A COMPLETE MONOTONICITY PROPERTY OF A FUNCTION INVOLVING THE (p,k)-DIGAMMA FUNCTION

KWARA NANTOMAH

Abstract. In this paper, we prove a complete monotonicity property of a certain function involving the (p,k)-digamma function. As a consequence, a sharp inequality is obtained. Also, some special cases of the established results are obtained. In the end, we point out few errors in some previous works.

1. Introduction

The digamma function, which is also known as the psi function is defined as follows (see [1, p. 258-259], [8, p. 139-140]).

$$\begin{aligned}\psi(x) = \frac{d}{dx}\ln\Gamma(x) &= -\gamma + \int_0^\infty \frac{e^{-t}-e^{-xt}}{1-e^{-t}}\,dt, \quad x>0,\\ &= -\gamma - \frac{1}{x} + \sum_{k=1}^{\infty}\frac{x}{k(k+x)}, \quad x>0,\end{aligned}$$

where, $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right)$ is the Euler-Mascheroni's constant and

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt,$$

is the classical Euler's gamma function. Also, it is well known in the literature that the integral

$$\frac{n!}{x^{n+1}} = \int_0^\infty t^n e^{-xt}\,dt, \tag{1}$$

holds for $x>0$ and $n\in\mathbb{N}_0$. See for instance [1, p. 255].

The (p,k)-digamma function, which is a two parameter deformation of the ordinary digamma function is defined as [7]

$$\psi_{p,k}(x) = \frac{d}{dx}\ln\Gamma_{p,k}(x) = \frac{1}{k}\ln(pk) - \int_0^\infty \frac{1-e^{-k(p+1)t}}{1-e^{-kt}}e^{-xt}\,dt, \quad x>0, \tag{2}$$

$$= \frac{1}{k}\ln(pk) - \sum_{n=0}^{p}\frac{1}{nk+x}, \quad x>0, \tag{3}$$

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where, $\Gamma_{p,k}(x)$ is the (p,k)-gamma function defined as [7]

$$\begin{aligned}\Gamma_{p,k}(x) &= \int_0^p t^{x-1}\left(1-\frac{t^k}{pk}\right)^p dt, \quad x>0,\\ &= \frac{(p+1)!k^{p+1}(pk)^{\frac{x}{k}-1}}{x(x+k)(x+2k)\dots(x+pk)}, \quad x>0.\end{aligned}$$

The function $\psi_{p,k}(x)$ satisfies the properties

$$\psi_{p,k}(x+k)-\psi_{p,k}(x) = \frac{1}{x}-\frac{1}{x+pk+k} \tag{4}$$

$$\psi_{p,k}(k) = \frac{1}{k}\left[\ln(pk)-H(p+1)\right],$$

where, $H(n)$ is the n-th harmonic number. It also satisfies the limit relations [7]

$$\begin{tikzcd} \psi_{p,k}(x) \arrow[r, "p\to\infty"] \arrow[d, "k\to1"'] & \psi_k(x) \arrow[d, "k\to1"] \\ \psi_p(x) \arrow[r, "p\to\infty"'] & \psi(x) \end{tikzcd}$$

where, $\psi_p(x)$ is the p-digamma function and $\psi_k(x)$ is the k-digamma functions defined as follows (see [3], [2], [6]).

$$\begin{aligned}\psi_p(x) &= \ln p - \sum_{n=0}^{p}\frac{1}{n+x}, \quad x>0\\ &= \ln p - \int_0^\infty \frac{1-e^{-(p+1)t}}{1-e^{-t}}e^{-xt}\,dt, \quad x>0,\end{aligned}$$

$$\begin{aligned}\psi_k(x) &= \frac{\ln k-\gamma}{k}-\frac{1}{x}+\sum_{n=1}^{\infty}\frac{x}{nk(nk+x)}, \quad x>0,\\ &= \frac{\ln k-\gamma}{k}+\sum_{n=0}^{\infty}\left(\frac{1}{nk+k}-\frac{1}{nk+x}\right), \quad x>0,\\ &= \int_0^\infty\left(\frac{2e^{-t}-e^{-kt}}{kt}-\frac{e^{-xt}}{1-e^{-kt}}\right)dt, \quad x>0.\end{aligned}$$

Differentiating m number of times of (2) and (3), gives

$$\begin{aligned}\psi_{p,k}^{(m)}(x) &= (-1)^{m+1}\int_0^\infty \frac{1-e^{-k(p+1)t}}{1-e^{-kt}}t^m e^{-xt}\,dt, \qquad (5)\\ &= (-1)^{m+1}m!\sum_{n=0}^{p}\frac{1}{(nk+x)^{m+1}},\end{aligned}$$

where $m\in\mathbb{N}$.

Definition 1.1. A function f is said to be *completely monotonic* on an interval I, if f has derivatives of all order and satisfies

$$(-1)^m f^{(m)}(x) \geq 0, \tag{6}$$

for all $x \in I$ and $m \in \mathbb{N}_0$. If the inequality (6) is strict, then f is said to be *strictly completely monotonic* on I.

Completely monotonic functions have some remarkable applications in various aspects of mathematics. In particular, they play a pivotal role in the theory of inequalities and special functions.
Qiu and Vuorinen [9] established among other things that, the function

$$h_1(x) = \psi\left(x + \frac{1}{2}\right) - \psi(x) - \frac{1}{2x}, \tag{7}$$

is strictly decreasing and convex on $(0, \infty)$. Motivated by this result, Mortici [5] proved a more generalized and deeper result which states that, the function

$$f_a(x) = \psi(x + a) - \psi(x) - \frac{a}{x}, \quad a \in (0, 1), \tag{8}$$

is strictly completely monotonic on $(0, \infty)$. Consequently, he obtained a sharp inequality for the function $\psi(x + a) - \psi(x)$. Also inspired by Mortici's results, Merovci [4] proved that for $p \in \mathbb{N}$, the function

$$f_a(x) = \psi_p(x + a) - \psi_p(x), \quad a \in (0, 1), \tag{9}$$

is strictly completely monotonic on $(0, \infty)$ and eventually obtained a sharp inequality for the function $\psi_p(x + a) - \psi_p(x)$. In this paper, the goal is to prove similar results involving the (p, k)-digamma function. We present our results in the following section.

2. Main Results

Theorem 2.1. *Let $p \in \mathbb{N}$, $k > 0$ and $a \in (0, 1)$. Then the function*

$$f_{a,p,k}(x) = \psi_{p,k}(x + ak) - \psi_{p,k}(x) - \frac{a}{x} + \frac{a}{x + pk + k}, \tag{10}$$

is strictly completely monotonic on $(0, \infty)$.

Proof. By repeated differentiation, we obtain

$$f^{(m)}_{a,p,k}(x) = \psi^{(m)}_{p,k}(x + ak) - \psi^{(m)}_{p,k}(x) - \frac{(-1)^m am!}{x^{m+1}} + \frac{(-1)^m am!}{(x + pk + k)^{m+1}},$$

for $m \in \mathbb{N}$. Next, by using relations (1) and (5), we have

$$\begin{aligned} &f^{(m)}_{a,p,k}(x) \\ &= (-1)^{m+1} \int_0^\infty \frac{1 - e^{-k(p+1)t}}{1 - e^{-kt}} t^m e^{-(x+ak)t}\, dt - (-1)^{m+1} \int_0^\infty \frac{1 - e^{-k(p+1)t}}{1 - e^{-kt}} t^m e^{-xt}\, dt \\ &\quad - (-1)^m a \int_0^\infty \frac{t^m e^{-xt}}{1 - e^{-t}}\, dt + (-1)^m a \int_0^\infty t^m e^{-(x+pk+k)t}\, dt. \end{aligned}$$

Then

$$
\begin{aligned}
&(-1)^m f_{a,p,k}^{(m)}(x)\\
&= -\int_0^\infty \frac{1-e^{-k(p+1)t}}{1-e^{-kt}} t^m e^{-(x+ak)t}\,dt + \int_0^\infty \frac{1-e^{-k(p+1)t}}{1-e^{-kt}} t^m e^{-xt}\,dt\\
&\quad - a\int_0^\infty \frac{t^m e^{-xt}}{1-e^{-t}}\,dt + a\int_0^\infty t^m e^{-(x+pk+k)t}\,dt\\
&= \int_0^\infty \frac{1-e^{-k(p+1)t}}{1-e^{-kt}} t^m e^{-xt}(1-e^{-akt})\,dt - a\int_0^\infty t^m e^{-xt}(1-e^{-k(p+1)t})\,dt\\
&= \int_0^\infty \left[\frac{1-e^{-akt}}{1-e^{-kt}} - a\right]\left(1-e^{-k(p+1)t}\right) t^m e^{-xt}\,dt\\
&= ak\int_0^\infty \left[\frac{1-e^{-akt}}{akt} - \frac{1-e^{-kt}}{kt}\right] \frac{1-e^{-k(p+1)t}}{1-e^{-kt}} t^{m+1} e^{-xt}\,dt\\
&> 0.
\end{aligned}
$$

Observe that, since the function $\frac{1-e^{-t}}{t}$ is strictly decreasing on $(0,\infty)$, then for $a \in (0,1)$ and $k > 0$, we have $\frac{1-e^{-akt}}{akt} > \frac{1-e^{-kt}}{kt}$. Hence the proof.

Corollary 2.2. *Let $p \in \mathbb{N}$, $k > 0$ and $a \in (0,1)$. Then the inequality*

$$
\begin{aligned}
\frac{a}{x} - \frac{a}{x+pk+k} < \psi_{p,k}(x+ak) - \psi_{p,k}(x) &\leq \psi_{p,k}(ak) - \psi_{p,k}(k)\\
&+ \frac{1}{k}\left(\frac{1}{a} - a\right) + \frac{1}{k}\left(\frac{a}{p+2} - \frac{1}{a+p+1}\right) + \frac{a}{x} - \frac{a}{x+pk+k}, \qquad (11)
\end{aligned}
$$

holds for $x \in [k,\infty)$.

Proof. Since $f_{a,p,k}(x)$ is completely monotonic, then it is decreasing. Then for $x \in [k,\infty)$ and by using (4), we obtain

$$
\begin{aligned}
0 = \lim_{x\to\infty} f_{a,p,k}(x) < f_{a,p,k}(x) &\leq f_{a,p,k}(k)\\
&= \psi_{p,k}(k+ak) - \psi_{p,k}(k) - \left(\frac{a}{k} - \frac{a}{k+pk+k}\right)\\
&= \psi_{p,k}(ak) - \psi_{p,k}(k) + \frac{1}{k}\left(\frac{1}{a} - a\right) + \frac{1}{k}\left(\frac{a}{p+2} - \frac{1}{a+p+1}\right),
\end{aligned}
$$

which yields inequality (11).

From Theorem 2.1 and Corollary 2.2, we deduce the following special cases.

Theorem 2.3. *Let $p \in \mathbb{N}$ and $a \in (0,1)$. Then the function*

$$
f_{a,p}(x) = \psi_p(x+a) - \psi_p(x) - \frac{a}{x} + \frac{a}{x+p+1}, \qquad (12)
$$

is strictly completely monotonic on $(0,\infty)$ *and the inequality*

$$\frac{a}{x}-\frac{a}{x+p+1}<\psi_p(x+a)-\psi_p(x)\leq \psi_p(a)-\psi_p(1)+\frac{1}{a}-a+\frac{a}{p+2}-\frac{1}{a+p+1}+\frac{a}{x}-\frac{a}{x+p+1}, \quad (13)$$

holds for $x\in[1,\infty)$.

Proof. Let $k=1$ in Theorem 2.1 and Corollary 2.2.

Theorem 2.4. *Let* $k>0$ *and* $a\in(0,1)$. *Then the function*

$$f_k(x)=\psi_k(x+ak)-\psi_k(x)-\frac{a}{x}, \quad (14)$$

is strictly completely monotonic on $(0,\infty)$ *and the inequality*

$$\frac{a}{x}<\psi_k(x+ak)-\psi_k(x)\leq \psi_k(ak)-\psi_k(k)+\frac{1}{k}\left(\frac{1}{a}-a\right)+\frac{a}{x}, \quad (15)$$

holds for $x\in[k,\infty)$.

Proof. Let $p\to\infty$ in Theorem 2.1 and Corollary 2.2.

Remark 2.5. By either letting $p\to\infty$ in Theorem 2.3 or letting $k=1$ in Theorem 2.4, we recover the main results of Mortici [5].

3. Concluding Remarks

We noticed the following errors in the previous works [4] and [5].
In [5], the author presented the inequality

$$0<\psi(x+a)-\psi(x)\leq \psi(a)+\gamma+\frac{1}{a}-a, \quad (16)$$

where $x\geq 1$. However, the correct form of (16) should be

$$\frac{a}{x}<\psi(x+a)-\psi(x)\leq \psi(a)+\gamma+\frac{1}{a}-a+\frac{a}{x}.$$

Also, in [4], the author gave the inequality

$$0<\psi_p(x+a)-\psi_p(x)\leq \frac{1}{x}-\frac{1}{x+p+1}, \quad (17)$$

where $x\geq 1$ and $p\in\mathbb{N}$. However, the correct form of (17) should be

$$0<\psi_p(x+a)-\psi_p(x)\leq \psi_p(a)-\psi_p(1)+\frac{1}{a}-\frac{1}{a+p+1}.$$

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Department of Mathematics, Faculty of Mathematical Sciences, University for Development Studies, Navrongo Campus, P. O. Box 24, Navrongo, UE/R, Ghana.

E-mail address: knantomah@uds.edu.gh