ON SOME OSTROWSKI TYPE RIEMANN-STIELTJES INTEGRAL INEQUALITIES FOR MONOTONIC NONDECREASING INTEGRANDS AND CONVEX INTEGRATORS

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Abstract. In this paper we obtain some inequalities for the Riemann-Stieltjes integral Ostrowski difference

$$\int_a^b f(t)\,du(t) - f(x)[u(b)-u(a)],$$

where f is a monotonic nondecreasing function on $[a,b]$, u is continuous convex on $[a,b]$ and $x \in (a,b)$. Some particular inequalities in the case of Riemann integral are provided as well.

1. Introduction

We recall the following Ostrowski type inequality for convex functions:

Theorem 1 (Dragomir, 2002 [5])**.** *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a convex function on $[a,b]$. Then for any $x\in(a,b)$ one has the inequality*

$$\frac{1}{2}\left[(b-x)^2 f'_+(x) - (x-a)^2 f'_-(x)\right] \le \int_a^b f(t)\,dt - (b-a)f(x) \le \frac{1}{2}\left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a)\right]. \tag{1.1}$$

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $x=a$ or $x=b$.

Corollary 1. *With the assumptions of Theorem 1 and if $x\in(a,b)$ is a point of differentiability for f, then*

$$\left(\frac{a+b}{2}-x\right)f'(x) \le \frac{1}{b-a}\int_a^b f(t)\,dt - f(x). \tag{1.2}$$

The following corollary provides both a sharper lower bound for the Hermite-Hadamard difference,

$$\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right),$$

which we know is nonnegative, and an upper bound [5].

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Corollary 2. *Let $f : [a, b] \to \mathbb{R}$ be a convex function on $[a, b]$. Then we have the inequality*

$$\begin{aligned}(1.3)\quad 0 &\leq \frac{1}{8}\left[f'_{+}\left(\frac{a+b}{2}\right) - f'_{-}\left(\frac{a+b}{2}\right)\right](b-a) \\ &\leq \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \leq \frac{1}{8}\left[f'_{-}(b) - f'_{+}(a)\right](b-a).\end{aligned}$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

For other related results see [7] and [8]. For more inequalities of Ostrowski type, see [1], [2]-[4], [9], [11], [12] and [13].

Motivated by the above results, we establish in this paper some inequalities for the Riemann-Stieltjes integral *Ostrowski difference*

$$\int_a^b f(t)\,du(t) - f(x)\left[u(b) - u(a)\right],$$

where f is a monotonic nondecreasing function on $[a, b]$ and u is convex and $x \in (a, b)$. Some applications for Riemann integral are given as well.

2. Main Results

We have the following main result:

Theorem 2. *Assume that $f : [a, b] \to \mathbb{R}$ is monotonic nondecreasing and $u : [a, b] \to \mathbb{R}$ is continuous convex on $[a, b]$. Then for $x \in (a, b)$ we have the inequalities*

$$\begin{aligned}(2.1)\quad & u'_{+}(a)\left[(x-a)f(x) - \int_a^x f(t)\,dt\right] + u'_{-}(b)\left[(b-x)f(x) - \int_x^b f(t)\,dt\right] \\ &\leq \left[u(b) - u(a)\right]f(x) - \int_a^b f(t)\,du(t) \\ &\leq \int_a^x \left[u'_{+}(t) - u'_{+}(a)\right](t-a)\,df(t) + \int_x^b \left[u'_{-}(t) - u'_{-}(b)\right](t-b)\,df(t) \\ &+ u'_{+}(a)\left[(x-a)f(x) - \int_a^x f(t)\,dt\right] + u'_{-}(b)\left[(b-x)f(x) - \int_x^b f(t)\,dt\right],\end{aligned}$$

provided the Riemann-Stieltjes integrals $\int_a^x u'_{+}(t)(t-a)\,df(t)$ and $\int_x^b u'_{-}(t)(t-b)\,df(t)$ exist.

This is equivalent to

$$\begin{aligned}(2.2)\quad 0 &\leq \left[u(b) - u(a)\right]f(x) \\ &- u'_{+}(a)\left[(x-a)f(x) - \int_a^x f(t)\,dt\right] - u'_{-}(b)\left[(b-x)f(x) - \int_x^b f(t)\,dt\right] \\ &- \int_a^b f(t)\,du(t) \\ &\leq \int_a^x \left[u'_{+}(t) - u'_{+}(a)\right](t-a)\,df(t) + \int_x^b \left[u'_{-}(t) - u'_{-}(b)\right](t-b)\,df(t).\end{aligned}$$

Proof. Using the integration by parts rule for the Riemann-Stieltjes integral, we have, see [3]

$$\begin{aligned} &\int_a^x [u(t)-u(a)]\,df(t)+\int_x^b [u(t)-u(b)]\,df(t) \quad (2.3)\\ &= [u(x)-u(a)]\,f(x)-\int_a^x f(t)\,du(t)+[u(b)-u(x)]\,f(x)-\int_x^b f(t)\,du(t)\\ &= [u(b)-u(a)]\,f(x)-\int_a^b f(t)\,du(t) \end{aligned}$$

for $x\in(a,b)$.

Using the gradient inequality we have

$$u(t)-u(a)\ge u'_+(a)(t-a) \text{ for } t\in[a,x]$$

and

$$u(b)-u(t)\le u'_-(b)(b-t) \text{ for } t\in[x,b].$$

Since f is monotonic nondecreasing and by using integration by parts we get

$$\begin{aligned} \int_a^x [u(t)-u(a)]\,df(t) &\ge u'_+(a)\int_a^x (t-a)\,df(t) \quad (2.4)\\ &= u'_+(a)\left[(x-a)\,f(x)-\int_a^x f(t)\,dt\right] \end{aligned}$$

and

$$\begin{aligned} \int_x^b [u(b)-u(t)]\,df(t) &\le u'_-(b)\int_x^b (b-t)\,df(t)\\ &= u'_-(b)\left[\int_x^b f(t)\,dt-(b-x)\,f(x)\right], \end{aligned}$$

which is equivalent to

$$\int_x^b [u(t)-u(b)]\,df(t)\ge u'_-(b)\left[(b-x)\,f(x)-\int_x^b f(t)\,dt\right] \quad (2.5)$$

for $x\in(a,b)$.

If we add (2.4) and (2.5) we get

$$\begin{aligned} &\int_a^x [u(t)-u(a)]\,df(t)+\int_x^b [u(t)-u(b)]\,df(t)\\ &\ge u'_+(a)\left[(x-a)\,f(x)-\int_a^x f(t)\,dt\right]+u'_-(b)\left[(b-x)\,f(x)-\int_x^b f(t)\,dt\right] \end{aligned}$$

and by (2.3) we get the first inequality in (2.1).

By the gradient inequality we also have

$$u(t)-u(a)\le u'_+(t)(t-a)=\left(u'_+(t)-u'_+(a)\right)(t-a)+u'_+(a)(t-a)$$

for $t\in[a,x]$ and

$$u(t)-u(b)\le u'_-(t)(t-b)=\left(u'_-(t)-u'_-(b)\right)(t-b)+u'_-(b)(t-b)$$

for $t\in[x,b]$.

These imply the integral inequalities

$$\begin{aligned}
(2.6)\quad & \int_a^x [u(t) - u(a)]\, df(t) \\
& \leq \int_a^x \left(u'_+(t) - u'_+(a)\right)(t-a)\, df(t) + u'_+(a) \int_a^x (t-a)\, f(t) \\
& = \int_a^x \left(u'_+(t) - u'_+(a)\right)(t-a)\, df(t) + u'_+(a) \left[(x-a) f(x) - \int_a^x f(t)\, dt\right]
\end{aligned}$$

and

$$\begin{aligned}
(2.7)\quad & \int_x^b [u(t) - u(b)]\, df(t) \\
& \leq \int_x^b \left(u'_-(t) - u'_-(b)\right)(t-b)\, df(t) + u'_-(b) \int_x^b (t-b)\, df(t) \\
& = \int_x^b \left(u'_-(t) - u'_-(b)\right)(t-b)\, df(t) + u'_-(b) \left[(b-x) f(x) - \int_x^b f(t)\, dt\right]
\end{aligned}$$

for $x \in (a,b)$.

If we add these inequalities, then we get

$$\begin{aligned}
& \int_a^x [u(t) - u(a)]\, df(t) + \int_x^b [u(t) - u(b)]\, df(t) \\
& \leq \int_a^x \left(u'_+(t) - u'_+(a)\right)(t-a)\, df(t) + \int_x^b \left(u'_-(t) - u'_-(b)\right)(t-b)\, df(t) \\
& + u'_+(a) \left[(x-a) f(x) - \int_a^x f(t)\, dt\right] + u'_-(b) \left[(b-x) f(x) - \int_x^b f(t)\, dt\right],
\end{aligned}$$

which together with (2.3) produces the second inequality in (2.1). □

Remark 1. *We observe that the Riemann-Stieltjes integrals* $\int_a^x u'_+(t)(t-a)\, df(t)$ *and* $\int_x^b u'_-(t)(t-b)\, df(t)$ *exist if either* f *is continuous on* $[a,b]$ *or* u *has a continuous derivative on an open interval incorporating* $[a,b]$.

In what follows we assume that all Riemann-Stieltjes integrals involved exist on those specific intervals.

Remark 2. *If we take* $x = \frac{a+b}{2}$ *in (2.2) we get*

$$\begin{aligned}
(2.8)\quad 0 & \leq [u(b) - u(a)]\, f\left(\frac{a+b}{2}\right) \\
& - u'_+(a) \left[\frac{1}{2}(b-a) f\left(\frac{a+b}{2}\right) - \int_a^{\frac{a+b}{2}} f(t)\, dt\right] \\
& - u'_-(b) \left[\frac{1}{2}(b-a) f\left(\frac{a+b}{2}\right) - \int_{\frac{a+b}{2}}^b f(t)\, dt\right] - \int_a^b f(t)\, du(t) \\
& \leq \int_a^{\frac{a+b}{2}} \left[u'_+(t) - u'_+(a)\right](t-a)\, df(t) + \int_{\frac{a+b}{2}}^b \left[u'_-(t) - u'_-(b)\right](t-b)\, df(t).
\end{aligned}$$

Corollary 3. *Assume that $g:[a,b]\to\mathbb{R}$ is continuous and nondecreasing on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing, then for $x\in(a,b)$,*

$$
\begin{aligned}
&g(a)\left[(x-a)f(x)-\int_a^x f(t)\,dt\right]+g(b)\left[(b-x)f(x)-\int_x^b f(t)\,dt\right] \\
&\qquad\le f(x)\int_a^b g(t)\,dt-\int_a^b f(t)\,g(t)\,dt \\
&\qquad\le \int_a^x [g(t)-g(a)](t-a)\,df(t)+\int_x^b [g(t)-g(b)](t-b)\,df(t) \\
&\qquad+g(a)\left[(x-a)f(x)-\int_a^x f(t)\,dt\right]+g(b)\left[(b-x)f(x)-\int_x^b f(t)\,dt\right].
\end{aligned}\tag{2.9}
$$

This is equivalent to

$$
\begin{aligned}
0&\le f(x)\int_a^b g(t)\,dt-g(a)\left[(x-a)f(x)-\int_a^x f(t)\,dt\right] \\
&\quad-g(b)\left[(b-x)f(x)-\int_x^b f(t)\,dt\right]-\int_a^b f(t)\,g(t)\,dt \\
&\le \int_a^x [g(t)-g(a)](t-a)\,df(t)+\int_x^b [g(t)-g(b)](t-b)\,df(t),
\end{aligned}\tag{2.10}
$$

for $x\in(a,b)$.

The proof follows from Theorem 2 by taking $u(t):=\int_a^t g(s)\,ds$ which is convex on $[a,b]$.

3. Inequalities for Riemann Integral

If $f(t)=t$, $t\in[a,b]$ and u is a convex function on $[a,b]$, then

$$
(x-a)f(x)-\int_a^x f(t)\,dt=(x-a)x-\int_a^x t\,dt=\frac{1}{2}(x-a)^2,
$$

$$
(b-x)f(x)-\int_x^b f(t)\,dt=(b-x)x-\int_x^b t\,dt=-\frac{1}{2}(b-x)^2,
$$

and

$$
\begin{aligned}
&[u(b)-u(a)]f(x)-\int_a^b f(t)\,du(t) \\
&\qquad=[u(b)-u(a)]x-\int_a^b t\,du(t) \\
&\qquad\qquad=\int_a^b u(t)\,dt-(x-a)u(a)-(b-x)u(b)
\end{aligned}
$$

for $x\in(a,b)$.

By utilising (2.1) we then get

$$\begin{aligned}(3.1)\quad &\frac{1}{2}(x-a)^2 u'_+(a) - \frac{1}{2}(b-x)^2 u'_-(b) \\ &\le \int_a^b u(t)\,dt - (x-a)u(a) - (b-x)u(b) \\ &\le \int_a^x \left[u'_+(t) - u'_+(a)\right](t-a)\,dt + \int_x^b \left[u'_-(t) - u'_-(b)\right](t-b)\,dt \\ &\quad + \frac{1}{2}(x-a)^2 u'_+(a) - \frac{1}{2}(b-x)^2 u'_-(b),\end{aligned}$$

namely

$$\begin{aligned}(3.2)\quad &\frac{1}{2}(b-x)^2 u'_-(b) - \frac{1}{2}(x-a)^2 u'_+(a) \\ &- \int_a^x \left[u'_+(t) - u'_+(a)\right](t-a)\,dt - \int_x^b \left[u'_-(t) - u'_-(b)\right](t-b)\,dt \\ &\le (x-a)u(a) + (b-x)u(b) - \int_a^b u(t)\,dt \\ &\le \frac{1}{2}(b-x)^2 u'_-(b) - \frac{1}{2}(x-a)^2 u'_+(a)\end{aligned}$$

for $x \in (a,b)$.

Now, by the monotonicity of the lateral derivatives of the convex function u and the fact that $u'_+(t) = u'_-(t)$ except a countable number of points in $[a,b]$ we have that

$$\begin{aligned}\int_a^x \left[u'_+(t) - u'_+(a)\right](t-a)\,dt &= \int_a^x \left[u'_-(t) - u'_+(a)\right](t-a)\,dt \\ &\le \left[u'_-(x) - u'_+(a)\right]\int_a^x (t-a)\,dt = \frac{1}{2}(x-a)^2\left[u'_-(x) - u'_+(a)\right]\end{aligned}$$

and

$$\begin{aligned}\int_x^b \left[u'_-(t) - u'_-(b)\right](t-b)\,dt &= \int_x^b \left[u'_-(b) - u'_+(t)\right](b-t)\,dt \\ &\le \left[u'_-(b) - u'_+(x)\right]\int_x^b (b-t)\,dt = \frac{1}{2}(b-x)^2\left[u'_-(b) - u'_+(x)\right],\end{aligned}$$

which, by addition, give

$$\begin{aligned}&\int_a^x \left[u'_+(t) - u'_+(a)\right](t-a)\,dt + \int_x^b \left[u'_-(t) - u'_-(b)\right](t-b)\,dt \\ &\le \frac{1}{2}(x-a)^2\left[u'_-(x) - u'_+(a)\right] + \frac{1}{2}(b-x)^2\left[u'_-(b) - u'_+(x)\right]\end{aligned}$$

for $x \in (a,b)$.

Therefore

$$\begin{aligned}(3.3)\quad & \frac{1}{2}(b-x)^2 u'_-(b) - \frac{1}{2}(x-a)^2 u'_+(a) \\ & - \int_a^x \left[u'_+(t) - u'_+(a)\right](t-a)\,dt - \int_x^b \left[u'_-(t) - u'_-(b)\right](t-b)\,dt \\ & \geq \frac{1}{2}(b-x)^2 u'_-(b) - \frac{1}{2}(x-a)^2 u'_+(a) \\ & - \frac{1}{2}(x-a)^2\left[u'_-(x) - u'_+(a)\right] - \frac{1}{2}(b-x)^2\left[u'_-(b) - u'_+(x)\right] \\ & = \frac{1}{2}(b-x)^2 u'_+(x) - \frac{1}{2}(x-a)^2 u'_-(x)\end{aligned}$$

for $x \in (a,b)$.

If we put together (3.2) and (3.3) we get for any convex function $u : [a,b] \to \mathbb{R}$

$$\begin{aligned}(3.4)\quad & \frac{1}{2}(b-x)^2 u'_+(x) - \frac{1}{2}(x-a)^2 u'_-(x) \leq \frac{1}{2}(b-x)^2 u'_-(b) - \frac{1}{2}(x-a)^2 u'_+(a) \\ & - \int_a^x \left[u'_+(t) - u'_+(a)\right](t-a)\,dt - \int_x^b \left[u'_-(t) - u'_-(b)\right](t-b)\,dt \\ & \leq (x-a)\,u(a) + (b-x)\,u(b) - \int_a^b u(t)\,dt \\ & \leq \frac{1}{2}(b-x)^2 u'_-(b) - \frac{1}{2}(x-a)^2 u'_+(a)\end{aligned}$$

for $x \in (a,b)$, see also [6].

If the function u is differentiable in $x \in (a,b)$, then we obtain from (3.4) that

$$\begin{aligned}(3.5)\quad & (b-a)\left(\frac{a+b}{2} - x\right) u'(x) \leq \frac{1}{2}(b-x)^2 u'_-(b) - \frac{1}{2}(x-a)^2 u'_+(a) \\ & - \int_a^x \left[u'_+(t) - u'_+(a)\right](t-a)\,dt - \int_x^b \left[u'_-(t) - u'_-(b)\right](t-b)\,dt \\ & \leq (x-a)\,u(a) + (b-x)\,u(b) - \int_a^b u(t)\,dt \\ & \leq \frac{1}{2}(b-x)^2 u'_-(b) - \frac{1}{2}(x-a)^2 u'_+(a)\end{aligned}$$

If in (3.4) we take $x = \frac{a+b}{2}$, then we get the Hermite-Hadamard type inequalities

$$\begin{aligned}(3.6)\quad & 0 \leq \frac{1}{8}\left[u'_+\left(\frac{a+b}{2}\right) - u'_-\left(\frac{a+b}{2}\right)\right](b-a)^2 \\ & \leq \frac{1}{8}(b-a)^2\left[u'_-(b) - u'_+(a)\right] \\ & - \int_a^{\frac{a+b}{2}} \left[u'_+(t) - u'_+(a)\right](t-a)\,dt - \int_{\frac{a+b}{2}}^b \left[u'_-(t) - u'_-(b)\right](t-b)\,dt \\ & \leq \frac{u(a) + u(b)}{2}(b-a) - \int_a^b u(t)\,dt \\ & \leq \frac{1}{8}(b-a)^2\left[u'_-(b) - u'_+(a)\right],\end{aligned}$$

for a convex function $u:[a,b]\to\mathbb{R}$.

The lower bound

$$\frac{1}{8}\left[u'_{+}\left(\frac{a+b}{2}\right)-u'_{-}\left(\frac{a+b}{2}\right)\right](b-a)^{2}$$

and the upper bound

$$\frac{1}{8}(b-a)^{2}\left[u'_{-}(b)-u'_{+}(a)\right]$$

for the trapezoid difference were obtained first in the paper [6]. The constant $\frac{1}{8}$ is best in both bounds.

If u is differentiable in $\frac{a+b}{2}$, then we get from (3.6) that

$$\begin{aligned}(3.7)\quad 0 &\le \frac{1}{8}(b-a)^{2}\left[u'_{-}(b)-u'_{+}(a)\right] \\ &\quad -\int_{a}^{\frac{a+b}{2}}\left[u'_{+}(t)-u'_{+}(a)\right](t-a)\,dt-\int_{\frac{a+b}{2}}^{b}\left[u'_{-}(t)-u'_{-}(b)\right](t-b)\,dt \\ &\le \frac{u(a)+u(b)}{2}(b-a)-\int_{a}^{b}u(t)\,dt \\ &\le \frac{1}{8}(b-a)^{2}\left[u'_{-}(b)-u'_{+}(a)\right].\end{aligned}$$

Now, if we take $u(t)=-\ln t$, $t\in[a,b]\subset(0,\infty)$ which is convex and f a monotonic nondecreasing function on $[a,b]$, then by (2.1) we get

$$\begin{aligned}(3.8)\quad &\frac{1}{a}\left[\int_{a}^{x}f(t)\,dt-(x-a)f(x)\right]+\frac{1}{b}\left[\int_{x}^{b}f(t)\,dt-(b-x)f(x)\right] \\ &\le \int_{a}^{b}\frac{f(t)}{t}dt+\frac{b-a}{ba}f(x) \\ &\le \frac{1}{a}\int_{a}^{x}\frac{1}{t}(t-a)^{2}\,df(t)+\frac{1}{b}\int_{x}^{b}\frac{1}{t}(t-b)^{2}\,df(t) \\ &\quad+\frac{1}{a}\left[\int_{a}^{x}f(t)\,dt-(x-a)f(x)\right]+\frac{1}{b}\left[\int_{x}^{b}f(t)\,dt-(b-x)f(x)\right],\end{aligned}$$

while from (2.2) we get

$$\begin{aligned}(3.9)\quad 0 &\le \frac{b-a}{ba}f(x)+\int_{a}^{b}\frac{f(t)}{t}dt \\ &\quad-\frac{1}{a}\left[\int_{a}^{x}f(t)\,dt-(x-a)f(x)\right]-\frac{1}{b}\left[\int_{x}^{b}f(t)\,dt-(b-x)f(x)\right] \\ &\le \frac{1}{a}\int_{a}^{x}\frac{1}{t}(t-a)^{2}\,df(t)+\frac{1}{b}\int_{x}^{b}\frac{1}{t}(t-b)^{2}\,df(t)\end{aligned}$$

for $x\in(a,b)$.

If $u(t)=t^p$ with $p\in(-\infty,0)\cup[1,\infty)$ and $t\in[a,b]$, then u is convex on $[a,b]$ and if $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing, then by (2.1) we get

$$p\left\{a^{p-1}\left[(x-a)f(x)-\int_a^x f(t)\,dt\right]+b^{p-1}\left[(b-x)f(x)-\int_x^b f(t)\,dt\right]\right\} \tag{3.10}$$
$$\leq (b^p-a^p)f(x)-p\int_a^b f(t)\,t^{p-1}dt$$
$$\leq p\left\{\int_a^x \left(t^{p-1}-a^{p-1}\right)(t-a)\,df(t)+\int_x^b \left(t^{p-1}-b^{p-1}\right)(t-b)\,df(t)\right\}$$
$$+p\left\{a^{p-1}\left[(x-a)f(x)-\int_a^x f(t)\,dt\right]+b^{p-1}\left[(b-x)f(x)-\int_x^b f(t)\,dt\right]\right\},$$

for $x\in(a,b)$, while from (2.2) we get

$$0\leq (b^p-a^p)f(x)-p\int_a^b f(t)\,t^{p-1}dt \tag{3.11}$$
$$-p\left\{a^{p-1}\left[(x-a)f(x)-\int_a^x f(t)\,dt\right]+b^{p-1}\left[(b-x)f(x)-\int_x^b f(t)\,dt\right]\right\}$$
$$\leq p\int_a^x \left(t^{p-1}-a^{p-1}\right)(t-a)\,df(t)+p\int_x^b \left(t^{p-1}-b^{p-1}\right)(t-b)\,df(t)$$

for $x\in(a,b)$.

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