ON SOME RIEMANN-STIELTJES INTEGRAL INEQUALITIES OF GENERALIZED TRAPEZOID TYPE WITH APPLICATIONS

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Abstract. In this paper we provide some bounds for the error in approximating the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by the generalized trapezoidal rule

$$[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a)$$

under various assumptions for the integrand f and the integrator u for which the above integral exists. Applications for continuous functions of selfadjoint operators and unitary operators in Hilbert spaces are provided as well.

1. Introduction

In [10], in order to approximate the *Riemann-Stieltjes integral* $\int_a^b f(t)\,du(t)$ by the *generalized trapezoid formula*

$$[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a), \qquad x\in[a,b], \tag{1.1}$$

the authors considered the error functional

$$T(f,u;a,b;x):=[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a)-\int_a^b f(t)\,du(t) \tag{1.2}$$

and proved that

$$|T(f,u;a,b;x)|\le H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(f), \qquad x\in[a,b], \tag{1.3}$$

provided that $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$ and u is of *r-H-Hölder type*, that is, $u:[a,b]\to\mathbb{R}$ satisfies the condition $|u(t)-u(s)|\le H|t-s|^r$ for any $t,s\in[a,b]$, where $r\in(0,1]$ and $H>0$ are given.

If $r=1$, namely u is *Lipschitzian* with the constant $L>0$, then by (1.3) we get

$$|T(f,u;a,b;x)|\le L\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f), \tag{1.4}$$

for $x\in[a,b]$, provided that $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$.

The dual case, namely, when f is of q-K-Hölder type and u is of bounded variation has been considered in [3] in which the authors obtained the bound:

$$|T(f,u;a,b;x)|\le K\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^q\bigvee_a^b(u) \tag{1.5}$$

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for any $x \in [a,b]$.

If $q=1$, namely, if f is Lipschitzian with the constant $M>0$, then by (1.5) we get

$$|T(f,u;a,b;x)| \leq M\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(u) \tag{1.6}$$

for any $x \in [a,b]$.

For other related results, see [7]-[8] and [11]-[12].

The case where f is monotonic and u is of r-H-Hölder type, which provides a refinement for (1.3), respectively the case where u is monotonic and f of q-K-Hölder type were considered by Cheung and Dragomir in [6], while the case where one function was of Hölder type and the other was Lipschitzian were considered in [2]. For other recent results in estimating the error $T(f,u;a,b,x)$ for absolutely continuous integrands f and integrators u of bounded variation, see [4] and [5].

2. Inequalities for Integrators of Bounded variation

Assume that u, $f:[a,b]\to\mathbb{C}$. If the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists, we write for simplicity, like in [1, p. 142] that $f\in\mathcal{R}_{\mathbb{C}}(u,[a,b])$, or $\mathcal{R}_{\mathbb{C}}(u)$ when the interval is implicitly known. If the functions u, f are real valued, then we write $f\in\mathcal{R}(u,[a,b])$, or $\mathcal{R}(u)$.

We start with the following identity of interest.

Lemma 1. *Let f, $u:[a,b]\to\mathbb{C}$ and $x\in[a,b]$ such that $f\in\mathcal{R}_{\mathbb{C}}(u,[a,b])$. Then for any γ, $\mu\in\mathbb{C}$,*

$$\begin{aligned}&[u(b)-\mu]f(b)+[\gamma-u(a)]f(a)+(\mu-\gamma)f(x)-\int_a^b f(t)\,du(t)\\&\qquad=\int_a^x[u(t)-\gamma]\,df(t)+\int_x^b[u(t)-\mu]\,df(t).\end{aligned} \tag{2.1}$$

In particular, for $\mu=\gamma$ we have

$$[u(b)-\gamma]f(b)+[\gamma-u(a)]f(a)-\int_a^b f(t)\,du(t)=\int_a^b[u(t)-\gamma]\,df(t). \tag{2.2}$$

Proof. Using integration by parts rule for the Riemann-Stieltjes integral, we have

$$\int_a^x[u(t)-\gamma]\,df(t)=[u(x)-\gamma]f(x)-[u(a)-\gamma]f(a)-\int_a^x f(t)\,du(t)$$

and

$$\int_x^b[u(t)-\mu]\,df(t)=[u(b)-\mu]f(b)-[u(x)-\mu]f(x)-\int_x^b f(t)\,du(t)$$

for any $x\in[a,b]$.

If we add these two equalities, we get

$$\begin{aligned}\int_a^x [u(t)-\gamma]\, df(t) + \int_x^b [u(t)-\mu]\, df(t) \\ &= [u(b)-\mu] f(b) + [\gamma - u(a)] f(a) + [\mu - u(x)] f(x) \\ &\quad + [u(x)-\gamma] f(x) - \int_a^x f(t)\, du(t) - \int_x^b f(t)\, du(t) \\ &= [u(b)-\mu] f(b) + [\gamma - u(a)] f(a) + (\mu-\gamma) f(x) - \int_a^b f(t)\, du(t)\end{aligned}$$

for any $x \in [a,b]$, which proves the desired equality (2.1). $\square$

From the equality (2.2) we have for $x \in [a,b]$ and $\gamma = u(x)$ that

$$T(f,u;a,b;x) = \int_a^b [u(t)-u(x)]\, df(t) \tag{2.3}$$

and in particular

$$T\left(f,u;a,b;\frac{a+b}{2}\right) = \int_a^b \left[u(t) - u\left(\frac{a+b}{2}\right)\right] df(t). \tag{2.4}$$

Also, if $p \in [a,b]$ is such that $u(p) = \frac{u(a)+u(b)}{2}$, then from (2.3) we get

$$\begin{aligned}T(f,u;a,b;p) = [u(b)-u(a)]\frac{f(b)+f(a)}{2} - \int_a^b f(t)\, du(t) \\ = \int_a^b [u(t)-u(p)]\, df(t).\end{aligned} \tag{2.5}$$

We have:

Theorem 1. *Assume that u, $f \in \mathcal{BV}_{\mathbb{C}}[a,b]$ (of bounded variations) and $f \in \mathcal{C}_{\mathbb{C}}[a,b]$. Then the Riemann-Stieltjes integral $\int_a^b f(t)\, du(t)$ exists and*

$$\begin{aligned}|T(f,u;a,b;x)| &\le \int_a^x \left(\bigvee_t^x (u)\right) d\left(\bigvee_a^t (f)\right) + \int_x^b \left(\bigvee_x^t (u)\right) d\left(\bigvee_x^t (f)\right) \\ &= \int_a^x \left(\bigvee_a^t (f)\right) d\left(\bigvee_a^t (u)\right) + \int_x^b \left(\bigvee_t^b (f)\right) d\left(\bigvee_x^t (u)\right) \\ &\le \bigvee_a^x (u) \bigvee_a^x (f) + \bigvee_x^b (u) \bigvee_x^b (f) \\ &\le \frac{1}{2} \times \begin{cases} \left[\bigvee_a^b (f) + \left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right] \bigvee_a^b (u), \\ \left[\bigvee_a^b (u) + \left|\bigvee_a^x (u) - \bigvee_x^b (u)\right|\right] \bigvee_a^b (f) \end{cases}\end{aligned} \tag{2.6}$$

for all $x \in [a,b]$.

Proof. It is well known that, if $p:[a,b]\to\mathbb{C}$ is continuous and $v:[a,b]\to\mathbb{C}$ of bounded variation, then

$$\left|\int_a^b p(t)\,dv(t)\right|\le\int_a^b|p(t)|\,d\left(\bigvee_a^t(u)\right)\le\max_{t\in[a,b]}|p(t)|\bigvee_a^b(u). \tag{2.7}$$

By making use of the equality (2.3) we have

$$\begin{aligned}|T(f,u;a,b;x)|&=\left|\int_a^x[u(t)-u(x)]\,df(t)+\int_x^b[u(t)-u(x)]\,df(t)\right|\\&\le\left|\int_a^x[u(t)-u(x)]\,df(t)\right|+\left|\int_x^b[u(t)-u(x)]\,df(t)\right|\\&\le\int_a^x|u(t)-u(x)|\,d\left(\bigvee_a^t(f)\right)+\int_x^b|u(t)-u(x)|\,d\left(\bigvee_x^t(f)\right)=:B(f,u,x).\end{aligned} \tag{2.8}$$

Since u is of bounded variation, we have

$$|u(t)-u(x)|\le\bigvee_t^x(u)\ \text{ for } t\in[a,x]$$

and

$$|u(t)-u(x)|\le\bigvee_x^t(u)\ \text{ for } t\in[x,b],$$

hence

$$\begin{aligned}B(f,u,x)&\le\int_a^x|u(t)-u(x)|\,d\left(\bigvee_a^t(f)\right)+\int_x^b|u(t)-u(x)|\,d\left(\bigvee_x^t(f)\right)\\&\le\int_a^x\left(\bigvee_t^x(u)\right)d\left(\bigvee_a^t(f)\right)+\int_x^b\left(\bigvee_x^t(u)\right)d\left(\bigvee_x^t(f)\right)\\&=\int_a^x\left(\bigvee_a^x(u)-\bigvee_a^t(u)\right)d\left(\bigvee_a^t(f)\right)+\int_x^b\left(\bigvee_x^t(u)\right)d\left(\bigvee_a^t(f)-\bigvee_a^x(u)\right)\\&=\int_a^x\left(\bigvee_a^x(u)-\bigvee_a^t(u)\right)d\left(\bigvee_a^t(f)\right)+\int_x^b\left(\bigvee_x^t(u)\right)d\left(\bigvee_a^t(f)\right)=:C(f,u,x).\end{aligned} \tag{2.9}$$

Using integration by parts, we have

$$\begin{aligned}&\int_a^x\left(\bigvee_a^x(u)-\bigvee_a^t(u)\right)d\left(\bigvee_a^t(f)\right)\\&=\left(\bigvee_a^x(u)-\bigvee_a^t(u)\right)\left(\bigvee_a^t(f)\right)\Bigg|_a^x+\int_a^x\bigvee_a^t(f)\,d\left(\bigvee_a^t(u)\right)\\&=\int_a^x\left(\bigvee_a^t(f)\right)d\left(\bigvee_a^t(u)\right)\end{aligned}$$

and

$$
\begin{aligned}
&\int_x^b \left(\bigvee_x^t (u)\right) d\left(\bigvee_a^t (f)\right) \\
&= \left(\bigvee_x^t (u)\right)\left(\bigvee_a^t (f)\right)\Bigg|_x^b - \int_x^b \left(\bigvee_a^t (f)\right) d\left(\bigvee_x^t (u)\right) \\
&= \left(\bigvee_x^b (u)\right)\left(\bigvee_a^b (f)\right) - \int_x^b \left(\bigvee_a^t (f)\right) d\left(\bigvee_x^t (u)\right) \\
&= \int_x^b \left(\bigvee_a^b (f) - \bigvee_a^t (f)\right) d\left(\bigvee_x^t (u)\right) = \int_x^b \left(\bigvee_t^b (f)\right) d\left(\bigvee_x^t (u)\right)
\end{aligned}
$$

that gives

$$
C(f,u,x) = \int_a^x \left(\bigvee_a^t (f)\right) d\left(\bigvee_a^t (u)\right) + \int_x^b \left(\bigvee_t^b (f)\right) d\left(\bigvee_x^t (u)\right).
$$

These prove the first inequality in (2.6) and the equality after that.

Using the properties of the total variation, we also have

$$
\begin{aligned}
&\int_a^x \left(\bigvee_t^x (u)\right) d\left(\bigvee_a^t (f)\right) + \int_x^b \left(\bigvee_x^t (u)\right) d\left(\bigvee_x^t (f)\right) \\
&\le \left(\bigvee_a^x (u)\right) \int_a^x d\left(\bigvee_a^t (f)\right) + \left(\bigvee_x^b (u)\right) \int_x^b d\left(\bigvee_x^t (f)\right) \\
&= \bigvee_a^x (u) \bigvee_a^x (f) + \bigvee_x^b (u) \bigvee_x^b (f),
\end{aligned}
$$

which proves the second inequality in (2.6).

The last part is obvious by the properties of maximum of two positive numbers.

□

Corollary 1. *With the assumptions of Theorem 1,*

(i) *If $q \in [a,b]$ is such that $\bigvee_a^q (f) = \bigvee_q^b (f)$, then*

$$
|T(f,u;a,b;q)| \le \frac{1}{2} \bigvee_a^b (u) \bigvee_a^b (f). \tag{2.10}
$$

(ii) *If $m \in [a,b]$ is such that $\bigvee_a^m (u) = \bigvee_m^b (u)$, then*

$$
|T(f,u;a,b;m)| \le \frac{1}{2} \bigvee_a^b (u) \bigvee_a^b (f). \tag{2.11}
$$

The case of monotonic integrands is as follows:

Corollary 2. *Assume that $u \in \mathcal{BV}_{\mathbb{C}}[a,b]$, $f \in \mathcal{M}^{\nearrow}[a,b]$ (monotonic nondecreasing) and $f \in \mathcal{C}_{\mathbb{C}}[a,b]$. Then the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and*

$$
\begin{aligned}
(2.12)\quad |T(f,u;a,b;x)| &\leq \int_a^x \left(\bigvee_t^x (u)\right) df(t) + \int_x^b \left(\bigvee_x^t (u)\right) df(t) \\
&= \int_a^x [f(t)-f(a)]\, d\left(\bigvee_a^t (u)\right) + \int_x^b [f(b)-f(t)]\, d\left(\bigvee_x^t (u)\right) \\
&\leq \bigvee_a^x (u)\,[f(x)-f(a)] + \bigvee_x^b (u)\,[f(b)-f(x)] \\
&\leq \begin{cases} \left[\frac{f(b)-f(a)}{2} + \left|f(x) - \frac{f(a)+f(b)}{2}\right|\right] \bigvee_a^b (u), \\[2ex] \frac{1}{2}\left[\bigvee_a^b (u) + \left|\bigvee_a^x (u) - \bigvee_x^b (u)\right|\right][f(b)-f(a)] \end{cases}
\end{aligned}
$$

for all $x \in [a,b]$.

Remark 1. *Under the assumptions of Corollary 2 and if $p \in [a,b]$ with $f(p) = \frac{f(a)+f(b)}{2}$, we have*

$$
(2.13)\qquad |T(f,u;a,b;p)| \leq \frac{1}{2}[f(b)-f(a)] \bigvee_a^b (u).
$$

Also, if $m \in [a,b]$ such that $\bigvee_a^m (u) = \bigvee_m^b (u)$, then

$$
(2.14)\qquad |T(f,u;a,b;m)| \leq \frac{1}{2}[f(b)-f(a)] \bigvee_a^b (u).
$$

We have:

Theorem 2. *Assume that $u \in \mathcal{BV}_{\mathbb{C}}[a,b]$ and f is Lipschitzian with the constant $L > 0$, namely*

$$
|f(t)-f(s)| \leq L\,|t-s| \text{ for all } t,\ s \in [a,b],
$$

then the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ *exists and*

$$
\begin{aligned}
(2.15)\quad |T(f,u;a,b;x)| &\le L\left[\int_a^x \left(\bigvee_t^x (u)\right) dt + \int_x^b \left(\bigvee_x^t (u)\right) dt\right] \\
&= L\left[\int_a^x (t-a)\, d\left(\bigvee_a^t (u)\right) + \int_x^b (b-t)\, d\left(\bigvee_x^t (u)\right)\right] \\
&\le L\left[(x-a)\bigvee_a^x (u) + (b-x)\bigvee_x^b (u)\right] \\
&\le L\times\begin{cases} \left[\frac{b-a}{2} + \left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b (u), \\ \\ \frac{1}{2}\left[\bigvee_a^b (u) + \left|\bigvee_a^x (u) - \bigvee_x^b (u)\right|\right](b-a) \end{cases}
\end{aligned}
$$

for $x\in[a,b]$.

Proof. It is well known that, if $p:[a,b]\to\mathbb{C}$ is Riemann integrable and $v:[a,b]\to\mathbb{C}$ Lipschitzian with the constant $L>0$, then

$$
\left|\int_a^b p(t)\,dv(t)\right| \le L\int_a^b |p(t)|\,dt \tag{2.16}
$$

By making use of the equality (2.3) we have

$$
\begin{aligned}
|T(f,u;a,b;x)| &= \left|\int_a^x [u(t)-u(x)]\,df(t) + \int_x^b [u(t)-u(x)]\,df(t)\right| \\
&\le \left|\int_a^x [u(t)-u(x)]\,df(t)\right| + \left|\int_x^b [u(t)-u(x)]\,df(t)\right| \\
&\le L\left[\int_a^x |u(t)-u(x)|\,dt + \int_x^b |u(t)-u(x)|\,dt\right] =: D(f,u,x)
\end{aligned}
$$

for $x\in[a,b]$.

Since u is of bounded variation, hence

$$
D(f,u,x) \le L\left[\int_a^x \left(\bigvee_t^x (u)\right) dt + \int_x^b \left(\bigvee_x^t (u)\right) dt\right]
$$

for $x\in[a,b]$, which proves the first inequality in (2.15).

Using the integration by parts, we have

$$\begin{aligned}
&\int_a^x \left(\bigvee_t^x (u)\right) dt + \int_x^b \left(\bigvee_x^t (u)\right) dt \\
&\quad = \left(\bigvee_t^x (u)\right) t \Big|_a^x - \int_a^x t d\left(\bigvee_t^x (u)\right) + \left(\bigvee_x^t (u)\right) t \Big|_x^b - \int_x^b t d\left(\bigvee_x^t (u)\right) \\
&\quad = -\left(\bigvee_a^x (u)\right) a - \int_a^x t d\left(\bigvee_a^x (u) - \bigvee_a^t (u)\right) + \left(\bigvee_x^b (u)\right) b - \int_x^b t d\left(\bigvee_x^t (u)\right) \\
&\quad = -\left(\bigvee_a^x (u)\right) a + \int_a^x t d\left(\bigvee_a^t (u)\right) + \int_x^b (b-t)\, d\left(\bigvee_x^t (u)\right) \\
&\quad = \int_a^x (t-a)\, d\left(\bigvee_a^t (u)\right) + \int_x^b (b-t)\, d\left(\bigvee_x^t (u)\right),
\end{aligned}$$

which proves the equality in (2.15).

The rest is obvious. □

Corollary 3. *With the assumptions of Theorem 2, we have*

$$\left|T\left(f, u; a, b; \frac{a+b}{2}\right)\right| \le \frac{1}{2} L (b-a) \bigvee_a^b (u). \tag{2.17}$$

If $m \in [a, b]$ such that $\bigvee_a^m (u) = \bigvee_m^b (u)$, then

$$|T(f, u; a, b; m)| \le \frac{1}{2} L (b-a) \bigvee_a^b (u). \tag{2.18}$$

3. Applications for Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. Let $A \in \mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda \in \mathbb{R}$ as follows

$$\varphi_\lambda (s) := \begin{cases} 1, & \text{for } -\infty < s \le \lambda, \\ \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$E_\lambda := \varphi_\lambda (A) \tag{3.1}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [13, p. 256]:

Theorem 3 (Spectral Representation Theorem)**.** *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a = \min\{\lambda \,|\, \lambda \in Sp(A)\} =: \min Sp(A)$ and $b = \max\{\lambda \,|\, \lambda \in Sp(A)\} =: \max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda \in \mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_\lambda \le E_{\lambda'}$ *for* $\lambda \le \lambda'$;
b) $E_{a-0} = 0, E_b = 1_H$ *and* $E_{\lambda+0} = E_\lambda$ *for all* $\lambda \in \mathbb{R}$;
c) *We have the representation*

$$A = \int_{a-0}^{b} \lambda dE_\lambda.$$

More generally, for every continuous complex-valued function φ *defined on* $\mathbb{R}$ *there exists a unique operator* $\varphi(A) \in \mathcal{B}(H)$ *such that for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *satisfying the inequality*

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi(\lambda_k') \left[E_{\lambda_k} - E_{\lambda_{k-1}} \right] \right\| \le \varepsilon$$

whenever

$$\begin{cases} \lambda_0 < a = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = b, \\ \\ \lambda_k - \lambda_{k-1} \le \delta \text{ for } 1 \le k \le n, \\ \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \le k \le n \end{cases}$$

this means that

$$\varphi(A) = \int_{a-0}^{b} \varphi(\lambda) dE_\lambda, \tag{3.2}$$

where the integral is of Riemann-Stieltjes type.

Corollary 4. *With the assumptions of Theorem 3 for* A, E_λ *and* φ *we have the representations*

$$\varphi(A)x = \int_{a-0}^{b} \varphi(\lambda) dE_\lambda x \quad \text{for all } x \in H$$

and

$$\langle \varphi(A)x, y \rangle = \int_{a-0}^{b} \varphi(\lambda) d\langle E_\lambda x, y \rangle \quad \text{for all } x,\ y \in H. \tag{3.3}$$

In particular,

$$\langle \varphi(A)x, x \rangle = \int_{a-0}^{b} \varphi(\lambda) d\langle E_\lambda x, x \rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(A)x\|^2 = \int_{a-0}^{b} |\varphi(\lambda)|^2 d\|E_\lambda x\|^2 \quad \text{for all } x \in H.$$

We need the following result that provides an upper bound for the total variation of the function $\mathbb{R} \ni \lambda \mapsto \langle E_\lambda x, y \rangle \in \mathbb{C}$ on an interval $[\alpha, \beta]$, see [9].

Lemma 2. *Let* $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ *be the spectral family of the bounded selfadjoint operator* A. *Then for any* $x,\ y \in H$ *and* $\alpha < \beta$ *we have the inequality*

$$\left[\bigvee_{\alpha}^{\beta} \left(\langle E_{(\cdot)}x, y \rangle \right) \right]^2 \le \langle (E_\beta - E_\alpha)x, x \rangle \langle (E_\beta - E_\alpha)y, y \rangle, \tag{3.4}$$

where $\bigvee_{\alpha}^{\beta} \left(\langle E_{(\cdot)}x, y \rangle \right)$ *denotes the total variation of the function* $\langle E_{(\cdot)}x, y \rangle$ *on* $[\alpha, \beta]$.

Remark 2. *For $\alpha = a - \varepsilon$ with $\varepsilon > 0$ and $\beta = b$ we get from (3.4) the inequality*

$$\bigvee_{a-\varepsilon}^{b} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \left\langle \left(1_H - E_{a-\varepsilon}\right) x, x\right\rangle^{1/2} \left\langle \left(1_H - E_{a-\varepsilon}\right) y, y\right\rangle^{1/2} \tag{3.5}$$

for any x, $y \in H$.

This implies, for any x, $y \in H$, that

$$\bigvee_{a-0}^{b} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \|x\| \|y\|, \tag{3.6}$$

where $\bigvee_{a-0}^{b} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$ *denotes the limit* $\lim_{\varepsilon \to 0+} \left[\bigvee_{a-\varepsilon}^{b} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right]$.

We can state the following result for functions of selfadjoint operators:

Theorem 4. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a = \min\{\lambda | \lambda \in Sp(A)\} =: \min Sp(A)$ and $b = \max\{\lambda | \lambda \in Sp(A)\} =: \max Sp(A)$. Also, assume that $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A and assume that $\varphi \in \mathcal{BV}_{\mathbb{C}}[a, b]$ and $\varphi \in \mathcal{C}_{\mathbb{C}}[a, b]$ where $[a, b] \subset \mathring{I}$ (the interior of I). Then for all $s \in [a, b]$*

$$\begin{aligned} &\left|\left\langle \left(1_H - E_s\right) x, y\right\rangle \varphi(b) + \left\langle E_s x, y\right\rangle \varphi(a) - \left\langle \varphi(A) x, y\right\rangle\right| \\ &\qquad \leq \frac{1}{2}\left[\bigvee_a^b (\varphi) + \left|\bigvee_a^s (\varphi) - \bigvee_s^b (\varphi)\right|\right] \bigvee_{a-0}^{b} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \\ &\qquad\qquad \leq \frac{1}{2}\left[\bigvee_a^b (\varphi) + \left|\bigvee_a^s (\varphi) - \bigvee_s^b (\varphi)\right|\right] \|x\| \|y\| \end{aligned} \tag{3.7}$$

for any x, $y \in H$.

Proof. Using the inequality (2.6) we have

$$\begin{aligned} &\Bigg|\left[\left\langle E_b x, y\right\rangle - \left\langle E_s x, y\right\rangle\right] \varphi(b) + \left[\left\langle E_s x, y\right\rangle - \left\langle E_{a-\varepsilon} x, y\right\rangle\right] \varphi(a - \varepsilon) \\ &\qquad - \int_{a-\varepsilon}^{b} \varphi(t)\, d\left\langle E_t x, y\right\rangle\Bigg| \\ &\qquad \leq \frac{1}{2}\left[\bigvee_{a-\varepsilon}^b (\varphi) + \left|\bigvee_{a-\varepsilon}^s (\varphi) - \bigvee_s^b (\varphi)\right|\right] \bigvee_{a-\varepsilon}^{b} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right), \end{aligned}$$

for small $\varepsilon > 0$ and for any x, $y \in H$.

Taking the limit over $\varepsilon \to 0+$ and using the continuity of φ and the Spectral Representation Theorem, we deduce the desired result (3.7). $\square$

Corollary 5. *With the assumptions of Theorem 4 and if* $q \in [a,b]$ *is such that* $\bigvee_a^q (\varphi) = \bigvee_q^b (\varphi)$, *then*

$$
(3.8) \quad \left| \langle (1_H - E_q)\, x, y \rangle \varphi(b) + \langle E_q x, y \rangle \varphi(a) - \langle \varphi(A)\, x, y \rangle \right| \\
\le \frac{1}{2} \bigvee_a^b (\varphi) \bigvee_{a-0}^b \left(\langle E_{(\cdot)} x, y \rangle \right) \le \frac{1}{2} \|x\| \|y\| \bigvee_a^b (\varphi)
$$

for any $x,\ y \in H$.

We also have:

Theorem 5. *Let* A *be a bounded selfadjoint operator on the Hilbert space* H *and let* $a = \min\{\lambda \,|\lambda \in Sp(A)\} =: \min Sp(A)$ *and* $b = \max\{\lambda \,|\lambda \in Sp(A)\} =: \max Sp(A)$. *Also, assume that* $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ *is the spectral family of the bounded selfadjoint operator* A *and assume that* φ *is Lipschitzian with the constant* $L > 0$ *on* $[a,b] \subset \mathring{I}$. *Then for all* $s \in [a,b]$

$$
(3.9) \quad \left| \langle (1_H - E_s)\, x, y \rangle \varphi(b) + \langle E_s x, y \rangle \varphi(a) - \langle \varphi(A)\, x, y \rangle \right| \\
\le L \left[\frac{1}{2}(b-a) + \left| s - \frac{a+b}{2} \right| \right] \bigvee_{a-0}^b \left(\langle E_{(\cdot)} x, y \rangle \right) \\
\le L \left[\frac{1}{2}(b-a) + \left| s - \frac{a+b}{2} \right| \right] \|x\| \|y\|
$$

for any $x,\ y \in H$.

In particular, we have

$$
(3.10) \quad \left| \left\langle \left(1_H - E_{\frac{a+b}{2}} \right) x, y \right\rangle \varphi(b) + \left\langle E_{\frac{a+b}{2}} x, y \right\rangle \varphi(a) - \langle \varphi(A)\, x, y \rangle \right| \\
\le \frac{1}{2} L (b-a) \bigvee_{a-0}^b \left(\langle E_{(\cdot)} x, y \rangle \right) \le \frac{1}{2} L (b-a) \|x\| \|y\|
$$

for any $x,\ y \in H$.

Remark 3. *The above results can provide particular inequalities of interest. For instance, if we take* $\varphi : [a,b] \subset (0,\infty) \to \mathbb{R}$, $\varphi(t) = \ln t$ *and* A *is a bounded selfadjoint operator on the Hilbert space* H *with* $a = \min\{\lambda \,|\lambda \in Sp(A)\}$ *and* $b = \max\{\lambda \,|\lambda \in Sp(A)\}$, *then by (3.7) we get*

$$
(3.11) \quad \left| \langle (1_H - E_s)\, x, y \rangle \ln b + \langle E_s x, y \rangle \ln a - \langle \ln A x, y \rangle \right| \\
\le \frac{1}{2} \left[\ln \left(\frac{b}{a} \right) + \left| \ln \left(\frac{s^2}{ab} \right) \right| \right] \bigvee_{a-0}^b \left(\langle E_{(\cdot)} x, y \rangle \right) \\
\le \frac{1}{2} \left[\ln \left(\frac{b}{a} \right) + \left| \ln \left(\frac{s^2}{ab} \right) \right| \right] \|x\| \|y\|
$$

for any $x,\ y \in H$.

In particular, if we take $s = G(a,b) := \sqrt{ab}$, *the geometric mean of* a *and* b, *then we get from (3.11) that*

$$\left|\left\langle\left(1_H - E_{G(a,b)}\right)x, y\right\rangle \ln b + \left\langle E_{G(a,b)}x, y\right\rangle \ln a - \langle \ln Ax, y\rangle\right| \le \frac{1}{2}\ln\left(\frac{b}{a}\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \le \frac{1}{2}\ln\left(\frac{b}{a}\right)\|x\|\,\|y\| \tag{3.12}$$

for any $x,\ y \in H$.

The function $\varphi : [a,b] \subset (0,\infty) \to \mathbb{R}$, $\varphi(t) = \ln t$ *is Lipschitzian on* $[a,b]$ *with constant* $L = \frac{1}{a} > 0$. *Then by (3.9) we get*

$$\begin{aligned}|\langle(1_H - E_s)x, y\rangle \ln b + \langle E_s x, y\rangle \ln a - \langle \ln Ax, y\rangle| &\le \frac{1}{a}\left[\frac{1}{2}(b-a) + \left|s - \frac{a+b}{2}\right|\right]\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \\ &\le \frac{1}{a}\left[\frac{1}{2}(b-a) + \left|s - \frac{a+b}{2}\right|\right]\|x\|\,\|y\|\end{aligned} \tag{3.13}$$

for any $x,\ y \in H$.

In particular, if we take $s = \frac{a+b}{2}$, *then we get from (3.13) that*

$$\left|\left\langle\left(1_H - E_{\frac{a+b}{2}}\right)x, y\right\rangle \ln b + \left\langle E_{\frac{a+b}{2}}x, y\right\rangle \ln a - \langle \ln Ax, y\rangle\right| \le \frac{1}{2}\left(\frac{b}{a} - 1\right)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \le \frac{1}{2}\left(\frac{b}{a} - 1\right)\|x\|\,\|y\| \tag{3.14}$$

for any $x,\ y \in H$.

4. Applications for Unitary Operators

A unitary operator is a bounded linear operator $U : H \to H$ on a Hilbert space H satisfying

$$U^*U = UU^* = 1_H$$

where U^* is the adjoint of U, and $1_H : H \to H$ is the identity operator. This property is equivalent to the following:

(i) U preserves the inner product $\langle\cdot,\cdot\rangle$ of the Hilbert space, i.e., for all vectors x and y in the Hilbert space, $\langle Ux, Uy\rangle = \langle x, y\rangle$ and
(ii) U is surjective.

The following result is well known [13, p. 275 - p. 276]:

Theorem 6 (Spectral Representation Theorem)**.** *Let* U *be a unitary operator on the Hilbert space* H. *Then there exists a family of projections* $\{P_\lambda\}_{\lambda\in[0,2\pi]}$, *called the spectral family of* U, *with the following properties*

a) $P_\lambda \le P_{\lambda'}$ *for* $\lambda \le \lambda'$;
b) $P_0 = 0, P_{2\pi} = 1_H$ *and* $P_{\lambda+0} = P_\lambda$ *for all* $\lambda \in [0,2\pi)$;
c) *We have the representation*

$$U = \int_0^{2\pi} \exp(i\lambda)\, dP_\lambda.$$

More generally, for every continuous complex-valued function φ *defined on the unit circle* $\mathcal{C}(0,1)$ *there exists a unique operator* $\varphi(U) \in \mathcal{B}(H)$ *such that for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *satisfying the inequality*

$$\left\| \varphi(U) - \sum_{k=1}^{n} \varphi\left(\exp\left(i\lambda_k'\right)\right)\left[P_{\lambda_k} - P_{\lambda_{k-1}}\right] \right\| \leq \varepsilon$$

whenever

$$\begin{cases} 0 = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = 2\pi, \\ \lambda_k - \lambda_{k-1} \leq \delta \text{ for } 1 \leq k \leq n, \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \leq k \leq n \end{cases}$$

this means that

$$\varphi(U) = \int_0^{2\pi} \varphi(\exp(i\lambda))\, dP_\lambda, \tag{4.1}$$

where the integral is of Riemann-Stieltjes type.

Corollary 6. *With the assumptions of Theorem 6 for* U, P_λ *and* φ *we have the representations*

$$\varphi(U)x = \int_0^{2\pi} \varphi(\exp(i\lambda))\, dP_\lambda x \quad \text{for all } x \in H$$

and

$$\langle \varphi(U)x, y\rangle = \int_0^{2\pi} \varphi(\exp(i\lambda))\, d\langle P_\lambda x, y\rangle \quad \text{for all } x, y \in H. \tag{4.2}$$

In particular,

$$\langle \varphi(U)x, x\rangle = \int_0^{2\pi} \varphi(\exp(i\lambda))\, d\langle P_\lambda x, x\rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(U)x\|^2 = \int_0^{2\pi} |\varphi(\exp(i\lambda))|^2\, d\|P_\lambda x\|^2 \quad \text{for all } x \in H.$$

On making use of an argument similar to the one in [9, Theorem 6], we have:

Lemma 3. *Let* $\{P_\lambda\}_{\lambda\in[0,2\pi]}$ *be the spectral family of the unitary operator* U *on the Hilbert space* H. *Then for any* x, $y \in H$ *and* $0 \leq \alpha < \beta \leq 2\pi$ *we have the inequality*

$$\bigvee_{\alpha}^{\beta}\left(\left\langle P_{(\cdot)}x, y\right\rangle\right) \leq \langle (P_\beta - P_\alpha)x, x\rangle^{1/2} \langle (P_\beta - P_\alpha)y, y\rangle^{1/2}, \tag{4.3}$$

where $\bigvee_{\alpha}^{\beta}\left(\left\langle P_{(\cdot)}x, y\right\rangle\right)$ *denotes the total variation of the function* $\left\langle P_{(\cdot)}x, y\right\rangle$ *on* $[\alpha, \beta]$.

In particular,

$$\bigvee_{0}^{2\pi}\left(\left\langle P_{(\cdot)}x, y\right\rangle\right) \leq \|x\|\,\|y\| \tag{4.4}$$

for any x, $y \in H$.

We have:

Theorem 7. *Let U be a unitary operator on the Hilbert space H and $\{P_\lambda\}_{\lambda\in[0,2\pi]}$ the spectral family of projections of U. Also, assume that $\varphi:\mathcal{C}(0,1)\to\mathbb{C}$ are continuous on $\mathcal{C}(0,1)$. If $\varphi\circ\exp(i\cdot)\in\mathcal{BV}_{\mathbb{C}}[0,2\pi]$, then for all $s\in[0,2\pi]$*

$$
\begin{aligned}
(4.5)\quad &|\varphi(1)\langle x,y\rangle-\langle\varphi(U)x,y\rangle| \\
&\le\frac{1}{2}\left[\bigvee_0^{2\pi}(\varphi\circ\exp(i\cdot))+\inf_{s\in[0,2\pi]}\left|\bigvee_0^{s}(\varphi\circ\exp(i\cdot))-\bigvee_s^{2\pi}(\varphi\circ\exp(i\cdot))\right|\right]\bigvee_0^{2\pi}\left(\langle P_{(\cdot)}x,y\rangle\right) \\
&\le\frac{1}{2}\left[\bigvee_0^{2\pi}(\varphi\circ\exp(i\cdot))+\inf_{s\in[0,2\pi]}\left|\bigvee_0^{s}(\varphi\circ\exp(i\cdot))-\bigvee_s^{2\pi}(\varphi\circ\exp(i\cdot))\right|\right]\|x\|\,\|y\|
\end{aligned}
$$

for any $x,\ y\in H$.

If there exists an $s\in[0,2\pi]$ such that

$$\bigvee_0^{s}(\varphi\circ\exp(i\cdot))=\bigvee_s^{2\pi}(\varphi\circ\exp(i\cdot)),$$

then

$$
\begin{aligned}
(4.6)\quad &|\varphi(1)\langle x,y\rangle-\langle\varphi(U)x,y\rangle| \\
&\le\frac{1}{2}\bigvee_0^{2\pi}(\varphi\circ\exp(i\cdot))\bigvee_0^{2\pi}\left(\langle P_{(\cdot)}x,y\rangle\right)\le\frac{1}{2}\bigvee_0^{2\pi}(\varphi\circ\exp(i\cdot))\|x\|\,\|y\|
\end{aligned}
$$

for any $x,\ y\in H$.

If $\varphi\circ\exp(i\cdot)$ is Lipschitzian with the constant $L>0$ on $[0,2\pi]$, then

$$(4.7)\qquad |\varphi(1)\langle x,y\rangle-\langle\varphi(U)x,y\rangle|\le\pi L\bigvee_0^{2\pi}\left(\langle P_{(\cdot)}x,y\rangle\right)\le\pi L(b-a)\|x\|\,\|y\|$$

for any $x,\ y\in H$.

Proof. From the inequality (3.7) we get

$$
\begin{aligned}
(4.8)\quad &\left|\langle(1_H-P_s)x,y\rangle\varphi\left(e^{2\pi i}\right)+\langle P_sx,y\rangle\varphi\left(e^0\right)-\langle\varphi(U)x,y\rangle\right| \\
&\le\frac{1}{2}\left[\bigvee_0^{2\pi}(\varphi\circ\exp(i\cdot))+\left|\bigvee_0^{s}(\varphi\circ\exp(i\cdot))-\bigvee_s^{2\pi}(\varphi\circ\exp(i\cdot))\right|\right]\bigvee_0^{2\pi}\left(\langle P_{(\cdot)}x,y\rangle\right) \\
&\le\frac{1}{2}\left[\bigvee_0^{2\pi}(\varphi\circ\exp(i\cdot))+\left|\bigvee_0^{s}(\varphi\circ\exp(i\cdot))-\bigvee_s^{2\pi}(\varphi\circ\exp(i\cdot))\right|\right]\|x\|\,\|y\|
\end{aligned}
$$

for any $x,\ y\in H$ and since $\varphi\left(e^{2\pi i}\right)=\varphi\left(e^0\right)=\varphi(1)$, hence by (4.8) we get

$$
\begin{aligned}
&|\langle x,y\rangle\varphi(1)-\langle\varphi(U)x,y\rangle| \\
&\le\frac{1}{2}\left[\bigvee_0^{2\pi}(\varphi\circ\exp(i\cdot))+\left|\bigvee_0^{s}(\varphi\circ\exp(i\cdot))-\bigvee_s^{2\pi}(\varphi\circ\exp(i\cdot))\right|\right]\bigvee_0^{2\pi}\left(\langle P_{(\cdot)}x,y\rangle\right) \\
&\le\frac{1}{2}\left[\bigvee_0^{2\pi}(\varphi\circ\exp(i\cdot))+\left|\bigvee_0^{s}(\varphi\circ\exp(i\cdot))-\bigvee_s^{2\pi}(\varphi\circ\exp(i\cdot))\right|\right]\|x\|\,\|y\|
\end{aligned}
$$

and by taking the infimum over $s\in[0,2\pi]$ we get (4.5).

The inequality (4.7) follows in a similar way from (3.9). $\square$

Remark 4. *If φ is differentiable, then*

$$(\varphi \circ \exp(it))' = \varphi'(\exp(it))(\exp(it))' = \varphi'(\exp(it))(\exp(it))\, i$$

and if the derivative is continuous, then

$$\begin{aligned}\bigvee_0^{2\pi}(\varphi \circ \exp(i\cdot)) &= \int_0^{2\pi}\left|(\varphi \circ \exp(it))'\right| dt = \int_0^{2\pi}|\varphi'(\exp(it))|\,|(\exp(it))\, i|\, dt \\ &= \int_0^{2\pi}|\varphi'(\exp(it))|\, dt.\end{aligned}$$

Similarly,

$$\bigvee_0^{s}(\varphi \circ \exp(i\cdot)) = \int_0^{s}|\varphi'(\exp(it))|\, dt \text{ and } \bigvee_s^{2\pi}(\varphi \circ \exp(i\cdot)) = \int_s^{2\pi}|\varphi'(\exp(it))|\, dt.$$

Since the function $\int_0^{\cdot}|\varphi'(\exp(it))|\, dt$ is continuous on $[0, 2\pi]$, hence there is an $s \in [0, 2\pi]$ such that

$$\bigvee_0^{s}(\varphi \circ \exp(i\cdot)) = \bigvee_s^{2\pi}(\varphi \circ \exp(i\cdot)),$$

and by (4.6) we get

$$\begin{aligned}(4.9)\quad & |\varphi(1)\langle x, y\rangle - \langle\varphi(U)x, y\rangle| \\ & \le \frac{1}{2}\int_0^{2\pi}|\varphi'(\exp(it))|\, dt \bigvee_0^{2\pi}\left(\left\langle P_{(\cdot)}x, y\right\rangle\right) \le \frac{1}{2}\|x\|\,\|y\|\int_0^{2\pi}|\varphi'(\exp(it))|\, dt,\end{aligned}$$

where U be a unitary operator on the Hilbert space H, $\{P_\lambda\}_{\lambda\in[0,2\pi]}$ is the spectral family of projections of U and x, $y \in H$.

We also have

$$\sup_{t\in[0,2\pi]}\left|(\varphi \circ \exp(it))'\right| = \sup_{t\in[0,2\pi]}|\varphi'(\exp(it))(\exp(it))\, i| = \sup_{z\in\mathcal{C}(0,1)}|\varphi'(z)|.$$

So if we assume that $L := \sup_{z\in\mathcal{C}(0,1)}|\varphi'(z)| < \infty$, then $\varphi \circ \exp(it)$ is Lipschitzian with the constant L. Then by (4.7) we get

$$\begin{aligned}(4.10)\quad |\varphi(1)\langle x, y\rangle - \langle\varphi(U)x, y\rangle| &\le \pi \sup_{z\in\mathcal{C}(0,1)}|\varphi'(z)| \bigvee_0^{2\pi}\left(\left\langle P_{(\cdot)}x, y\right\rangle\right) \\ &\le \pi \sup_{z\in\mathcal{C}(0,1)}|\varphi'(z)|\,\|x\|\,\|y\|,\end{aligned}$$

where U be a unitary operator on the Hilbert space H, $\{P_\lambda\}_{\lambda\in[0,2\pi]}$ is the spectral family of projections of U and x, $y \in H$.

If we take , for instance, $\varphi(z) = z^n$ with $n \in \mathbb{N}$, then by both (4.9) and (4.10) we get

$$|\langle x, y\rangle - \langle U^n x, y\rangle| \le n\pi dt \bigvee_0^{2\pi}\left(\left\langle P_{(\cdot)}x, y\right\rangle\right) \le n\pi\|x\|\,\|y\|,$$

where U be a unitary operator on the Hilbert space H, $\{P_\lambda\}_{\lambda\in[0,2\pi]}$ is the spectral family of projections of U and x, $y \in H$.

We can give a more interesting example as follows:

Example 1. *For $a \neq \pm 1, 0$ consider the function $\varphi : \mathcal{C}(0,1) \to \mathbb{C}$, $\varphi_a(z) = \frac{1}{1-az}$. Observe that*

$$|\varphi_a(z) - \varphi_a(w)| = \frac{|a||z-w|}{|1-az||1-aw|} \tag{4.11}$$

for any $z, w \in \mathcal{C}(0,1)$.

If $z = e^{it}$ with $t \in [0, 2\pi]$, then we have

$$\begin{aligned} |1-az|^2 &= 1 - 2a \operatorname{Re}(\bar{z}) + a^2 |z|^2 = 1 - 2a\cos t + a^2 \\ &\geq 1 - 2|a| + a^2 = (1-|a|)^2 \end{aligned}$$

therefore

$$\frac{1}{|1-az|} \leq \frac{1}{|1-|a||} \text{ and } \frac{1}{|1-aw|} \leq \frac{1}{|1-|a||} \tag{4.12}$$

for any $z, w \in \mathcal{C}(0,1)$.

Utilising (4.11) and (4.12) we deduce

$$|\varphi_a(z) - \varphi_a(w)| \leq \frac{|a|}{(1-|a|)^2} |z-w| \tag{4.13}$$

for any $z, w \in \mathcal{C}(0,1)$, showing that the function φ_a is Lipschitzian with the constant $L_a = \frac{|a|}{(1-|a|)^2}$ on the circle $\mathcal{C}(0,1)$.

If we take $z = e^{it}$ and $w = e^{is}$ with $t, s \in [0, 2\pi]$ in (4.13) we get

$$\left|\varphi_a\left(e^{it}\right) - \varphi_a\left(e^{is}\right)\right| \leq \frac{|a|}{(1-|a|)^2} \left|e^{it} - e^{is}\right| \tag{4.14}$$

Since

$$\begin{aligned} \left|e^{is} - e^{it}\right|^2 &= \left|e^{is}\right|^2 - 2\operatorname{Re}\left(e^{i(s-t)}\right) + \left|e^{it}\right|^2 \\ &= 2 - 2\cos(s-t) = 4\sin^2\left(\frac{s-t}{2}\right) \end{aligned}$$

for any $t, s \in \mathbb{R}$, hence

$$\left|e^{is} - e^{it}\right| = 2\left|\sin\left(\frac{s-t}{2}\right)\right| \leq |s-t| \tag{4.15}$$

for $t, s \in [0, 2\pi]$.

Therefore by (4.14) and (4.15) we get

$$\left|\varphi_a\left(e^{it}\right) - \varphi_a\left(e^{is}\right)\right| \leq \frac{|a|}{(1-|a|)^2} |s-t| \tag{4.16}$$

for $t, s \in [0, 2\pi]$, which shows that $\varphi_a\left(e^{i\cdot}\right)$ is Lipschitzian with the constant $L = \frac{|a|}{(1-|a|)^2} > 0$ on $[0, 2\pi]$.

If we use the inequality (4.7) for φ_a we get

$$\begin{aligned} &\left|(1-a)^{-1}\langle x, y\rangle - \left\langle (1_H - aU)^{-1} x, y\right\rangle\right| \\ &\qquad \leq \pi \frac{|a|}{(1-|a|)^2} \bigvee_0^{2\pi}\left(\left\langle P_{(\cdot)} x, y\right\rangle\right) \leq \pi \frac{|a|}{(1-|a|)^2} \|x\| \|y\|, \end{aligned} \tag{4.17}$$

where U be a unitary operator on the Hilbert space H, $\{P_\lambda\}_{\lambda \in [0, 2\pi]}$ is the spectral family of projections of U and $x, y \in H$.

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