OSTROWSKI TYPE RIEMANN-STIELTJES INTEGRAL INEQUALITIES FOR CONVEX INTEGRANDS AND NONDECREASING INTEGRATORS

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Abstract. In this paper we obtain some inequalities for the Ostrowski difference

$$\int_a^b f(t)\,du(t) - f(x)[u(b)-u(a)],$$

where f is a convex function on $[a,b]$, u is monotonic nondecreasing and $x \in (a,b)$. In the case of Riemann integral, namely for $u(t)=t$, some particular inequalities are given. Applications for functions of selfadjoint operators on complex Hilbert spaces with examples are provided as well.

1. Introduction

We recall the following Ostrowski type inequality for convex functions:

Theorem 1 (Dragomir, 2002 [5])**.** *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a convex function on $[a,b]$. Then for any $x\in(a,b)$ one has the inequality*

$$\frac{1}{2}\left[(b-x)^2 f'_+(x) - (x-a)^2 f'_-(x)\right] \le \int_a^b f(t)\,dt - (b-a)f(x) \tag{1.1}$$
$$\le \frac{1}{2}\left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a)\right].$$

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $x=a$ or $x=b$.

Corollary 1. *With the assumptions of Theorem 1 and if $x\in(a,b)$ is a point of differentiability for f, then*

$$\left(\frac{a+b}{2}-x\right)f'(x) \le \frac{1}{b-a}\int_a^b f(t)\,dt - f(x). \tag{1.2}$$

The following corollary provides both a sharper lower bound for the Hermite-Hadamard difference,

$$\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right),$$

which we know is nonnegative, and an upper bound [5].

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Corollary 2. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then we have the inequality*

$$0\leq\frac{1}{8}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right](b-a)\leq\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)\leq\frac{1}{8}\left[f'_{-}(b)-f'_{+}(a)\right](b-a).\tag{1.3}$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

For other related results see [6] and [7]. For more inequalities of Ostrowski type, see [1], [3]-[4], [8], [10], [12] and [14].

Motivated by the above results, we establish in this paper some inequalities for the *Ostrowski difference*

$$\int_a^b f(t)\,du(t)-f(x)\left[u(b)-u(a)\right],$$

where f is a convex function on $[a,b]$, u is monotonic nondecreasing and $x\in(a,b)$. In the case of Riemann integral, namely for $u(t)=t$, some particular inequalities are given. Applications for functions of selfadjoint operators on complex Hilbert spaces with examples are provided as well.

2. The Main Results

We start with the following inequality for convex integrands and monotonic nondecreasing integrators:

Theorem 2. *Assume that $f:[a,b]\to\mathbb{R}$ is continuous convex on $[a,b]$ and $u:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then*

$$\begin{aligned}0&\leq\int_a^b f(t)\,du(t)-f(x)\left[u(b)-u(a)\right]\\&\quad-f'_{+}(x)\left[(b-x)u(b)-\int_x^b u(t)\,dt\right]\\&\quad-f'_{-}(x)\left[(x-a)u(a)-\int_a^x u(t)\,dt\right]\\&\leq\int_x^b(t-x)\left[f'_{+}(t)-f'_{+}(x)\right]du(t)+\int_a^x(x-t)\left[f'_{-}(x)-f'_{-}(t)\right]du(t)\end{aligned}\tag{2.1}$$

for $x\in(a,b)$, provided that the Riemann-Stieltjes integrals in the right member exist.

If f is differentiable in $x\in(a,b)$, then we have the simpler inequality

$$\begin{aligned}0&\leq\int_a^b f(t)\,du(t)-f(x)\left[u(b)-u(a)\right]\\&\quad-f'(x)\left[(b-x)u(b)+(x-a)u(a)-\int_a^b u(t)\,dt\right]\\&\leq\int_x^b(t-x)\left[f'_{+}(t)-f'(x)\right]du(t)+\int_a^x(x-t)\left[f'(x)-f'_{-}(t)\right]du(t),\end{aligned}\tag{2.2}$$

provided that the Riemann-Stieltjes integrals in the right member exist.

Proof. We have

$$\int_a^b [f(t)-f(x)]\,du(t) = \int_a^b f(t)\,du(t) - f(x)[u(b)-u(a)] \tag{2.3}$$

for $x \in [a,b]$.

Also

$$\begin{aligned}\int_a^b [f(t)-f(x)]\,du(t) &= \int_a^x [f(t)-f(x)]\,du(t) + \int_x^b [f(t)-f(x)]\,du(t) \\ &= \int_x^b [f(t)-f(x)]\,du(t) - \int_a^x [f(x)-f(t)]\,du(t) \\ &=: B(f,u;x)\end{aligned} \tag{2.4}$$

for $x \in (a,b)$.

Since f is convex, hence by the *gradient inequality*, we have

$$f(t)-f(x) \geq (t-x)\,f'_+(x) \text{ for } t \in [x,b]$$

and

$$f(x)-f(t) \leq (x-t)\,f'_-(x) \text{ for } [a,x].$$

Since u is monotonic nondecreasing, it follows by using integration by parts that

$$\begin{aligned}\int_x^b [f(t)-f(x)]\,du(t) &\geq f'_+(x)\int_x^b (t-x)\,du(t) \\ &= f'_+(x)\left[(b-x)\,u(b) - \int_x^b u(t)\,dt\right]\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\int_a^x [f(x)-f(t)]\,du(t) &\leq f'_-(x)\int_a^x (x-t)\,du(t) \\ &= f'_-(x)\left[\int_a^x u(t)\,dt - (x-a)\,u(a)\right],\end{aligned}$$

which is equivalent to

$$-\int_a^x [f(x)-f(t)]\,du(t) \geq f'_-(x)\left[(x-a)\,u(a) - \int_a^x u(t)\,dt\right] \tag{2.6}$$

for $x \in (a,b)$.

Now, if we add (2.5) with (2.6) we get

$$\begin{aligned}&B(f,u;x) \\ &\quad \geq f'_+(x)\left[(b-x)\,u(b) - \int_x^b u(t)\,dt\right] + f'_-(x)\left[(x-a)\,u(a) - \int_a^x u(t)\,dt\right],\end{aligned}$$

and by (2.3) and (2.4) we get the first inequality in (2.1).

By the gradient inequality we also have

$$f(t)-f(x) \leq (t-x)\,f'_+(t) \text{ for } t \in [x,b]$$

and

$$f(x)-f(t) \geq (x-t)\,f'_-(t) \text{ for } [a,x].$$

Since u is monotonic nondecreasing, it follows that

$$\begin{aligned}(2.7)\quad \int_x^b [f(t)-f(x)]\,du(t) &\leq \int_x^b (t-x)\,f'_+(t)\,du(t)\\ &= \int_x^b (t-x)\left[f'_+(t)-f'_+(x)\right]du(t)+f'_+(x)\int_x^b (t-x)\,du(t)\\ &= \int_x^b (t-x)\left[f'_+(t)-f'_+(x)\right]du(t)+f'_+(x)\left[(b-x)\,u(b)-\int_x^b u(t)\,dt\right]\end{aligned}$$

and

$$\int_a^x [f(x)-f(t)]\,du(t) \geq \int_a^x (x-t)\,f'_-(t)\,du(t),$$

which gives

$$\begin{aligned}(2.8)\quad -\int_a^x [f(x)-f(t)]\,du(t) &\leq \int_a^x (t-x)\,f'_-(t)\,du(t)\\ &= \int_a^x (t-x)\left[f'_-(t)-f'_-(x)\right]du(t)+f'_-(x)\int_a^x (t-x)\,du(t)\\ &= \int_a^x (t-x)\left[f'_-(t)-f'_-(x)\right]du(t)+f'_-(x)\left[(x-a)\,u(a)-\int_a^x u(t)\,dt\right]\end{aligned}$$

for $x\in(a,b)$.

By making use of (2.3) and (2.4) we get the second inequality in (2.1). □

Remark 1. *We observe that the Riemann-Stieltjes integrals from the right member of (2.1) and (2.2) exist if either u is assumed to be continuous or the derivative f' exists and is continuous on (a,b).*

Corollary 3. *With the assumptions of Theorem 2, we have*

$$\begin{aligned}(2.9)\quad 0 &\leq \int_a^b f(t)\,du(t)-f\left(\frac{a+b}{2}\right)[u(b)-u(a)]\\ &\quad - f'_+\left(\frac{a+b}{2}\right)\left[\frac{1}{2}(b-a)\,u(b)-\int_{\frac{a+b}{2}}^b u(t)\,dt\right]\\ &\quad - f'_-\left(\frac{a+b}{2}\right)\left[\frac{1}{2}(b-a)\,u(a)-\int_a^{\frac{a+b}{2}} u(t)\,dt\right]\\ &\leq \int_{\frac{a+b}{2}}^b \left(t-\frac{a+b}{2}\right)\left[f'_+(t)-f'_+\left(\frac{a+b}{2}\right)\right]du(t)\\ &\quad + \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-t\right)\left[f'_-\left(\frac{a+b}{2}\right)-f'_-(t)\right]du(t).\end{aligned}$$

If f is differentiable in $\frac{a+b}{2}$, then

$$\begin{aligned} (2.10)\quad 0 \le & \int_a^b f(t)\,du(t) - f\left(\frac{a+b}{2}\right)[u(b)-u(a)] \\ & - f'\left(\frac{a+b}{2}\right)\left[\frac{u(b)+u(a)}{2}(b-a) - \int_a^b u(t)\,dt\right] \\ & \le \int_{\frac{a+b}{2}}^b \left(t-\frac{a+b}{2}\right)\left[f'_+(t) - f'\left(\frac{a+b}{2}\right)\right] du(t) \\ & \quad + \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-t\right)\left[f'\left(\frac{a+b}{2}\right) - f'_-(t)\right] du(t). \end{aligned}$$

Since most of the convex functions used in applications are smooth, we can state the following result:

Corollary 4. *Let I an interval and $\mathring{I}$, the interior of I. Assume that $f : I \to \mathbb{R}$ is convex on I, differentiable and with the derivative f' continuous on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If $u : [a,b] \to \mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then*

$$\begin{aligned} (2.11)\quad 0 \le & \int_a^b f(t)\,du(t) - f(x)[u(b)-u(a)] \\ & - f'(x)\left[(b-x)u(b) + (x-a)u(a) - \int_a^b u(t)\,dt\right] \\ & \le \int_a^b (t-x)[f'(t)-f'(x)]\,du(t) \\ & \le \begin{cases} \left[\frac{1}{2}(b-a) + \left|x-\frac{a+b}{2}\right|\right] \int_a^b |f'(t)-f'(x)|\,du(t); \\ \left(\int_a^b |t-x|^p\,du(t)\right)^{1/p} \left(\int_a^b |f'(t)-f'(x)|^q\,du(t)\right)^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}[f'(b)-f'(a)] + \left|f'(x) - \frac{f'(a)+f'(b)}{2}\right|\right] \int_a^b |t-x|\,du(t) \end{cases} \end{aligned}$$

for all $x \in [a,b]$.

In particular,

$$\begin{aligned} (2.12)\quad 0 \le & \int_a^b f(t)\,du(t) - f\left(\frac{a+b}{2}\right)[u(b)-u(a)] \\ & - f'\left(\frac{a+b}{2}\right)\left[\frac{u(b)+u(a)}{2}(b-a) - \int_a^b u(t)\,dt\right] \\ & \le \int_a^b \left(t-\frac{a+b}{2}\right)\left[f'(t) - f'\left(\frac{a+b}{2}\right)\right] du(t) \end{aligned}$$

$$\leq \begin{cases} \frac{1}{2}(b-a)\int_a^b \left|f'(t)-f'\left(\frac{a+b}{2}\right)\right| du(t); \\ \\ \left(\int_a^b \left|t-\frac{a+b}{2}\right|^p du(t)\right)^{1/p} \left(\int_a^b \left|f'(t)-f'\left(\frac{a+b}{2}\right)\right|^q du(t)\right)^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\frac{1}{2}\left[f'(b)-f'(a)\right]+\left|f'\left(\frac{a+b}{2}\right)-\frac{f'(a)+f'(b)}{2}\right|\right]\int_a^b \left|t-\frac{a+b}{2}\right| du(t). \end{cases}$$

Proof. The first two inequalities are obvious from (2.2) written for differentiable functions.

We have, by Hölder's inequality for Riemann-Stieltjes integral of monotonic nondecreasing integrators, that

$$\begin{aligned} 0 &\leq \int_a^b (t-x)\left[f'(t)-f'(x)\right] du(t) = \left|\int_a^b (t-x)\left[f'(t)-f'(x)\right] du(t)\right| \\ &\leq \int_a^b \left|(t-x)\left[f'(t)-f'(x)\right]\right| du(t) = \int_a^b |t-x|\left|f'(t)-f'(x)\right| du(t) \\ &\leq \begin{cases} \max_{t\in[a,b]}|t-x|\int_a^b \left|f'(t)-f'(x)\right| du(t); \\ \\ \left(\int_a^b |t-x|^p du(t)\right)^{1/p}\left(\int_a^b \left|f'(t)-f'(x)\right|^q du(t)\right)^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \max_{t\in[a,b]}\left|f'(t)-f'(x)\right|\int_a^b |t-x|\, du(t), \end{cases} \end{aligned}$$

which proves the last part of (2.11). □

Remark 2. *We observe that, if $m\in[a,b]$ such that $f'(m)=\frac{f'(a)+f'(b)}{2}$, then by (2.11) we get*

$$\begin{aligned} (2.13)\quad 0 \leq \int_a^b f(t)\,du(t) - f(m)\left[u(b)-u(a)\right] \\ - f'(m)\left[(b-m)\,u(b)+(m-a)\,u(a)-\int_a^b u(t)\,dt\right] \\ \leq \int_a^b (t-m)\left[f'(t)-f'(m)\right] du(t) \end{aligned}$$

$$\leq \begin{cases} \left[\frac{1}{2}(b-a)+\left|m-\frac{a+b}{2}\right|\right]\int_a^b \left|f'(t)-f'(m)\right| du(t); \\ \\ \left(\int_a^b |t-m|^p du(t)\right)^{1/p}\left(\int_a^b \left|f'(t)-f'(m)\right|^q du(t)\right)^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \frac{1}{2}\left[f'(b)-f'(a)\right]\int_a^b |t-m|\, du(t). \end{cases}$$

Further, we consider some inequalities with positive weights in the Riemann-Stieltjes integral:

Corollary 5. *Assume that f is as in Corollary 4. If $g:[a,b]\to[0,\infty)$ is continuous and $v:[a,b]\to\mathbb{R}$ is monotonic nondecreasing, then*

$$0\leq\int_a^b f(t)\,g(t)\,dv(t)-f(x)\int_a^b g(t)\,dv(t)-f'(x)\int_a^b(t-x)\,g(t)\,dv(t) \tag{2.14}$$
$$\leq\int_a^b(t-x)\left[f'(t)-f'(x)\right]g(t)\,dv(t)$$
$$\leq\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\int_a^b\left|f'(t)-f'(x)\right|g(t)\,dv(t); \\ \\ \left(\int_a^b|t-x|^p\,g(t)\,dv(t)\right)^{1/p}\left(\int_a^b\left|f'(t)-f'(x)\right|^q g(t)\,dv(t)\right)^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\frac{1}{2}\left[f'(b)-f'(a)\right]+\left|f'(x)-\frac{f'(a)+f'(b)}{2}\right|\right]\int_a^b|t-x|\,g(t)\,dv(t).\end{cases}$$

Proof. First we observe that, using integration by parts we have

$$(b-x)\,u(b)+(x-a)\,u(a)-\int_a^b u(t)\,dt=\int_a^b(t-x)\,du(t).$$

Using the properties of Riemann-Stieltjes integral with integrators u given by an integral, namely, if

$$u(t)=\int_a^t g(s)\,dv(s),$$

which is monotonic nondecreasing on $[a,b]$, then

$$\int_a^b f(t)\,du(t)=\int_a^b f(t)\,g(t)\,dv(t),\quad\int_a^b(t-x)\,du(t)=\int_a^b(t-x)\,g(t)\,dv(t),$$

$$\int_a^b(t-x)\left[f'(t)-f'(x)\right]du(t)=\int_a^b(t-x)\left[f'(t)-f'(x)\right]g(t)\,dv(t),$$

$$\int_a^b\left|f'(t)-f'(x)\right|du(t)=\int_a^b\left|f'(t)-f'(x)\right|g(t)\,dv(t),$$

$$\int_a^b|t-x|^p\,du(t)=\int_a^b|t-x|^p\,g(t)\,dv(t),$$

$$\int_a^b\left|f'(t)-f'(x)\right|^q du(t)=\int_a^b\left|f'(t)-f'(x)\right|^q g(t)\,dv(t)$$

and

$$\int_a^b|t-x|\,du(t)=\int_a^b|t-x|\,g(t)\,dv(t).$$

By utilising the inequality (2.11) we then get (2.14). □

Remark 3. *If we take in (2.14) $x = \frac{a+b}{2}$, then we get the mid-point inequality*

$$
\begin{aligned}
(2.15)\quad 0 &\leq \int_a^b f(t)\, g(t)\, dv(t) - f\left(\frac{a+b}{2}\right) \int_a^b g(t)\, dv(t) \\
&\quad - f'\left(\frac{a+b}{2}\right) \int_a^b \left(t - \frac{a+b}{2}\right) g(t)\, dv(t) \\
&\leq \int_a^b \left(t - \frac{a+b}{2}\right) \left[f'(t) - f'\left(\frac{a+b}{2}\right)\right] g(t)\, dv(t) \\
&\leq \begin{cases}
\frac{1}{2}(b-a) \int_a^b \left|f'(t) - f'\left(\frac{a+b}{2}\right)\right| g(t)\, dv(t)\,; \\[2ex]
\left(\int_a^b \left|t - \frac{a+b}{2}\right|^p g(t)\, dv(t)\right)^{1/p} \left(\int_a^b \left|f'(t) - f'\left(\frac{a+b}{2}\right)\right|^q g(t)\, dv(t)\right)^{1/q}, \\
p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\[2ex]
\left[\frac{1}{2}\left[f'(b) - f'(a)\right] + \left|f'\left(\frac{a+b}{2}\right) - \frac{f'(a)+f'(b)}{2}\right|\right] \int_a^b \left|t - \frac{a+b}{2}\right| g(t)\, dv(t)\,.
\end{cases}
\end{aligned}
$$

Also, if $m \in [a,b]$ is such that $f'(m) = \frac{f'(a)+f'(b)}{2}$, then by (2.14) we get

$$
\begin{aligned}
(2.16)\quad 0 &\leq \int_a^b f(t)\, g(t)\, dv(t) - f(m) \int_a^b g(t)\, dv(t) \\
&\quad - f'(m) \int_a^b (t-m)\, g(t)\, dv(t) \\
&\leq \int_a^b (t-m) \left[f'(t) - f'(m)\right] g(t)\, dv(t) \\
&\leq \begin{cases}
\left[\frac{1}{2}(b-a) + \left|m - \frac{a+b}{2}\right|\right] \int_a^b \left|f'(t) - f'(m)\right| g(t)\, dv(t)\,; \\[2ex]
\left(\int_a^b |t-m|^p g(t)\, dv(t)\right)^{1/p} \left(\int_a^b \left|f'(t) - f'(m)\right|^q g(t)\, dv(t)\right)^{1/q}, \\
p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\[2ex]
\frac{1}{2}\left[f(b) - f(a)\right] \int_a^b |t-m|\, g(t)\, dv(t)\,.
\end{cases}
\end{aligned}
$$

3. Inequalities for Riemann Integral

If in (2.14), (2.15) and (2.16) we take $v(t) = t$, then we get the weighted integral inequality for the Riemann integral

$$
\begin{aligned}
(3.1)\quad 0 &\leq \int_a^b f(t)\, g(t)\, dt - f(x) \int_a^b g(t)\, dt - f'(x) \int_a^b (t-x)\, g(t)\, dt \\
&\leq \int_a^b (t-x) \left[f'(t) - f'(x)\right] g(t)\, dt
\end{aligned}
$$

$$\leq \begin{cases} \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\int_a^b |f'(t)-f'(x)|\,g(t)\,dt; \\ \left(\int_a^b |t-x|^p g(t)\,dt\right)^{1/p}\left(\int_a^b |f'(t)-f'(x)|^q g(t)\,dt\right)^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}[f'(b)-f'(a)]+\left|f'(x)-\frac{f'(a)+f'(b)}{2}\right|\right]\int_a^b |t-x|\,g(t)\,dt, \end{cases}$$

$$\begin{aligned} (3.2)\quad 0 &\leq \int_a^b f(t)\,g(t)\,dt - f\left(\frac{a+b}{2}\right)\int_a^b g(t)\,dt \\ &\qquad - f'\left(\frac{a+b}{2}\right)\int_a^b \left(t-\frac{a+b}{2}\right) g(t)\,dt \\ &\leq \int_a^b \left(t-\frac{a+b}{2}\right)\left[f'(t)-f'\left(\frac{a+b}{2}\right)\right] g(t)\,dt \\ &\leq \begin{cases} \frac{1}{2}(b-a)\int_a^b \left|f'(t)-f'\left(\frac{a+b}{2}\right)\right| g(t)\,dt; \\ \left(\int_a^b \left|t-\frac{a+b}{2}\right|^p g(t)\,dt\right)^{1/p}\left(\int_a^b \left|f'(t)-f'\left(\frac{a+b}{2}\right)\right|^q g(t)\,dt\right)^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}[f(b)-f(a)]+\left|f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{2}\right|\right]\int_a^b \left|t-\frac{a+b}{2}\right| g(t)\,dt \end{cases} \end{aligned}$$

and

$$\begin{aligned} (3.3)\quad 0 &\leq \int_a^b f(t)\,g(t)\,dt - f(m)\int_a^b g(t)\,dt \\ &\qquad - f'(m)\int_a^b (t-m)\,g(t)\,dt \\ &\leq \int_a^b (t-m)\,[f'(t)-f'(m)]\,g(t)\,dt \\ &\leq \begin{cases} \left[\frac{1}{2}(b-a)+\left|m-\frac{a+b}{2}\right|\right]\int_a^b |f'(t)-f'(m)|\,g(t)\,dt; \\ \left(\int_a^b |t-m|^p g(t)\,dv(t)\right)^{1/p}\left(\int_a^b |f'(t)-f'(m)|^q g(t)\,dt\right)^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}[f'(b)-f'(a)]\int_a^b |t-m|\,g(t)\,dt. \end{cases} \end{aligned}$$

Now, by taking the weight g to be uniform, namely $g(t)=1$, $t\in[a,b]$, then we get from (3.1)-(3.3) the following inequality for the differentiable convex function

$f:[a,b]\to\mathbb{R}$

$$
\begin{aligned}
(3.4)\quad 0 &\le \int_a^b f(t)\,dt - f(x)(b-a) - f'(x)(b-a)\left(\frac{a+b}{2}-x\right) \\
&\le \int_a^b (t-x)\left[f'(t)-f'(x)\right]dt \\
&\le \begin{cases}
\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\int_a^b \left|f'(t)-f'(x)\right|dt; \\[2ex]
\frac{1}{(p+1)^{1/p}}\left((b-x)^{p+1}+(x-a)^{p+1}\right)^{1/p}\left(\int_a^b \left|f'(t)-f'(x)\right|^q dt\right)^{1/q}, \\
p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex]
\left[\frac{1}{2}\left[f'(b)-f'(a)\right]+\left|f'(x)-\frac{f'(a)+f'(b)}{2}\right|\right]\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right],
\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
(3.5)\quad 0 &\le \int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a) \\
&\le \int_a^b \left(t-\frac{a+b}{2}\right)\left[f'(t)-f'\left(\frac{a+b}{2}\right)\right]dt \\
&\le \begin{cases}
\frac{1}{2}(b-a)\int_a^b \left|f'(t)-f'\left(\frac{a+b}{2}\right)\right|dt; \\[2ex]
\frac{1}{2(p+1)^{1/p}}(b-a)^{1+1/p}\left(\int_a^b \left|f'(t)-f'\left(\frac{a+b}{2}\right)\right|^q dt\right)^{1/q}, \\
p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex]
\frac{1}{4}(b-a)^2\left[\frac{1}{2}\left[f(b)-f(a)\right]+\left|f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{2}\right|\right]
\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
(3.6)\quad 0 &\le \int_a^b f(t)\,dt - f(m)(b-a) - f'(m)(b-a)\left(\frac{a+b}{2}-m\right) \\
&\le \int_a^b (t-m)\left[f'(t)-f'(m)\right]dt \\
&\le \begin{cases}
\left[\frac{1}{2}(b-a)+\left|m-\frac{a+b}{2}\right|\right]\int_a^b \left|f'(t)-f'(m)\right|dt; \\[2ex]
\frac{1}{(p+1)^{1/p}}\left((b-m)^{p+1}+(m-a)^{p+1}\right)^{1/p}\left(\int_a^b \left|f'(t)-f'(m)\right|^q dt\right)^{1/q}, \\
p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex]
\frac{1}{2}\left[f'(b)-f'(a)\right]\left[\frac{1}{4}(b-a)^2+\left(m-\frac{a+b}{2}\right)^2\right].
\end{cases}
\end{aligned}
$$

Using Čebyšev's inequality for functions with the same monotonicity, we have for differentiable convex functions that

$$\begin{aligned}0 &\leq \frac{1}{b-a}\int_a^b (t-x)\left[f'(t)-f'(x)\right]dt \\ &\qquad - \frac{1}{b-a}\int_a^b (t-x)\,dt\frac{1}{b-a}\int_a^b \left[f'(t)-f'(x)\right]dt \\ &= \frac{1}{b-a}\int_a^b (t-x)\left[f'(t)-f'(x)\right]dt - \left(\frac{a+b}{2}-x\right)\left(\frac{f(b)-f(a)}{b-a}-f'(x)\right)\end{aligned}$$

Using Ostrowski' s inequality [13] we also have

$$\begin{aligned}0 &\leq \frac{1}{b-a}\int_a^b (t-x)\left[f'(t)-f'(x)\right]dt - \left(\frac{a+b}{2}-x\right)\left(\frac{f(b)-f(a)}{b-a}-f'(x)\right) \\ &\leq \frac{1}{8}(b-a)\max_{t\in[a,b]}\left[\left|\frac{d}{dt}(t-x)\right|\right]\left[\max_{t\in[a,b]}\left[f'(t)-f'(x)\right]-\min_{t\in[a,b]}\left[f'(t)-f'(x)\right]\right] \\ &\qquad = \frac{1}{8}(b-a)\left[f'_-(b)-f'_+(a)\right],\end{aligned}$$

which implies that

$$\int_a^b (t-x)\left[f'(t)-f'(x)\right]dt \leq \left(\frac{a+b}{2}-x\right)\left(f(b)-f(a)-f'(x)(b-a)\right) + \frac{1}{8}(b-a)^2\left[f'_-(b)-f'_+(a)\right] \tag{3.7}$$

for $x\in(a,b)$.

By using (3.4) we get

$$\begin{aligned}0 &\leq \int_a^b f(t)\,dt - f(x)(b-a) - f'(x)(b-a)\left(\frac{a+b}{2}-x\right) \\ &\leq \int_a^b (t-x)\left[f'(t)-f'(x)\right]dt \\ &\leq \left(\frac{a+b}{2}-x\right)\left(f(b)-f(a)-f'(x)(b-a)\right)+\frac{1}{8}(b-a)^2\left[f'_-(b)-f'_+(a)\right]\end{aligned} \tag{3.8}$$

for $x\in(a,b)$.

In particular, we have

$$\begin{aligned}0 &\leq \int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a) \\ &\leq \int_a^b \left(t-\frac{a+b}{2}\right)\left[f'(t)-f'\left(\frac{a+b}{2}\right)\right]dt \leq \frac{1}{8}(b-a)^2\left[f'_-(b)-f'_+(a)\right].\end{aligned} \tag{3.9}$$

If we use the first and last inequality in (3.8) and add $f'(x)(b-a)\left(\frac{a+b}{2}-x\right)$, then we get the Ostrowski type inequality

$$f'(x)(b-a)\left(\frac{a+b}{2}-x\right) \le \int_a^b f(t)\,dt - f(x)(b-a) \le \left(\frac{a+b}{2}-x\right)[f(b)-f(a)] + \frac{1}{8}(b-a)^2\left[f'_-(b)-f'_+(a)\right] \tag{3.10}$$

for $x \in (a,b)$.

In particular, we get the Hermite-Hadamard type inequalities [5], see also (1.3)

$$0 \le \int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a) \le \frac{1}{8}(b-a)^2\left[f'_-(b)-f'_+(a)\right], \tag{3.11}$$

in which the constant $\frac{1}{8}$ is best.

Further, assume that $f:[a,b]\to\mathbb{R}$ is convex and twice differentiable on (a,b) and $\|f''\|_\infty := \sup_{t\in(a,b)}|f''(t)| < \infty$. By using Čebyšev's inequality [2] we have

$$\begin{aligned} 0 &\le \frac{1}{b-a}\int_a^b (t-x)[f'(t)-f'(x)]\,dt - \left(\frac{a+b}{2}-x\right)\left(\frac{f(b)-f(a)}{b-a}-f'(x)\right) \\ &\le \frac{1}{12}(b-a)^2 \sup_{t\in[a,b]}\left[\left|\frac{d}{dt}(t-x)\right|\right]\sup_{t\in[a,b]}\left[\left|\frac{d}{dt}[f'(t)-f'(x)]\right|\right] \\ &= \frac{1}{12}(b-a)^2\|f''\|_\infty, \end{aligned}$$

namely

$$\int_a^b (t-x)[f'(t)-f'(x)]\,dt \le \left(\frac{a+b}{2}-x\right)(f(b)-f(a)-f'(x)(b-a)) + \frac{1}{12}(b-a)^3\|f''\|_\infty \tag{3.12}$$

for $x \in (a,b)$.

By using (3.4) we get

$$\begin{aligned} 0 &\le \int_a^b f(t)\,dt - f(x)(b-a) - f'(x)(b-a)\left(\frac{a+b}{2}-x\right) \\ &\le \int_a^b (t-x)[f'(t)-f'(x)]\,dt \\ &\le \left(\frac{a+b}{2}-x\right)(f(b)-f(a)-f'(x)(b-a)) + \frac{1}{12}(b-a)^3\|f''\|_\infty \end{aligned} \tag{3.13}$$

for $x \in (a,b)$.

In particular, we have

$$\begin{aligned} 0 &\le \int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a) \\ &\le \int_a^b \left(t-\frac{a+b}{2}\right)\left[f'(t)-f'\left(\frac{a+b}{2}\right)\right]dt \le \frac{1}{12}(b-a)^3\|f''\|_\infty. \end{aligned} \tag{3.14}$$

4. Applications for Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H; \langle\cdot,\cdot\rangle)$. Let $A \in \mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda \in \mathbb{R}$ as follows

$$\varphi_\lambda(s) := \begin{cases} 1, \text{ for } -\infty < s \leq \lambda, \\ \\ 0, \text{ for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$E_\lambda := \varphi_\lambda(A) \tag{4.1}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [11, p. 256]:

Theorem 3 (Spectral Representation Theorem)**.** *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a = \min\{\lambda \,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $b = \max\{\lambda \,|\lambda \in Sp(A)\} =: \max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_\lambda \leq E_{\lambda'}$ *for* $\lambda \leq \lambda'$;
b) $E_{a-0} = 0, E_b = I$ *and* $E_{\lambda+0} = E_\lambda$ *for all* $\lambda \in \mathbb{R}$;
c) *We have the representation*

$$A = \int_{a-0}^{b} \lambda dE_\lambda.$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ there exists a unique operator $\varphi(A) \in \mathcal{B}(H)$ such that for every $\varepsilon > 0$ there exists a $\delta > 0$ satisfying the inequality

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi\left(\lambda_k'\right)\left[E_{\lambda_k} - E_{\lambda_{k-1}}\right] \right\| \leq \varepsilon$$

whenever

$$\begin{cases} \lambda_0 < a = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = b, \\ \\ \lambda_k - \lambda_{k-1} \leq \delta \text{ for } 1 \leq k \leq n, \\ \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \leq k \leq n \end{cases}$$

this means that

$$\varphi(A) = \int_{a-0}^{b} \varphi(\lambda)\, dE_\lambda, \tag{4.2}$$

where the integral is of Riemann-Stieltjes type.

Corollary 6. *With the assumptions of Theorem 3 for A, E_λ and φ we have the representations*

$$\varphi(A)x = \int_{a-0}^{b} \varphi(\lambda)\, dE_\lambda x \quad \text{for all } x \in H$$

and

$$\langle \varphi (A) x, y\rangle = \int_{a-0}^{b} \varphi (\lambda) \, d \langle E_\lambda x, y\rangle \quad \text{for all } x, \ y \in H. \tag{4.3}$$

In particular,

$$\langle \varphi (A) x, x\rangle = \int_{a-0}^{b} \varphi (\lambda) \, d \langle E_\lambda x, x\rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi (A) x\|^2 = \int_{a-0}^{b} |\varphi (\lambda)|^2 \, d \|E_\lambda x\|^2 \quad \text{for all } x \in H.$$

We need the following result that provides an upper bound for the total variation of the function $\mathbb{R} \ni \lambda \mapsto \langle E_\lambda x, y\rangle \in \mathbb{C}$ on an interval $[\alpha, \beta]$, see [9].

Lemma 1. *Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A. Then for any x, $y \in H$ and $\alpha < \beta$ we have the inequality*

$$\left[\bigvee_{\alpha}^{\beta} \left(\left\langle E_{(\cdot)} x, y\right\rangle\right)\right]^2 \le \langle (E_\beta - E_\alpha) x, x\rangle \langle (E_\beta - E_\alpha) y, y\rangle, \tag{4.4}$$

where $\bigvee_{\alpha}^{\beta} \left(\left\langle E_{(\cdot)} x, y\right\rangle\right)$ *denotes the total variation of the function* $\left\langle E_{(\cdot)} x, y\right\rangle$ *on* $[\alpha, \beta]$.

Remark 4. *For $\alpha = a - \varepsilon$ with $\varepsilon > 0$ and $\beta = b$ we get from (4.4) the inequality*

$$\bigvee_{a-\varepsilon}^{b} \left(\left\langle E_{(\cdot)} x, y\right\rangle\right) \le \langle (I - E_{a-\varepsilon}) x, x\rangle^{1/2} \langle (I - E_{a-\varepsilon}) y, y\rangle^{1/2} \tag{4.5}$$

for any x, $y \in H$.

This implies, for any x, $y \in H$, that

$$\bigvee_{a-0}^{b} \left(\left\langle E_{(\cdot)} x, y\right\rangle\right) \le \|x\| \, \|y\|, \tag{4.6}$$

where $\bigvee_{a-0}^{b} \left(\left\langle E_{(\cdot)} x, y\right\rangle\right)$ *denotes the limit* $\lim_{\varepsilon\to 0+} \left[\bigvee_{a-\varepsilon}^{b} \left(\left\langle E_{(\cdot)} x, y\right\rangle\right)\right]$.

We can state the following result for functions of selfadjoint operators:

Theorem 4. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a = \min\{\lambda \,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $b = \max\{\lambda \,|\lambda \in Sp(A)\} =: \max Sp(A)$. Also, assume that $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A and φ, $\psi : I \to \mathbb{C}$ are continuous on I, $[a, b] \subset \mathring{I}$ (the interior of I). If φ is differentiable convex on $[a, b]$ with a continuous derivative on $\mathring{I}$ and*

ψ is nonnegative on $[a,b]$, then for $s\in(a,b)$

$$
\begin{aligned}
(4.7)\quad 0 &\le \langle \varphi(A)\psi(A)x,x\rangle - \varphi(s)\langle \psi(A)x,x\rangle - \varphi'(s)\langle (A-s1_H)\psi(A)x,x\rangle \\
&\le \langle (A-s1_H)[\varphi'(A)-\varphi'(s)1_H]\psi(A)x,x\rangle \\
&\le \begin{cases} \left[\frac{1}{2}(b-a)+\left|s-\frac{a+b}{2}\right|\right]\langle |\varphi'(A)-\varphi'(s)1_H|\psi(A)x,x\rangle ; \\ \\ \langle |A-s1_H|^p\psi(A)x,x\rangle^{1/p}\left(\langle |\varphi'(A)-\varphi'(s)1_H|^q\psi(A)x,x\rangle\right)^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\frac{1}{2}[\varphi'(b)-\varphi'(a)]+\left|\varphi'(s)-\frac{\varphi'(a)+\varphi'(b)}{2}\right|\right]\langle |A-s1_H|\psi(A)x,x\rangle , \end{cases}
\end{aligned}
$$

for all $x\in H$.

Proof. Using the inequality (2.14) we have for small $\varepsilon>0$, and for any $x\in H$ that

$$
\begin{aligned}
0 &\le \int_{a-\varepsilon}^{b}\varphi(t)\psi(t)\,d\langle E_t x,x\rangle - \varphi(s)\int_{a-\varepsilon}^{b}\psi(t)\,d\langle E_t x,x\rangle \\
&\qquad - \varphi'(s)\int_{a-\varepsilon}^{b}(t-s)\psi(t)\,d\langle E_t x,x\rangle \\
&\le \int_{a-\varepsilon}^{b}(t-s)[\varphi'(t)-\varphi'(s)]\psi(t)\,d\langle E_t x,x\rangle \\
&\le \begin{cases} \left[\frac{1}{2}(b-a)+\left|s-\frac{a+b}{2}\right|\right]\int_{a-\varepsilon}^{b}|\varphi'(t)-\varphi'(s)|\psi(t)\,d\langle E_t x,x\rangle ; \\ \\ \left(\int_{a-\varepsilon}^{b}|t-s|^p\psi(t)\,d\langle E_t x,x\rangle\right)^{1/p}\left(\int_{a-\varepsilon}^{b}|\varphi'(t)-\varphi'(s)|^q\psi(t)\,d\langle E_t x,x\rangle\right)^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\frac{1}{2}[\varphi'(b)-\varphi'(a)]+\left|\varphi'(s)-\frac{\varphi'(a)+\varphi'(b)}{2}\right|\right]\int_{a-\varepsilon}^{b}|t-s|\psi(t)\,d\langle E_t x,x\rangle . \end{cases}
\end{aligned}
$$

Taking the limit over $\varepsilon\to 0+$ and using the continuity of φ, ψ and the Spectral Representation Theorem, we deduce the desired result (4.7). □

Corollary 7. *With the assumptions of Theorem 4 we have*

$$
\begin{aligned}
(4.8)\quad 0 &\le \langle \varphi(A)\psi(A)x,x\rangle - \varphi\left(\frac{a+b}{2}\right)\langle \psi(A)x,x\rangle \\
&\qquad - \varphi'\left(\frac{a+b}{2}\right)\left\langle \left(A-\frac{a+b}{2}1_H\right)\psi(A)x,x\right\rangle \\
&\le \left\langle \left(A-\frac{a+b}{2}1_H\right)\left[\varphi'(A)-\varphi'\left(\frac{a+b}{2}\right)1_H\right]\psi(A)x,x\right\rangle \\
&\le \begin{cases} \frac{1}{2}(b-a)\left\langle \left|\varphi'(A)-\varphi'\left(\frac{a+b}{2}\right)1_H\right|\psi(A)x,x\right\rangle ; \\ \\ \left\langle \left|A-\frac{a+b}{2}1_H\right|^p\psi(A)x,x\right\rangle^{1/p}\left(\left\langle \left|\varphi'(A)-\varphi'\left(\frac{a+b}{2}\right)1_H\right|^q\psi(A)x,x\right\rangle\right)^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\frac{1}{2}[\varphi'(b)-\varphi'(a)]+\left|\varphi'\left(\frac{a+b}{2}\right)-\frac{\varphi'(a)+\varphi'(b)}{2}\right|\right]\left\langle \left|A-\frac{a+b}{2}1_H\right|\psi(A)x,x\right\rangle , \end{cases}
\end{aligned}
$$

for all $x \in H$.

If $m \in [a,b]$ *is such that* $\varphi'(m) = \frac{\varphi'(a)+\varphi'(b)}{2}$, *then*

$$
\begin{aligned}
(4.9)\quad 0 &\le \langle \varphi(A)\psi(A)x,x\rangle - \varphi(m)\langle \psi(A)x,x\rangle - \varphi'(m)\langle (A-m1_H)\psi(A)x,x\rangle \\
&\le \langle (A-m1_H)[\varphi'(A)-\varphi'(m)1_H]\psi(A)x,x\rangle \\
&\le \begin{cases} \left[\frac{1}{2}(b-a)+\left|m-\frac{a+b}{2}\right|\right]\langle |\varphi'(A)-\varphi'(m)1_H|\psi(A)x,x\rangle; \\ \\ \langle |A-m1_H|^p\psi(A)x,x\rangle^{1/p}\left(\langle |\varphi'(A)-\varphi'(m)1_H|^q\psi(A)x,x\rangle\right)^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \frac{1}{2}[\varphi'(b)-\varphi'(a)]\langle |A-m1_H|\psi(A)x,x\rangle, \end{cases}
\end{aligned}
$$

for all $x \in H$.

Consider the function $\varphi(t) = -\ln t$ with $t \in [a,b] \subset 0$ and A a bounded selfadjoint operator on the Hilbert space H with $a = \min\{\lambda \,|\lambda \in Sp(A)\}$ and $b = \max\{\lambda \,|\lambda \in Sp(A)\}$. If $\psi(t) = t^r$ with r a real number. Then by (4.7) we have

$$
\begin{aligned}
(4.10)\quad 0 &\le \ln(s)\langle A^r x,x\rangle + s^{-1}\langle (A-s1_H)A^r x,x\rangle - \langle A^r \ln Ax,x\rangle \\
&\le \left\langle (A-s1_H)\left(s^{-1}1_H - A^{-1}\right)A^r x,x\right\rangle \\
&\le \begin{cases} \left[\frac{1}{2}(b-a)+\left|s-\frac{a+b}{2}\right|\right]\left\langle \left|A^{-1}-s^{-1}1_H\right|A^r x,x\right\rangle; \\ \\ \langle |A-s1_H|^p A^r x,x\rangle^{1/p}\left(\left\langle \left|A^{-1}-s^{-1}1_H\right|^q A^r x,x\right\rangle\right)^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\frac{1}{2}\frac{b-a}{ab}+\left|\frac{1}{s}-\frac{a+b}{2ab}\right|\right]\langle |A-s1_H|A^r x,x\rangle, \end{cases}
\end{aligned}
$$

for all $x \in H$.

If we take $s = \frac{a+b}{2}$ in (4.10), then we get

$$
\begin{aligned}
(4.11)\quad 0 &\le \ln\left(\frac{a+b}{2}\right)\langle A^r x,x\rangle + \left(\frac{a+b}{2}\right)^{-1}\left\langle \left(A-\frac{a+b}{2}1_H\right)A^r x,x\right\rangle \\
&- \langle A^r \ln Ax,x\rangle \le \left\langle \left(A-\frac{a+b}{2}1_H\right)\left(\left(\frac{a+b}{2}\right)^{-1}1_H - A^{-1}\right)A^r x,x\right\rangle \\
&\le \begin{cases} \frac{1}{2}(b-a)\left\langle \left|A^{-1}-\left(\frac{a+b}{2}\right)^{-1}1_H\right|A^r x,x\right\rangle; \\ \\ \left\langle \left|A-\frac{a+b}{2}1_H\right|^p A^r x,x\right\rangle^{1/p}\left(\left\langle \left|A^{-1}-\left(\frac{a+b}{2}\right)^{-1}1_H\right|^q A^r x,x\right\rangle\right)^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \frac{b-a}{a(a+b)}\left\langle \left|A-\frac{a+b}{2}1_H\right|A^r x,x\right\rangle, \end{cases}
\end{aligned}
$$

for all $x \in H$.

If we take $s=\frac{2ab}{a+b}$ in (4.10), then we get

$$0\leq\ln\left(\frac{2ab}{a+b}\right)\langle A^{r}x,x\rangle+\left(\frac{2ab}{a+b}\right)^{-1}\left\langle\left(A-\frac{2ab}{a+b}1_{H}\right)A^{r}x,x\right\rangle \tag{4.12}$$

$$-\langle A^{r}\ln Ax,x\rangle\leq\left\langle\left(A-\frac{2ab}{a+b}1_{H}\right)\left(\left(\frac{2ab}{a+b}\right)^{-1}1_{H}-A^{-1}\right)A^{r}x,x\right\rangle$$

$$\leq\begin{cases}\frac{b(b-a)}{a+b}\left\langle\left|A^{-1}-\left(\frac{2ab}{a+b}\right)^{-1}1_{H}\right|A^{r}x,x\right\rangle;\\ \\ \left\langle\left|A-\frac{2ab}{a+b}1_{H}\right|^{p}A^{r}x,x\right\rangle^{1/p}\left(\left\langle\left|A^{-1}-\left(\frac{2ab}{a+b}\right)^{-1}1_{H}\right|^{q}A^{r}x,x\right\rangle\right)^{1/q},\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \frac{b-a}{2ab}\left\langle\left|A-s1_{H}\right|A^{r}x,x\right\rangle,\end{cases}$$

for all $x\in H$.

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