes integral* $\int_a^b f(t)\,du(t)$ *exists and*

$$\begin{aligned}
(2.9)\quad & |T(f,u;a,b;\alpha)| \\
&\leq (1-\alpha)\int_a^b \left(\bigvee_a^t (u)\right) df(t) + \alpha \int_a^b \left(\bigvee_t^b (u)\right) df(t) \\
&= (1-\alpha)\int_a^b [f(b)-f(t)]\, d\left(\bigvee_a^t (u)\right) + \alpha \int_a^b [f(t)-f(a)]\, d\left(\bigvee_a^t (u)\right) \\
&\leq \left[\frac{1}{2} + \left|\alpha - \frac{1}{2}\right|\right][f(b)-f(a)]\bigvee_a^b (u),
\end{aligned}$$

for all $\alpha \in [0,1]$.

In particular

$$|T(f,u;a,b)| \leq \frac{1}{2}[f(b)-f(a)]\bigvee_a^b (u). \tag{2.10}$$

3. Inequalities for Lipschitzian Integrands

The function $f : [a,b] \to \mathbb{C}$ is called *Lipschitzian* with the constant $L > 0$ if

$$|f(t) - f(s)| \leq L|t-s| \text{ for all } t,\ s \in [a,b].$$

For the case of Lipschitzian integrators, we have:

Theorem 5. *Assume that* $u \in \mathcal{BV}_{\mathbb{C}}[a,b]$ *and* f *is Lipschitzian with the constant* $L > 0$. *Then the Riemann-Stieltjes integral* $\int_a^b f(t)\,du(t)$ *exists and*

$$\begin{aligned}
(3.1)\quad & |T(f,u;a,b;\alpha)| \\
&\leq L\left[(1-\alpha)\int_a^b \left(\bigvee_a^t (u)\right) dt + \alpha \int_a^b \left(\bigvee_t^b (u)\right) dt\right] \\
&= L\left[(1-\alpha)\int_a^b (b-t)\, d\left(\bigvee_a^t (u)\right) + \alpha \int_a^b (t-a)\, d\left(\bigvee_a^t (u)\right)\right] \\
&\leq \left[\frac{1}{2} + \left|\alpha - \frac{1}{2}\right|\right] L(b-a)\bigvee_a^b (u),
\end{aligned}$$

for all $\alpha \in [0,1]$.

In particular

$$|T(f,u;a,b)| \leq \frac{1}{2}(b-a)L\bigvee_a^b (u). \tag{3.2}$$

Proof. It is well known that if $p \in \mathcal{R}(u,[a,b])$, where $u \in \mathcal{L}_{L,\mathbb{C}}[a,b]$, namely u is Lipschitzian with the constant u, then we have

$$\left|\int_a^b p(t)\,dv(t)\right| \leq L\int_a^b |p(t)|\,dt. \tag{3.3}$$

By the inequality (2.8) we have

$$
\begin{aligned}
(3.4)\quad & |T(f,u;a,b;\alpha)| \\
& \leq (1-\alpha)\left|\int_a^b [u(t)-u(a)]\,df(t)\right| + \alpha\left|\int_a^b [u(t)-u(b)]\,df(t)\right| \\
& \leq L\left[(1-\alpha)\int_a^b |u(t)-u(a)|\,dt + \alpha\int_a^b |u(t)-u(b)|\,dt\right] =: C(f,u;\alpha).
\end{aligned}
$$

Since $u \in \mathcal{BV}_{\mathbb{C}}[a,b]$, hence

$$
\begin{aligned}
C(f,u;\alpha) &\leq L\left[(1-\alpha)\int_a^b \left(\bigvee_a^t (u)\right) dt + \alpha\int_a^b \left(\bigvee_t^b (u)\right) dt\right] \\
&= L(1-\alpha)\left[\left(\bigvee_a^t (u)\right) t\Big|_a^b - \int_a^b t\,d\left(\bigvee_a^t (u)\right)\right] \\
&\quad + L\alpha\left[\left(\bigvee_t^b (u)\right) t\Big|_a^b - \int_a^b t\,d\left(\bigvee_t^b (u)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= L(1-\alpha)\left[\left(\bigvee_a^b (u)\right) b - \int_a^b t\,d\left(\bigvee_a^t (u)\right)\right] \\
&\quad + L\alpha\left[-\left(\bigvee_a^b (u)\right) a - \int_a^b t\,d\left(\bigvee_a^b (u) - \bigvee_a^t (u)\right)\right] \\
&= L(1-\alpha)\left[\int_a^b (b-t)\,d\left(\bigvee_a^t (u)\right)\right] + L\alpha\left[\left(\int_a^b t-a\right) d\left(\bigvee_a^t (u)\right)\right],
\end{aligned}
$$

which proves the second part of (3.1).

The last part is obvious. □

4. Applications for Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Let $A \in \mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda \in \mathbb{R}$ as follows

$$
\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \leq \lambda, \\ \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}
$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$
E_\lambda := \varphi_\lambda(A) \tag{4.1}
$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [9, p. 256]:

Theorem 6 (Spectral Representation Theorem)**.** *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a = \min\{\lambda \,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $b = \max\{\lambda \,|\lambda \in Sp(A)\} =: \max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_\lambda \leq E_{\lambda'}$ *for* $\lambda \leq \lambda'$;
b) $E_{a-0} = 0, E_b = 1_H$ *and* $E_{\lambda+0} = E_\lambda$ *for all* $\lambda \in \mathbb{R}$*;*
c) *We have the representation*

$$A = \int_{a-0}^{b} \lambda dE_\lambda.$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ there exists a unique operator $\varphi(A) \in \mathcal{B}(H)$ such that for every $\varepsilon > 0$ there exists a $\delta > 0$ satisfying the inequality

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi(\lambda_k') \left[E_{\lambda_k} - E_{\lambda_{k-1}}\right] \right\| \leq \varepsilon$$

whenever

$$\begin{cases} \lambda_0 < a = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = b, \\ \lambda_k - \lambda_{k-1} \leq \delta \text{ for } 1 \leq k \leq n, \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \leq k \leq n \end{cases}$$

this means that

$$\varphi(A) = \int_{a-0}^{b} \varphi(\lambda)\, dE_\lambda, \tag{4.2}$$

where the integral is of Riemann-Stieltjes type.

Corollary 2. *With the assumptions of Theorem 6 for A, E_λ and φ we have the representations*

$$\varphi(A)\, x = \int_{a-0}^{b} \varphi(\lambda)\, dE_\lambda x \quad \text{for all } x \in H$$

and

$$\langle \varphi(A)\, x, y\rangle = \int_{a-0}^{b} \varphi(\lambda)\, d\langle E_\lambda x, y\rangle \quad \text{for all } x,\ y \in H. \tag{4.3}$$

In particular,

$$\langle \varphi(A)\, x, x\rangle = \int_{a-0}^{b} \varphi(\lambda)\, d\langle E_\lambda x, x\rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(A)\, x\|^2 = \int_{a-0}^{b} |\varphi(\lambda)|^2\, d\|E_\lambda x\|^2 \quad \text{for all } x \in H.$$

We need the following result that provides an upper bound for the total variation of the function $\mathbb{R} \ni \lambda \mapsto \langle E_\lambda x, y\rangle \in \mathbb{C}$ on an interval $[\alpha, \beta]$, see [7].

Lemma 2. *Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A. Then for any $x,\ y \in H$ and $\alpha < \beta$ we have the inequality*

$$\left[\bigvee_{\alpha}^{\beta}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right]^2 \leq \left\langle \left(E_\beta - E_\alpha\right)x, x\right\rangle \left\langle \left(E_\beta - E_\alpha\right)y, y\right\rangle, \tag{4.4}$$

where $\bigvee_{\alpha}^{\beta}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$ denotes the total variation of the function $\left\langle E_{(\cdot)}x, y\right\rangle$ on $[\alpha,\beta]$.

Remark 1. *For $\alpha = a - \varepsilon$ with $\varepsilon > 0$ and $\beta = b$ we get from (4.4) the inequality*

$$\bigvee_{a-\varepsilon}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \left\langle \left(1_H - E_{a-\varepsilon}\right)x, x\right\rangle^{1/2} \left\langle \left(1_H - E_{a-\varepsilon}\right)y, y\right\rangle^{1/2} \tag{4.5}$$

for any $x,\ y \in H$.

This implies, for any $x,\ y \in H$, that

$$\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \|x\|\,\|y\|, \tag{4.6}$$

where $\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$ denotes the limit $\lim_{\varepsilon\to 0+}\left[\bigvee_{a-\varepsilon}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right]$.

We can state the following result for functions of selfadjoint operators:

Theorem 7. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a = \min\{\lambda\,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $b = \max\{\lambda\,|\lambda \in Sp(A)\} =: \max Sp(A)$. Also, assume that $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A and assume that $\varphi \in \mathcal{BV}_{\mathbb{C}}[a,b]$ and $\varphi \in \mathcal{C}_{\mathbb{C}}[a,b]$ where $[a,b] \subset \mathring{I}$ (the interior of I). Then for all $\alpha \in [0,1]$*

$$\begin{aligned} &\left|\left[(1-\alpha)\varphi(b) + \alpha\varphi(a)\right]\langle x, y\rangle - \langle \varphi(A)x, y\rangle\right| \\ &\quad\leq \left[\frac{1}{2} + \left|\alpha - \frac{1}{2}\right|\right]\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\bigvee_{a}^{b}(\varphi) \leq \left[\frac{1}{2} + \left|\alpha - \frac{1}{2}\right|\right]\|x\|\,\|y\|\bigvee_{a}^{b}(\varphi) \end{aligned} \tag{4.7}$$

for any $x,\ y \in H$.

In particular,

$$\begin{aligned} &\left|\frac{\varphi(b) + \varphi(a)}{2}\langle x, y\rangle - \langle \varphi(A)x, y\rangle\right| \\ &\qquad\leq \frac{1}{2}\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\bigvee_{a}^{b}(\varphi) \leq \frac{1}{2}\|x\|\,\|y\|\bigvee_{a}^{b}(\varphi) \end{aligned} \tag{4.8}$$

for any $x,\ y \in H$.

Proof. Using the inequality (2.5) we have for $\alpha \in [0,1]$ that

$$\begin{aligned} &\Bigg|\left[(1-\alpha)\varphi(b) + \alpha\varphi(a-\varepsilon)\right]\left[\langle E_b x, y\rangle - \langle E_{a-\varepsilon}x, y\rangle\right] \\ &\qquad - \int_{a-\varepsilon}^{b}\varphi(t)\,d\langle E_t x, y\rangle\Bigg| \leq \left[\frac{1}{2} + \left|\alpha - \frac{1}{2}\right|\right]\bigvee_{a-\varepsilon}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\bigvee_{a}^{b}(\varphi), \end{aligned}$$

for small $\varepsilon > 0$ and for any $x,\, y \in H$.

Taking the limit over $\varepsilon \to 0+$ and using the continuity of φ and the Spectral Representation Theorem, we deduce the desired result (4.7). □

We also have:

Theorem 8. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a = \min\{\lambda\,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $b = \max\{\lambda\,|\lambda \in Sp(A)\} =: \max Sp(A)$. Also, assume that $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A and assume that φ is Lipschitzian with the constant $L > 0$ on $[a,b] \subset \mathring{I}$. Then for all $\alpha \in [0,1]$*

$$\begin{aligned} &|[(1-\alpha)\varphi(b) + \alpha\varphi(a)]\langle x,y\rangle - \langle\varphi(A)x,y\rangle| \\ &\leq \left[\frac{1}{2} + \left|\alpha - \frac{1}{2}\right|\right] L(b-a) \bigvee_{a-0}^{b}(\langle E_{(\cdot)}x,y\rangle) \leq \left[\frac{1}{2} + \left|\alpha - \frac{1}{2}\right|\right] L(b-a)\|x\|\,\|y\| \end{aligned} \tag{4.9}$$

for any $x,\, y \in H$.

In particular,

$$\begin{aligned} &\left|\frac{\varphi(b)+\varphi(a)}{2}\langle x,y\rangle - \langle\varphi(A)x,y\rangle\right| \\ &\leq \frac{1}{2}L(b-a)\bigvee_{a-0}^{b}(\langle E_{(\cdot)}x,y\rangle) \leq \frac{1}{2}L(b-a)\|x\|\,\|y\| \end{aligned} \tag{4.10}$$

for any $x,\, y \in H$.

The proof follows by the inequality (3.1).

Remark 2. *The above results can provide particular inequalities of interest. For instance, if we take $\varphi : [a,b] \subset (0,\infty) \to \mathbb{R}$, $\varphi(t) = \ln t$ and A is a bounded selfadjoint operator on the Hilbert space H with $a = \min\{\lambda\,|\lambda \in Sp(A)\}$ and $b = \max\{\lambda\,|\lambda \in Sp(A)\}$, then by (4.7) we get for $\alpha \in [0,1]$ that*

$$\begin{aligned} &|\langle x,y\rangle \ln G_\alpha(a,b) - \langle \ln Ax,y\rangle| \\ &\leq \left[\frac{1}{2} + \left|\alpha - \frac{1}{2}\right|\right] \ln\left(\frac{b}{a}\right) \bigvee_{a-0}^{b}(\langle E_{(\cdot)}x,y\rangle) \leq \left[\frac{1}{2} + \left|\alpha - \frac{1}{2}\right|\right] \ln\left(\frac{b}{a}\right)\|x\|\,\|y\| \end{aligned} \tag{4.11}$$

for any $x,\, y \in H$, where $G_\alpha(a,b) := b^{1-\alpha}a^\alpha$.

In particular,

$$\begin{aligned} &|\langle x,y\rangle \ln G(a,b) - \langle \ln Ax,y\rangle| \\ &\leq \frac{1}{2}\ln\left(\frac{b}{a}\right)\bigvee_{a-0}^{b}(\langle E_{(\cdot)}x,y\rangle) \leq \frac{1}{2}\ln\left(\frac{b}{a}\right)\|x\|\,\|y\| \end{aligned} \tag{4.12}$$

for any $x,\, y \in H$, where $G_\alpha(a,b) := \sqrt{ab}$.

The function $\varphi : [a,b] \subset (0,\infty) \to \mathbb{R}$, $\varphi(t) = \ln t$ *is Lipschitzian on* $[a,b]$ *with constant* $L = \frac{1}{a} > 0$. *Then by (4.9) we get*

$$|\langle x,y\rangle \ln G_\alpha(a,b) - \langle \ln Ax, y\rangle| \le \left[\frac{1}{2} + \left|\alpha - \frac{1}{2}\right|\right]\left(\frac{b}{a} - 1\right)\bigvee_{a-0}^{b}\left(\langle E_{(\cdot)}x,y\rangle\right) \le \left[\frac{1}{2} + \left|\alpha - \frac{1}{2}\right|\right]\left(\frac{b}{a} - 1\right)\|x\|\,\|y\| \tag{4.13}$$

for any x, $y \in H$.

In particular,

$$|\langle x,y\rangle \ln G(a,b) - \langle \ln Ax, y\rangle| \le \frac{1}{2}\left(\frac{b}{a} - 1\right)\bigvee_{a-0}^{b}\left(\langle E_{(\cdot)}x,y\rangle\right) \le \frac{1}{2}\left(\frac{b}{a} - 1\right)\|x\|\,\|y\| \tag{4.14}$$

for any x, $y \in H$.

References

[1] T. M. Apostol, *Mathematical Analysis,* Addison-Wesley Publishing Company, Second Edition, 1981.

[2] N. S. Barnett, W. S. Cheung, S. S. Dragomir, A. Sofo, Ostrowski and trapezoid type inequalities for the Stieltjes integral with Lipschitzian integrands or integrators, *Computers & Mathematics with Applications*, 57(2009), Issue 2, 195-201.

[3] P. Cerone, W. S. Cheung, S. S. Dragomir, On Ostrowski type inequalities for Stieltjes integrals with absolutely continuous integrands and integrators of bounded variation, *Computers & Mathematics with Applications*, **54**(2007), Issue 2, 183-191.

[4] S. S. Dragomir, Some inequalities for Riemann-Stieltjes integral and applications, in: A. Rubinov and B. Glover (eds.), *Optimization and Related Topics*, 197-235, Kluwer Academic Publishers, 2001.

[5] S. S. Dragomir, Some inequalities of midpoint and trapezoid type for the Riemann–Stieltjes integral, *Nonlinear Analysis*, **47**(2001), Issue 4, 2333-2340.

[6] S. S. Dragomir, Approximating the Riemann–Stieltjes integral in terms of generalised trapezoidal rules, *Nonlinear Analysis: Theory, Methods & Applications*, **71**(2009), e62-e72.

[7] S. S. Dragomir, Some inequalities for continuous functions of selfadjoint operators in Hilbert spaces, *Acta Math Vietnam* (2014) **39**:287–303, DOI 10.1007/s40306-014-0061-4. Preprint *RGMIA Res. Rep. Coll.* **15**(2012), Art. 16.

[8] S. S. Dragomir, Approximating the Riemann-Stieltjes integral by a trapezoidal quadrature rule with applications, *Mathematical and Computer Modelling* **54** (2011) 243–260.

[9] G. Helmberg, *Introduction to Spectral Theory in Hilbert Space*, John Wiley & Sons, Inc. -New York, 1969.

[10] P. R. Mercer, Hadamard's inequality and trapezoid rules for the Riemann–Stieltjes integral, *J. Math. Anal. Appl.* **344** (2008) 921–926.

[11] M. Munteanu, Quadrature formulas for the generalized Riemann-Stieltjes integral. *Bull. Braz. Math. Soc.* (N.S.) **38** (2007), no. 1, 39–50.

Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir/

DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences,, School of Computer Science and Applied Mathematics, University of the Witwatersrand, Private Bag-3, Wits-2050, Johannesburg, South Africa