TRAPEZOID TYPE RIEMANN-STIELTJES INTEGRAL INEQUALITIES FOR CONVEX INTEGRANDS AND NONDECREASING INTEGRATORS

SILVESTRU SEVER DRAGOMIR

Abstract. In this paper we obtain some inequalities for the trapezoid difference

$$[u(x)-u(a)]f(a)+[u(b)-u(x)]f(b)-\int_a^b f(t)\,du(t)$$

where f is a convex function on $[a,b]$, u is monotonic nondecreasing and $x\in(a,b)$. In the case of Riemann integral, namely for $u(t)=t$, some particular inequalities are given.

1. Introduction

We start with the following result concerning two inequalities of trapezoid type for convex functions obtained in [6]:

Theorem 1. *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a convex function on $[a,b]$. Then for any $x\in[a,b]$ one has the inequality*

$$\frac{1}{2}\left[(b-x)^2 f'_+(x)-(x-a)^2 f'_-(x)\right] \leq (x-a)f(a)+(b-x)f(b)-\int_a^b f(t)\,dt \leq \frac{1}{2}\left[(b-x)^2 f'_-(b)-(x-a)^2 f'_+(a)\right]. \tag{1.1}$$

The constant $\frac{1}{2}$ is sharp in both inequalities.
The second inequality also holds for $x=a$ or $x=b$.

We have have a simpler first inequality in the case of differentiability:

Corollary 1. *With the assumptions of Lemma 1 and if $x\in(a,b)$ is a point of differentiability for f, then*

$$\left(\frac{a+b}{2}-x\right)(b-a)f'(x)\leq(x-a)f(a)+(b-x)f(b)-\int_a^b f(t)\,dt. \tag{1.2}$$

Now, recall that the following inequality, which is well known in the literature as the Hermite-Hadamard inequality for convex functions, holds

$$f\left(\frac{a+b}{2}\right)(b-a)\leq\int_a^b f(t)\,dt\leq\frac{f(a)+f(b)}{2}(b-a). \tag{1.3}$$

1991 *Mathematics Subject Classification.* 26D15, 41A55, 47A63.

Key words and phrases. Riemann-Stieltjes integral, Trapezoid inequality.

RGMIA Res. Rep. Coll. 21 (2018), Art. 97, 13 pp. Received 19/08/18

The following corollary provides some sharp bounds for the trapezoid difference

$$\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt.$$

Corollary 2. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then we have the inequality*

$$\begin{aligned}(1.4)\quad 0 &\le \frac{1}{8}\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right](b-a)^2\\ &\le \frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt\\ &\le \frac{1}{8}\left[f'_-(b)-f'_+(a)\right](b-a)^2.\end{aligned}$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

For various trapezoid type inequalities involving Riemann-Stieltjes integral, see [1]-[12] and [8]-[16].

Motivated by the above results, in this paper we obtain some inequalities for the Riemann-Stieltjes integral trapezoid difference

$$[u(x)-u(a)]f(a)+[u(b)-u(x)]f(b)-\int_a^b f(t)\,du(t)$$

where f is a convex function on $[a,b]$, u is monotonic nondecreasing and $x\in(a,b)$. In the case of Riemann integral, namely for $u(t)=t$, some particular inequalities are also given.

2. The Main Results

Theorem 2. *Assume that $f:[a,b]\to\mathbb{R}$ is continuous convex on $[a,b]$ and $u:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then*

$$\begin{aligned}(2.1)\quad &-\int_x^b (b-t)\left[f'_-(b)-f'_-(t)\right]du(t)-\int_a^x (t-a)\left[f'_+(t)-f_+(a)\right]du(t)\\ &+f'_-(b)\left[\int_x^b u(t)\,dt-(b-x)\,u(x)\right]+f_+(a)\left[\int_a^x u(t)\,dt-(x-a)\,u(x)\right]\\ &\le [u(x)-u(a)]f(a)+[u(b)-u(x)]f(b)-\int_a^b f(t)\,du(t)\\ &\le f'_-(b)\left[\int_x^b u(t)\,dt-(b-x)\,u(x)\right]+f'_+(a)\left[\int_a^x u(t)\,dt-(x-a)\,u(x)\right]\end{aligned}$$

for $x\in(a,b)$, provided the Riemann-Stieltjes integrals $\int_x^b (b-t)\,f'_-(t)\,du(t)$ and $\int_a^x (t-a)\,f'_+(t)\,du(t)$ exist.

This is equivalent to the inequality

$$0 \leq \int_a^b f(t)\,du(t) - [u(x) - u(a)]\,f(a) - [u(b) - u(x)]\,f(b) \tag{2.2}$$
$$+ f'_-(b)\left[\int_x^b u(t)\,dt - (b-x)\,u(x)\right] + f_+(a)\left[\int_a^x u(t)\,dt - (x-a)\,u(x)\right]$$
$$\leq \int_x^b (b-t)\left[f'_-(b) - f'_-(t)\right]du(t) + \int_a^x (t-a)\left[f'_+(t) - f_+(a)\right]du(t)$$

for $x \in (a,b)$.

Proof. For any $x \in (a,b)$ we have

$$B(f,u,x) := \int_x^b [f(b) - f(t)]\,du(t) - \int_a^x [f(t) - f(a)]\,du(t) \tag{2.3}$$
$$= [u(x) - u(a)]\,f(a) + [u(b) - u(x)]\,f(b) - \int_a^b f(t)\,du(t).$$

Since f is convex, hence by the *gradient inequality*, we have

$$f(b) - f(t) \leq (b-t)\,f'_-(b) \text{ for } t \in [x,b]$$

and

$$f(t) - f(a) \geq (t-a)\,f'_+(a) \text{ for } [a,x].$$

Since u is monotonic nondecreasing, it follows by using integration by parts that

$$\int_x^b [f(b) - f(t)]\,du(t) \leq f'_-(b)\int_x^b (b-t)\,du(t) \tag{2.4}$$
$$= f'_-(b)\left[\int_x^b u(t)\,dt - (b-x)\,u(x)\right]$$

and

$$\int_a^x [f(t) - f(a)]\,du(t) \geq f'_+(a)\int_a^x (t-a)\,du(t)$$
$$= f'_+(a)\left[(x-a)\,u(x) - \int_a^x u(t)\,dt\right],$$

which is equivalent to

$$-\int_a^x [f(t) - f(a)]\,du(t) \leq f'_+(a)\left[\int_a^x u(t)\,dt - (x-a)\,u(x)\right] \tag{2.5}$$

for any $x \in (a,b)$.

If we add (2.4) with (2.5), then we get

$$B(f,u,x)$$
$$\leq f'_-(b)\left[\int_x^b u(t)\,dt - (b-x)\,u(x)\right] + f'_+(a)\left[\int_a^x u(t)\,dt - (x-a)\,u(x)\right]$$

for any $x \in (a,b)$.

This proves the second inequality in (2.1).

By the gradient inequality we also have

$$f(b)-f(t)\geq(b-t)f'_{-}(t)\text{ for }t\in[x,b]$$

and

$$f(t)-f(a)\leq(t-a)f'_{+}(t)\text{ for }t\in[a,x].$$

This imply that

$$\int_{x}^{b}[f(b)-f(t)]\,du(t)\geq\int_{x}^{b}(b-t)f'_{-}(t)\,du(t)$$

and

$$-\int_{a}^{x}[f(t)-f(a)]\,du(t)\geq-\int_{a}^{x}(t-a)f'_{+}(t)\,du(t)$$

for any $x\in(a,b)$.

If we add these two inequalities, we get

$$\begin{aligned}
&B(f,u,x)\\
&\geq-\int_{x}^{b}(t-b)f'_{-}(t)\,du(t)-\int_{a}^{x}(t-a)f'_{+}(t)\,du(t)\\
&=-\int_{x}^{b}(t-b)\left[f'_{-}(t)-f'_{-}(b)\right]du(t)-\int_{a}^{x}(t-a)\left[f'_{+}(t)-f_{+}(a)\right]du(t)\\
&-f'_{-}(b)\int_{x}^{b}(t-b)\,du(t)-f_{+}(a)\int_{a}^{x}(t-a)\,du(t)\\
&\\
&=-\int_{x}^{b}(b-t)\left[f'_{-}(b)-f'_{-}(t)\right]du(t)-\int_{a}^{x}(t-a)\left[f'_{+}(t)-f_{+}(a)\right]du(t)\\
&+f'_{-}(b)\int_{x}^{b}(b-t)\,du(t)-f_{+}(a)\int_{a}^{x}(t-a)\,du(t)\\
&=f_{-}(b)\left[\int_{x}^{b}u(t)\,dt-(b-x)\,u(x)\right]+f_{+}(a)\left[\int_{a}^{x}u(t)\,dt-(x-a)\,u(x)\right]\\
&-\int_{x}^{b}(b-t)\left[f'_{-}(b)-f'_{-}(t)\right]du(t)-\int_{a}^{x}(t-a)\left[f'_{+}(t)-f_{+}(a)\right]du(t),
\end{aligned}$$

which proves the first inequality in (2.1). □

Remark 1. *We observe that a sufficient condition for the Riemann-Stieltjes integrals* $\int_{x}^{b}(b-t)f'_{-}(t)\,du(t)$ *and* $\int_{a}^{x}(t-a)f'_{+}(t)\,du(t)$ *to exist is either* u *is monotonic nondecreasing and continuous on* $[a,b]$ *or* f *is convex and has a continuous derivative on an open interval containing* $[a,b]$.

In what follows, we assume that the involved Riemann-Stieltjes integrals exist.

Remark 2. *If we take* $x=\frac{a+b}{2}$ *in (2.2), then we get*

$$0\leq\int_a^b f(t)\,du(t)-\left[u\left(\frac{a+b}{2}\right)-u(a)\right]f(a)-\left[u(b)-u\left(\frac{a+b}{2}\right)\right]f(b)$$
$$+f'_-(b)\left[\int_{\frac{a+b}{2}}^b u(t)\,dt-\frac{1}{2}(b-a)\,u\left(\frac{a+b}{2}\right)\right]$$
$$+f_+(a)\left[\int_a^{\frac{a+b}{2}} u(t)\,dt-\frac{1}{2}(b-a)\,u\left(\frac{a+b}{2}\right)\right]$$
$$\leq\int_{\frac{a+b}{2}}^b (b-t)\left[f'_-(b)-f'_-(t)\right]du(t)+\int_a^{\frac{a+b}{2}}(t-a)\left[f'_+(t)-f_+(a)\right]du(t). \tag{2.6}$$

If $q\in(a,b)$ *is such that* $u(q)=\frac{u(a)+u(b)}{2}$*, then by (2.2) we get*

$$0\leq\int_a^b f(t)\,du(t)-[u(b)-u(a)]\frac{f(a)+f(b)}{2}$$
$$+f'_-(b)\left[\int_q^b u(t)\,dt-(b-q)\,u(q)\right]+f_+(a)\left[\int_a^q u(t)\,dt-(q-a)\,u(q)\right]$$
$$\leq\int_q^b (b-t)\left[f_-(b)-f'_-(t)\right]du(t)+\int_a^q (t-a)\left[f'_+(t)-f_+(a)\right]du(t). \tag{2.7}$$

Corollary 3. *With the assumptions of Theorem 2, we have*

$$0\leq\int_a^b f(t)\,du(t)-[u(x)-u(a)]\,f(a)-[u(b)-u(x)]\,f(b)$$
$$+f'_-(b)\left[\int_x^b u(t)\,dt-(b-x)\,u(x)\right]+f_+(a)\left[\int_a^x u(t)\,dt-(x-a)\,u(x)\right]$$
$$\leq\begin{cases}(b-x)\int_x^b\left[f'_-(b)-f'_-(t)\right]du(t)\\ \\ \left(\int_x^b(b-t)^p\,du(t)\right)^{1/p}\left(\int_x^b\left[f'_-(b)-f'_-(t)\right]^q du(t)\right)^{1/q}\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \\ \left[f'_-(b)-f'_-(x)\right]\int_x^b(b-t)\,du(t)\end{cases}$$
$$+\begin{cases}(x-a)\int_a^x\left[f'_+(t)-f_+(a)\right]du(t)\\ \\ \left(\int_a^x(t-a)^p\,du(t)\right)^{1/p}\left(\int_a^x\left[f'_+(t)-f_+(a)\right]^q du(t)\right)^{1/q}\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \\ \left[f'_+(x)-f_+(a)\right]\int_a^x(t-a)\,du(t)\end{cases} \tag{2.8}$$

for $x\in(a,b)$.

Moreover, if f *is differentiable, then* f'_- *and* f'_+ *can be replaced by* f' *for the interior points.*

Proof. Since f is convex on $[a,b]$, hence f'_- and f'_+ are monotonic nondecreasing and by Hölder's inequality for Riemann-Stieltjes integral with monotonic nondecreasing integrators

$$\int_x^b (b-t)\left[f'_-(b)-f'_-(t)\right]du(t)+\int_a^x (t-a)\left[f'_+(t)-f'_+(a)\right]du(t)$$
$$\leq \begin{cases} (b-x)\int_x^b \left[f'_-(b)-f'_-(t)\right]du(t), \\ \\ \left(\int_x^b (b-t)^p\,du(t)\right)^{1/p}\left(\int_x^b \left[f'_-(b)-f'_-(t)\right]^q du(t)\right)^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left[f'_-(b)-f'_-(x)\right]\int_x^b (b-t)\,du(t) \end{cases}$$
$$+\begin{cases} (x-a)\int_a^x \left[f'_+(t)-f'_+(a)\right]du(t) \\ \\ \left(\int_a^x (t-a)^p\,du(t)\right)^{1/p}\left(\int_a^x \left[f'_+(t)-f'_+(a)\right]^q du(t)\right)^{1/q} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left[f'_+(x)-f_+(a)\right]\int_a^x (t-a)\,du(t) \end{cases}$$

for $x\in(a,b)$. □

Remark 3. *If f is differentiable convex on (a,b), then*

$$\int_x^b \left[f'_-(b)-f'(t)\right]du(t)=f'_-(b)\left[u(b)-u(x)\right]-\int_x^b f'(t)\,du(t)$$

and

$$\int_a^x \left[f'(t)-f'_+(a)\right]du(t)=\int_a^x f'(t)\,du(t)-f'_+(a)\left[u(x)-u(a)\right]$$

then

$$C(f,u,x):=\int_x^b \left[f'_-(b)-f'(t)\right]du(t)+\int_a^x \left[f'(t)-f'_+(a)\right]du(t) \tag{2.9}$$
$$=f'_-(b)\left[u(b)-u(x)\right]-f'_+(a)\left[u(x)-u(a)\right]+\int_a^x f'(t)\,du(t)-\int_x^b f'(t)\,du(t)$$
$$=f'_-(b)\left[u(b)-u(x)\right]-f'_+(a)\left[u(x)-u(a)\right]+\int_a^b \operatorname{sgn}(x-t)\,f'(t)\,du(t)$$

and

$$(2.10)\quad D(f,u,x):=\max\left\{\int_x^b\left[f'_-(b)-f'(t)\right]du(t),\int_a^x\left[f'(t)-f'_+(a)\right]du(t)\right\}$$
$$=\frac{1}{2}\left\{\int_x^b\left[f'_-(b)-f'(t)\right]du(t)+\int_a^x\left[f'(t)-f'_+(a)\right]du(t)\right\}$$
$$+\frac{1}{2}\left|\int_x^b\left[f'_-(b)-f'(t)\right]du(t)-\int_a^x\left[f'(t)-f'_+(a)\right]du(t)\right|$$
$$=\frac{1}{2}\left\{f'_-(b)\left[u(b)-u(x)\right]-f'_+(a)\left[u(x)-u(a)\right]+\int_a^b\operatorname{sgn}(x-t)f'(t)\,du(t)\right\}$$
$$+\frac{1}{2}\left|f'_-(b)\left[u(b)-u(x)\right]-\int_x^b f'(t)\,du(t)-\int_a^x f'(t)\,du(t)+f'_+(a)\left[u(x)-u(a)\right]\right|$$
$$=\frac{1}{2}\left\{f'_-(b)\left[u(b)-u(x)\right]-f'_+(a)\left[u(x)-u(a)\right]+\int_a^b\operatorname{sgn}(x-t)f'(t)\,du(t)\right\}$$
$$+\frac{1}{2}\left|f'_-(b)\left[u(b)-u(x)\right]+f'_+(a)\left[u(x)-u(a)\right]-\int_a^b f'(t)\,du(t)\right|$$

for $x\in(a,b)$.

Therefore,

$$(b-x)\int_x^b\left[f'_-(b)-f'_-(t)\right]du(t)+(x-a)\int_a^x\left[f'_+(t)-f'_+(a)\right]du(t)\le\begin{cases}\max\{b-x,x-a\}\,C(f,u,x)\\[2mm](b-a)\,D(f,u,x)\end{cases}$$

and by (2.8) we get

$$(2.11)\quad 0\le\int_a^b f(t)\,du(t)-\left[u(x)-u(a)\right]f(a)-\left[u(b)-u(x)\right]f(b)$$
$$+f'_-(b)\left[\int_x^b u(t)\,dt-(b-x)\,u(x)\right]+f'_+(a)\left[\int_a^x u(t)\,dt-(x-a)\,u(x)\right]$$
$$\le\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]C(f,u,x),\\[2mm](b-a)\,D(f,u,x),\end{cases}$$

where $C(f,u,x)$ *is defined by (2.9) while* $D(f,u,x)$ *is defined by (2.10).*

We also have

$$\int_x^b(b-t)\,du(t)=\int_x^b u(t)\,dt-(b-x)\,u(x)$$

and

$$\int_a^x(t-a)\,du(t)=(x-a)\,u(x)-\int_a^x u(t)\,dt$$

for $x\in(a,b)$.

Then

$$E(f,u,x) := (x-a)u(x) - \int_a^x u(t)\,dt + \int_x^b u(t)\,dt - (b-x)u(x) \tag{2.12}$$
$$= (2x-a-b)u(x) + \int_x^b u(t)\,dt - \int_a^x u(t)\,dt$$
$$= (2x-a-b)u(x) + \int_a^b \operatorname{sgn}(t-x)\,u(t)\,dt$$

and

$$F(f,u,x) := \max\left\{(x-a)u(x) - \int_a^x u(t)\,dt, \int_x^b u(t)\,dt - (b-x)u(x)\right\} \tag{2.13}$$
$$= \frac{1}{2}\left\{(2x-a-b)u(x) + \int_a^b \operatorname{sgn}(t-x)\,u(t)\,dt\right\}$$
$$+ \frac{1}{2}\left|(x-a)u(x) - \int_a^x u(t)\,dt - \int_x^b u(t)\,dt + (b-x)u(x)\right|$$
$$= \left(x - \frac{a+b}{2}\right)u(x) + \frac{1}{2}\int_a^b \operatorname{sgn}(t-x)\,u(t)\,dt$$
$$+ \frac{1}{2}\left|(b-a)u(x) - \int_a^b u(t)\,dt\right|$$

Therefore,

$$\left[f'_-(b) - f'(x)\right]\int_x^b (b-t)\,du(t) + \left[f'(x) - f'_+(a)\right]\int_a^x (t-a)\,du(t)$$
$$\leq \begin{cases} \max\left\{f'_-(b) - f'(x), f'(x) - f'_+(a)\right\} E(f,u,x) \\ \\ \left[f'_-(b) - f'_+(a)\right] F(f,u,x) \end{cases}$$

and by (2.8) we get

$$0 \leq \int_a^b f(t)\,du(t) - [u(x) - u(a)]f(a) - [u(b) - u(x)]f(b) \tag{2.14}$$
$$+ f'_-(b)\left[\int_x^b u(t)\,dt - (b-x)u(x)\right] + f'_+(a)\left[\int_a^x u(t)\,dt - (x-a)u(x)\right]$$
$$\leq \begin{cases} \left[\frac{1}{2}\left(f'_-(b) - f'_+(a)\right) + \left|f'(x) - \frac{f'_+(a)+f'_-(b)}{2}\right|\right] E(f,u,x), \\ \\ \left[f'_-(b) - f'_+(a)\right] F(f,u,x), \end{cases}$$

for $x \in (a,b)$, where $E(f,u,x)$ is defined by (2.12) while $F(f,u,x)$ is defined by (2.13).

Corollary 4. *Let I an interval and $\mathring{I}$, the interior of I. Assume that $f : I \to \mathbb{R}$ is convex on I, differentiable and with the derivative f' continuous on $\mathring{I}$ and $[a,b] \subset \mathring{I}$.*

If $g:[a,b]\to[0,\infty)$ *is continuous and* $v:[a,b]\to\mathbb{R}$ *is monotonic nondecreasing, then*

$$\begin{aligned}(2.15)\quad 0 &\leq \int_a^b f(t)\,g(t)\,dv(t)-f(a)\int_a^x g(t)\,dv(t)-f(b)\int_x^b g(t)\,dv(t)\\ &\quad+f'_-(b)\int_x^b (b-t)\,g(t)\,dv(t)+f_+(a)\int_a^x (t-a)\,g(t)\,dv(t)\\ &\leq \int_x^b (b-t)\left[f'_-(b)-f'(t)\right]g(t)\,dv(t)+\int_a^x (t-a)\left[f'(t)-f_+(a)\right]g(t)\,dv(t)\\ &\leq (b-x)\left[f'_-(b)-f'(x)\right]\int_x^b g(t)\,dv(t)+(x-a)\left[f'(x)-f_+(a)\right]\int_a^x g(t)\,dv(t)\\ &\leq \max\left\{(b-x)\left[f'_-(b)-f'(x)\right],(x-a)\left[f'(x)-f_+(a)\right]\right\}\int_a^b g(t)\,dv(t)\end{aligned}$$

for $x\in(a,b)$.

Proof. Using the properties of Riemann-Stieltjes integral with integrators u given by an integral, namely, if

$$u(t)=\int_a^t g(s)\,dv(s),$$

which is monotonic nondecreasing on $[a,b]$, then

$$\int_a^b f(t)\,du(t)=\int_a^b f(t)\,g(t)\,dv(t),\quad \int_x^b (b-t)\,du(t)=\int_x^b (b-t)\,g(t)\,dv(t),$$

$$\int_a^x (t-a)\,du(t)=\int_a^x (t-a)\,g(t)\,dv(t)$$

$$\int_x^b (b-t)\left[f'_-(b)-f'(t)\right]du(t)=\int_x^b (b-t)\left[f'_-(b)-f'(t)\right]g(t)\,dv(t)$$

and

$$\int_a^x (t-a)\left[f'(t)-f_+(a)\right]du(t)=\int_a^x (t-a)\left[f'(t)-f_+(a)\right]g(t)\,dv(t)$$

and by (2.2) for differentiable functions, we get the first part of (2.15). The second part is obvious. □

3. Inequalities for Riemann Integral

Consider the function $u(t)=t$, $t\in[a,b]$. Then

$$\int_x^b u(t)\,dt-(b-x)\,u(x)=\frac{1}{2}\left(b^2-x^2\right)-(b-x)\,x=\frac{1}{2}(b-x)^2,$$

$$\int_a^x u(t)\,dt-(x-a)\,u(x)=\frac{1}{2}\left(x^2-a^2\right)-(x-a)\,x=-\frac{1}{2}(x-a)^2,$$

$$\begin{aligned}\int_x^b (b-t)\left[f'_-(b)-f'(t)\right]dt&=f'_-(b)\int_x^b (b-t)\,dt-\int_x^b (b-t)\,f'(t)\,dt\\ &=\frac{1}{2}f'_-(b)\,(b-x)^2+(b-x)\,f(x)-\int_x^b f(t)\,dt\end{aligned}$$

and

$$\int_a^x (t-a)\left[f'(t)-f_+(a)\right]du(t) = \int_a^x (t-a) f'(t)\,dt - f_+(a)\int_a^x (t-a)\,dt$$
$$= (x-a) f(x) - \frac{1}{2} f_+(a)(x-a)^2 - \int_a^x f(t)\,dt$$

for $x \in (a,b)$.

By utilising the inequality (2.1) for convex functions, we have

$$-\frac{1}{2} f'_-(b)(b-x)^2 - (b-x) f(x) + \int_x^b f(t)\,dt - (x-a) f(x)$$
$$+\frac{1}{2} f_+(a)(x-a)^2 + \int_a^x f(t)\,dt + \frac{1}{2} f'_-(b)(b-x)^2 - \frac{1}{2} f_+(a)(x-a)^2$$
$$\leq (x-a) f(a) + (b-x) f(b) - \int_a^b f(t)\,dt$$
$$\leq \frac{1}{2} f'_-(b)(b-x)^2 - \frac{1}{2} f'_+(a)(x-a)^2,$$

which is equivalent to

$$\int_a^b f(t)\,dt - (b-a) f(x) \tag{3.1}$$
$$\leq (x-a) f(a) + (b-x) f(b) - \int_a^b f(t)\,dt$$
$$\leq \frac{1}{2}\left[f'_-(b)(b-x)^2 - f'_+(a)(x-a)^2\right]$$

for $x \in (a,b)$.

In particular, if we take in (3.1) $x = \frac{a+b}{2}$, then we get

$$0 \leq \int_a^b f(t)\,dt - (b-a) f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)\,dt \tag{3.2}$$
$$\leq \frac{1}{8}(b-a)^2\left[f'_-(b) - f'_+(a)\right],$$

with $\frac{1}{8}$ as best possible constant.

If we write the inequality (2.2) for differentiable convex functions and $u(t) = t$, then we get

$$0 \leq \int_a^b f(t)\,dt - (x-a) f(a) - (b-x) f(b) \tag{3.3}$$
$$+\frac{1}{2}\left[f'_-(b)(b-x)^2 - f_+(a)(x-a)^2\right]$$
$$\leq \int_x^b (b-t)\left[f'_-(b) - f'_-(t)\right]dt + \int_a^x (t-a)\left[f'_+(t) - f_+(a)\right]dt$$

for $x \in (a,b)$, which, in particular, gives

$$(3.4)\quad 0 \le \int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a) + \frac{1}{8}\left[f'_-(b) - f_+(a)\right](b-a)^2$$
$$\le \int_{\frac{a+b}{2}}^b (b-t)\left[f'_-(b) - f'_-(t)\right]dt + \int_a^{\frac{a+b}{2}} (t-a)\left[f'_+(t) - f_+(a)\right]dt.$$

Since f is convex, hence we have

$$0 \le f'_-(b) - f'_-(t) \le f'_-(b) - f'_-(x) \text{ for } t \in [x,b]$$

and

$$0 \le f'_+(t) - f_+(a) \le f'_+(x) - f_+(a) \text{ for } t \in [a,x],$$

which gives

$$\int_x^b (b-t)\left[f'_-(b) - f'_-(t)\right]dt + \int_a^x (t-a)\left[f'_+(t) - f_+(a)\right]dt$$
$$\le \left[f'_-(b) - f'_-(x)\right]\int_x^b (b-t)\,dt + \left[f'_+(x) - f_+(a)\right]\int_a^x (t-a)\,dt$$
$$= \frac{1}{2}\left\{\left[f'_-(b) - f'_-(x)\right](b-x)^2 + \left[f'_+(x) - f_+(a)\right](x-a)^2\right\}$$

for $x \in (a,b)$.

By (3.3) we then get

$$(3.5)\quad 0 \le \int_a^b f(t)\,dt - (x-a)f(a) - (b-x)f(b)$$
$$+ \frac{1}{2}\left[f'_-(b)(b-x)^2 - f_+(a)(x-a)^2\right]$$
$$\le \frac{1}{2}\left\{\left[f'_-(b) - f'_-(x)\right](b-x)^2 + \left[f'_+(x) - f_+(a)\right](x-a)^2\right\}$$

for $x \in (a,b)$.

Now, if there exists the constants L_a, $L_b > 0$ and $q > -1$ such that

$$0 \le f'_-(b) - f'_-(t) \le L_b(b-t)^q \text{ for } t \in [x,b]$$

and

$$0 \le f'_+(t) - f_+(a) \le L_a(t-a)^q \text{ for } t \in [a,x],$$

then

$$\int_x^b (b-t)\left[f'_-(b) - f'_-(t)\right]dt + \int_a^x (t-a)\left[f'_+(t) - f_+(a)\right]dt$$
$$\le L_b\int_x^b (b-t)^{q+1}\,dt + L_a\int_a^x (t-a)^{q+1}\,dt$$
$$= \frac{1}{q+2}\left[L_b(b-x)^{q+2} + L_a(x-a)^{q+2}\right]$$

and by (3.3) we get

$$0\leq\int_{a}^{b}f(t)\,dt-(x-a)\,f(a)-(b-x)\,f(b)+\frac{1}{2}\left[f_{-}'(b)\,(b-x)^{2}-f_{+}(a)\,(x-a)^{2}\right]\leq\frac{1}{q+2}\left[L_{b}\,(b-x)^{q+2}+L_{a}\,(x-a)^{q+2}\right]\tag{3.6}$$

for $x\in(a,b)$.

If we take $x=\frac{a+b}{2}$ in (3.6), then we get

$$0\leq\int_{a}^{b}f(t)\,dt-\frac{f(a)+f(b)}{2}(b-a)+\frac{1}{8}\left[f_{-}'(b)-f_{+}(a)\right](b-a)^{2}\leq\frac{L_{b}+L_{a}}{(q+2)\,2^{q+2}}(b-a)^{q+2}.\tag{3.7}$$

If $q=1$ and $L=L_{b}=L_{a}$, then (3.6) becomes

$$0\leq\int_{a}^{b}f(t)\,dt-(x-a)\,f(a)-(b-x)\,f(b)+\frac{1}{2}\left[f_{-}'(b)\,(b-x)^{2}-f_{+}(a)\,(x-a)^{2}\right]\leq\frac{L}{3}\left[(b-x)^{3}+(x-a)^{3}\right]\tag{3.8}$$

for $x\in(a,b)$ and in particular

$$0\leq\int_{a}^{b}f(t)\,dt-\frac{f(a)+f(b)}{2}(b-a)+\frac{1}{8}\left[f_{-}'(b)-f_{+}(a)\right](b-a)^{2}\leq\frac{L}{12}(b-a)^{3}.\tag{3.9}$$

References

[1] M. W. Alomari, New sharp Ostrowski-type inequalities and generalized trapezoid-type inequalities for Riemann-Stieltjes integrals and their applications. *Ukrainian Math. J.* **65** (2013), no. 7, 995–1018.

[2] A. Asanov, M. Haluk Chelik and A. Chalish, Approximating the Stieltjes integral by using the generalized trapezoid rule. *Matematiche* (Catania) **66** (2011), no. 2, 13–21.

[3] N. S. Barnett, W.-S. Cheung, S. S. Dragomir and A. Sofo, Ostrowski and trapezoid type inequalities for the Stieltjes integral with Lipschitzian integrands or integrators. *Comput. Math. Appl.* **57** (2009), no. 2, 195–201.

[4] P. Cerone and S. S. Dragomir, Sharp error bounds in approximating the Riemann-Stieltjes integral by a generalised trapezoid formula and applications. *J. Inequal. Appl.* **2013**, 2013:53, 11 pp.

[5] S. S. Dragomir, Some inequalities of midpoint and trapezoid type for the Riemann-Stieltjes integral. *Proceedings of the Third World Congress of Nonlinear Analysts*, Part 4 (Catania, 2000). *Nonlinear Anal.* **47** (2001), no. 4, 2333–2340.

[6] S. S. Dragomir, An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products. *J. Inequal. Pure Appl. Math.* **3** (2002), no. 3, Article 35, 8 pp.

[7] S. S. Dragomir, Approximating the Riemann-Stieltjes integral in terms of generalised trapezoidal rules. *Nonlinear Anal.* **71** (2009), no. 12, e62–e72.

[8] S. S. Dragomir, Approximating the Riemann-Stieltjes integral by a trapezoidal quadrature rule with applications. *Math. Comput. Modelling* **54** (2011), no. 1-2, 243–260.
[9] S. S. Dragomir, Trapezoidal type inequalities for Riemann-Stieltjes integral via Čebyšev functional with applications. *Nihonkai Math. J.* **26** (2015), no. 1, 47–69.
[10] S. S. Dragomir, C. Buşe, M. V Boldea and L. Braescu,A generalization of the trapezoidal rule for the Riemann-Stieltjes integral and applications. *Nonlinear Anal. Forum* **6** (2001), no. 2, 337–351.
[11] S. S. Dragomir, Y. J. Cho, Y. H. Kim, On the trapezoid inequality for the Riemann-Stieltjes integral with Hölder continuous integrands and bounded variation integrators. *Inequality theory and applications.* Vol. **5**, 71–79, Nova Sci. Publ., New York, 2007.
[12] S. S. Dragomir and I. Fedotov, Approximating the Stieltjes integral via a weighted trapezoidal rule with applications. *Math. Comput. Modelling* **57** (2013), no. 3-4, 602–611.
[13] P. R. Mercer, Hadamard's inequality and trapezoid rules for the Riemann-Stieltjes integral. J. Math. Anal. Appl. 344 (2008), no. 2, 921–926.
[14] W. Zhao and Z. Zhang, Derivative-based trapezoid rule for the Riemann-Stieltjes integral. *Math. Probl. Eng.* **2014**, Art. ID 874651, 6 pp.
[15] W. Zhao, Z. Zhang and Z. Ye, Midpoint derivative-based trapezoid rule for the Riemann-Stieltjes integral. *Ital. J. Pure Appl. Math.* No. **33** (2014), 369–376.
[16] W. Zhao, Z. Zhang and Z. Ye, Composite trapezoid rule for the Riemann-Stieltjes integral and its Richardson extrapolation formula. Ital. J. Pure Appl. Math. No. 35 (2015), 311–318

Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir/

DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences,, School of Computer Science and Applied Mathematics, University of the Witwatersrand, Private Bag-3, Wits-2050, Johannesburg, South Africa