NEW TRAPEZOID TYPE RIEMANN-STIELTJES INTEGRAL INEQUALITIES FOR MONOTONIC INTEGRANDS AND CONVEX INTEGRATORS

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Abstract. In this paper we obtain some inequalities for the trapezoid difference

$$[u(x)-u(a)]f(a)+[u(b)-u(x)]f(b)-\int_a^b f(t)\,du(t)$$

where f is a monotonic nondecreasing function on $[a,b]$, u is continuous convex on $[a,b]$ and $x\in(a,b)$. Some particular inequalities for the Riemann integral are also given.

1. Introduction

We start with the following result concerning two inequalities of trapezoid type for convex functions obtained in [6]:

Theorem 1. *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a convex function on $[a,b]$. Then for any $x\in[a,b]$ one has the inequality*

$$\begin{aligned}(1.1)\quad \frac{1}{2}\left[(b-x)^2 f'_+(x)-(x-a)^2 f'_-(x)\right] &\le (x-a)f(a)+(b-x)f(b)-\int_a^b f(t)\,dt\\ &\le \frac{1}{2}\left[(b-x)^2 f'_-(b)-(x-a)^2 f'_+(a)\right].\end{aligned}$$

The constant $\frac{1}{2}$ is sharp in both inequalities.
The second inequality also holds for $x=a$ or $x=b$.

We have a simpler first inequality in the case of differentiability:

Corollary 1. *With the assumptions of Lemma 1 and if $x\in(a,b)$ is a point of differentiability for f, then*

$$(1.2)\qquad \left(\frac{a+b}{2}-x\right)(b-a)f'(x)\le(x-a)f(a)+(b-x)f(b)-\int_a^b f(t)\,dt.$$

Now, recall that the following inequality, which is well known in the literature as the Hermite-Hadamard inequality for convex functions, holds

$$(1.3)\qquad f\left(\frac{a+b}{2}\right)(b-a)\le\int_a^b f(t)\,dt\le\frac{f(a)+f(b)}{2}(b-a).$$

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The following corollary provides some sharp bounds for the trapezoid difference

$$\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt.$$

Corollary 2. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then we have the inequality*

$$(1.4)\quad 0\le\frac{1}{8}\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right](b-a)^2 \le\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt \le\frac{1}{8}\left[f'_-(b)-f'_+(a)\right](b-a)^2.$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

For various trapezoid type inequalities involving Riemann-Stieltjes integral, see [1]-[12] and [8]-[16].

Motivated by the above results, in this paper we obtain some inequalities for the Riemann-Stieltjes integral trapezoid difference

$$[u(x)-u(a)]f(a)+[u(b)-u(x)]f(b)-\int_a^b f(t)\,du(t)$$

where f is a convex function on $[a,b]$, u is monotonic nondecreasing and $x\in(a,b)$. In the case of Riemann integral, namely for $u(t)=t$, some particular inequalities are also given.

2. Main Results

We have the following main result:

Theorem 2. *Assume that $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing and $u:[a,b]\to\mathbb{R}$ is continuous convex on $[a,b]$. Then for $x\in(a,b)$ we have the inequalities*

$$(2.1)\quad u'_+(x)\left[(b-x)f(b)-\int_x^b f(t)\,dt\right]+u'_-(x)\left[(x-a)f(a)-\int_a^x f(t)\,dt\right]$$
$$\le[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a)-\int_a^b f(t)\,du(t)$$
$$\le\int_a^x(t-x)\left[u'_+(t)-u'_-(x)\right]df(t)+\int_x^b(t-x)\left[u'_-(t)-u'_+(x)\right]df(t)$$
$$+u'_+(x)\left[(b-x)f(b)-\int_x^b f(t)\,dt\right]+u'_-(x)\left[(x-a)f(a)-\int_a^x f(t)\,dt\right]$$

provided the Riemann-Stieltjes integrals $\int_a^x u'_+(t)(t-x)\,df(t)$ and $\int_x^b u'_-(t)(t-x)\,df(t)$ exist.

This is equivalent to

$$(2.2)\quad 0 \leq [u(b) - u(x)] f(b) + [u(x) - u(a)] f(a) - u'_{+}(x) \left[(b-x) f(b) - \int_x^b f(t)\, dt \right] - u'_{-}(x) \left[(x-a) f(a) - \int_a^x f(t)\, dt \right] - \int_a^b f(t)\, du(t) \leq \int_a^x (t-x) \left[u'_{+}(t) - u_{-}(x) \right] df(t) + \int_x^b (t-x) \left[u'_{-}(t) - u'_{+}(x) \right] df(t)$$

for $x \in (a,b)$.

Proof. Using the integration by parts rule for the Riemann-Stieltjes integral, we have

$$(2.3)\quad \int_a^b [u(t) - u(x)]\, df(t) = [u(b) - u(x)] f(b) + [u(x) - u(a)] f(a) - \int_a^b f(t)\, du(t)$$

for all $x \in [a,b]$.

We also have

$$(2.4)\quad \int_a^b [u(t) - u(x)]\, df(t) = \int_a^x [u(t) - u(x)]\, df(t) + \int_x^b [u(t) - u(x)]\, df(t)$$

for all $x \in (a,b)$.

Using the gradient inequality for the convex function u we have

$$u(x) - u(t) \leq (x-t)\, u'_{-}(x) \text{ for } t \in [a,x]$$

and

$$u(t) - u(x) \geq (t-x)\, u'_{+}(x) \text{ for } t \in [x,b].$$

Since f is monotonic nondecreasing and by using integration by parts we get

$$(2.5)\quad \int_x^b [u(t) - u(x)]\, df(t) \geq u'_{+}(x) \int_x^b (t-x)\, df(t) = u'_{+}(x) \left[(b-x) f(b) - \int_x^b f(t)\, dt \right]$$

and

$$\begin{aligned} \int_a^x [u(x) - u(t)]\, df(t) &\leq u'_{-}(x) \int_a^x (x-t)\, df(t) \\ &= u'_{-}(x) \left[\int_a^x f(t)\, dt - (x-a) f(a) \right], \end{aligned}$$

which is equivalent to

$$(2.6)\quad \int_a^x [u(x) - u(t)]\, df(t) \geq u'_{-}(x) \left[(x-a) f(a) - \int_a^x f(t)\, dt \right]$$

for all $x \in (a,b)$.

If we add (2.5) and (2.6), then we get

$$\begin{aligned}&\int_a^x [u(t)-u(x)]\,df(t)+\int_x^b [u(t)-u(x)]\,df(t)\\&\quad\geq u'_+(x)\left[(b-x)f(b)-\int_x^b f(t)\,dt\right]+u'_-(x)\left[(x-a)f(a)-\int_a^x f(t)\,dt\right],\end{aligned}$$

which together with (2.3) and (2.4) provide the first inequality in (2.1).

Using the gradient inequality we also have

$$u(x)-u(t)\geq (x-t)\,u'_+(t)\ \text{ for } t\in[a,x]$$

and

$$u(t)-u(x)\leq (t-x)\,u'_-(t)\ \text{ for } t\in[x,b].$$

Since f is monotonic nondecreasing and by using integration by parts we get

$$\int_a^x [u(x)-u(t)]\,df(t)\geq \int_a^x (x-t)\,u'_+(t)\,df(t) \tag{2.7}$$

and

$$\begin{aligned}\int_x^b [u(t)-u(x)]\,df(t)&\leq \int_x^b (t-x)\,u'_-(t)\,df(t)\\&=\int_x^b (t-x)\left[u'_-(t)-u'_+(x)\right]df(t)+u'_+(x)\int_x^b (t-x)\,df(t)\\&=\int_x^b (t-x)\left[u'_-(t)-u'_+(x)\right]df(t)\\&\qquad+u'_+(x)\left[(b-x)f(b)-\int_x^b f(t)\,dt\right].\end{aligned}\tag{2.8}$$

From (2.7) we get

$$\begin{aligned}\int_a^x [u(t)-u(x)]\,df(t)&\leq \int_a^x (t-x)\,u'_+(t)\,df(t)\\&=\int_a^x (t-x)\left[u'_+(t)-u_-(x)\right]df(t)+u'_-(x)\int_a^x (t-x)\,df(t)\\&=\int_a^x (t-x)\left[u'_+(t)-u'_-(x)\right]df(t)\\&\qquad+u_-(x)\left[(x-a)f(a)-\int_a^x f(t)\,dt\right].\end{aligned}\tag{2.9}$$

If we add (2.8) and (2.9) we get

$$\begin{aligned}&\int_x^b [u(t)-u(x)]\,df(t)+\int_a^x [u(t)-u(x)]\,df(t)\\&\quad\leq \int_x^b (t-x)\left[u'_-(t)-u'_+(x)\right]df(t)+u'_+(x)\left[(b-x)f(b)-\int_x^b f(t)\,dt\right]\\&\quad+\int_a^x (t-x)\left[u'_+(t)-u'_-(x)\right]df(t)+u'_-(x)\left[(x-a)f(a)-\int_a^x f(t)\,dt\right],\end{aligned}$$

which together with (2.3) and (2.4) give the second inequality in (2.1). □

Corollary 3. *With the assumptions of Theorem 2 and if u is differentiable in x, then from (2.1) we get*

$$
\begin{aligned}
(2.10)\quad & u'(x)\left[(b-x)f(b)+(x-a)f(a)-\int_a^b f(t)\,dt\right] \\
&\le [u(b)-u(x)]f(b)+[u(x)-u(a)]f(a)-\int_a^b f(t)\,du(t) \\
&\le \int_a^x (t-x)\left[u'_+(t)-u'(x)\right]df(t)+\int_x^b (t-x)\left[u'_-(t)-u'(x)\right]df(t) \\
&\quad + u'(x)\left[(b-x)f(b)+(x-a)f(a)-\int_a^b f(t)\,dt\right],
\end{aligned}
$$

and, equivalently,

$$
\begin{aligned}
(2.11)\quad & 0\le [u(b)-u(x)]f(b)+[u(x)-u(a)]f(a) \\
&- u'(x)\left[(b-x)f(b)+(x-a)f(a)-\int_a^b f(t)\,dt\right]-\int_a^b f(t)\,du(t) \\
&\le \int_a^x (t-x)\left[u'_+(t)-u'(x)\right]df(t)+\int_x^b (t-x)\left[u'_-(t)-u'(x)\right]df(t),
\end{aligned}
$$

Remark 1. *If we take $x=\frac{a+b}{2}$, in (2.1) and (2.2) we get*

$$
\begin{aligned}
(2.12)\quad & u'_+\left(\frac{a+b}{2}\right)\left[\frac{1}{2}(b-a)f(b)-\int_{\frac{a+b}{2}}^b f(t)\,dt\right] \\
&+ u'_-\left(\frac{a+b}{2}\right)\left[\frac{1}{2}(b-a)f(a)-\int_a^{\frac{a+b}{2}} f(t)\,dt\right] \\
&\le \left[u(b)-u\left(\frac{a+b}{2}\right)\right]f(b)+\left[u\left(\frac{a+b}{2}\right)-u(a)\right]f(a)-\int_a^b f(t)\,du(t) \\
&\le \int_a^{\frac{a+b}{2}}\left(t-\frac{a+b}{2}\right)\left[u'_+(t)-u'_-\left(\frac{a+b}{2}\right)\right]df(t) \\
&+ \int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)\left[u'_-(t)-u'_+\left(\frac{a+b}{2}\right)\right]df(t) \\
&+ u'_+\left(\frac{a+b}{2}\right)\left[\frac{1}{2}(b-a)f(b)-\int_{\frac{a+b}{2}}^b f(t)\,dt\right] \\
&+ u'_-\left(\frac{a+b}{2}\right)\left[\frac{1}{2}(b-a)f(a)-\int_a^{\frac{a+b}{2}} f(t)\,dt\right]
\end{aligned}
$$

provided the Riemann-Stieltjes integrals $\int_a^{\frac{a+b}{2}} u'_+(t)\left(t-\frac{a+b}{2}\right)df(t)$ and $\int_{\frac{a+b}{2}}^b u'_-(t)\left(t-\frac{a+b}{2}\right)df(t)$ exist.

This is equivalent to

$$
\begin{aligned}
(2.13)\quad 0 &\le \left[u(b)-u\left(\frac{a+b}{2}\right)\right] f(b)+\left[u\left(\frac{a+b}{2}\right)-u(a)\right] f(a) \\
&\quad - u'_{+}\left(\frac{a+b}{2}\right)\left[\frac{1}{2}(b-a) f(b)-\int_{\frac{a+b}{2}}^{b} f(t)\, dt\right] \\
&\quad - u'_{-}\left(\frac{a+b}{2}\right)\left[\frac{1}{2}(b-a) f(a)-\int_{a}^{\frac{a+b}{2}} f(t)\, dt\right] \\
&\quad - \int_a^b f(t)\, du(t) \\
&\le \int_a^{\frac{a+b}{2}}\left(t-\frac{a+b}{2}\right)\left[u'_{+}(t)-u'_{-}\left(\frac{a+b}{2}\right)\right] df(t) \\
&\quad + \int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)\left[u'_{-}(t)-u'_{+}\left(\frac{a+b}{2}\right)\right] df(t).
\end{aligned}
$$

If u is differentiable in $\frac{a+b}{2}$, then by (2.10) we get

$$
\begin{aligned}
(2.14)\quad & u'\left(\frac{a+b}{2}\right)\left[\frac{f(b)+f(a)}{2}(b-a)-\int_a^b f(t)\, dt\right] \\
&\le \left[u(b)-u\left(\frac{a+b}{2}\right)\right] f(b)+\left[u\left(\frac{a+b}{2}\right)-u(a)\right] f(a)-\int_a^b f(t)\, du(t) \\
&\le \int_a^{\frac{a+b}{2}}\left(t-\frac{a+b}{2}\right)\left[u'_{+}(t)-u'\left(\frac{a+b}{2}\right)\right] df(t) \\
&\quad + \int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)\left[u'_{-}(t)-u'\left(\frac{a+b}{2}\right)\right] df(t) \\
&\quad + u'\left(\frac{a+b}{2}\right)\left[\frac{f(b)+f(a)}{2}(b-a)-\int_a^b f(t)\, dt\right],
\end{aligned}
$$

and, equivalently

$$
\begin{aligned}
(2.15)\quad 0 &\le \left[u(b)-u\left(\frac{a+b}{2}\right)\right] f(b)+\left[u\left(\frac{a+b}{2}\right)-u(a)\right] f(a) \\
&\quad - u'\left(\frac{a+b}{2}\right)\left[\frac{f(b)+f(a)}{2}(b-a)-\int_a^b f(t)\, dt\right]-\int_a^b f(t)\, du(t) \\
&\le \int_a^{\frac{a+b}{2}}\left(t-\frac{a+b}{2}\right)\left[u'_{+}(t)-u'\left(\frac{a+b}{2}\right)\right] df(t) \\
&\quad + \int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)\left[u'_{-}(t)-u'\left(\frac{a+b}{2}\right)\right] df(t).
\end{aligned}
$$

Corollary 4. *Assume that $g:[a,b]\to\mathbb{R}$ is continuous and nondecreasing on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing, then for $x\in(a,b)$,*

$$\begin{aligned}(2.16)\quad 0 &\le f(b)\int_x^b g(t)\,dt + f(a)\int_a^x g(t)\,dt \\ &\quad - g(x)\left[(b-x)f(b)+(x-a)f(a)-\int_a^b f(t)\,dt\right]-\int_a^b f(t)\,g(t)\,dt \\ &\le \int_a^b (t-x)\left[g(t)-g(x)\right]df(t),\end{aligned}$$

and, in particular, for $x=\frac{a+b}{2}$

$$\begin{aligned}(2.17)\quad 0 &\le f(b)\int_{\frac{a+b}{2}}^b g(t)\,dt + f(a)\int_a^{\frac{a+b}{2}} g(t)\,dt \\ &\quad - g\left(\frac{a+b}{2}\right)\left[\frac{f(b)+f(a)}{2}(b-a)-\int_a^b f(t)\,dt\right]-\int_a^b f(t)\,g(t)\,dt \\ &\le \int_a^b \left(t-\frac{a+b}{2}\right)\left[g(t)-g\left(\frac{a+b}{2}\right)\right]df(t).\end{aligned}$$

The proof follows from Theorem 2 by taking $u(t):=\int_a^t g(s)\,ds$ which is convex on $[a,b]$.

3. Inequalities for Riemann Integral

If we take $f(t)=t$, $t\in[a,b]$ in (2.1) we get for a convex function $u:[a,b]\to\mathbb{R}$ that

$$\begin{aligned}(3.1)\quad & u'_+(x)\left[(b-x)b-\int_x^b tdt\right]+u'_-(x)\left[(x-a)a-\int_a^x tdt\right] \\ &\le [u(b)-u(x)]b+[u(x)-u(a)]a-\int_a^b tdu(t) \\ &\le \int_a^x (t-x)\left[u'_+(t)-u'_-(x)\right]dt+\int_x^b (t-x)\left[u'_-(t)-u'_+(x)\right]dt \\ &\quad + u'_+(x)\left[(b-x)b-\int_x^b tdt\right]+u'_-(x)\left[(x-a)a-\int_a^x tdt\right]\end{aligned}$$

for $x\in(a,b)$.

Observe that

$$(b-x)b-\int_x^b tdt=(b-x)b-\frac{1}{2}\left(b^2-x^2\right)=\frac{1}{2}(b-x)^2,$$

$$(x-a)a-\int_a^x tdt=(x-a)a-\frac{1}{2}\left(x^2-a^2\right)=-\frac{1}{2}(x-a)^2$$

and

$$
\begin{aligned}
&[u(b)-u(x)]\,b+[u(x)-u(a)]\,a-\int_a^b t\,du(t)\\
&=[u(b)-u(x)]\,b+[u(x)-u(a)]\,a-\left(bu(b)-au(a)-\int_a^b u(t)\,dt\right)\\
&=\int_a^b u(t)\,dt-u(x)(b-a)
\end{aligned}
$$

for $x\in(a,b)$.

Using (3.1) we get

$$
\begin{aligned}
\frac{1}{2}(b-x)^2u'_+(x)-\frac{1}{2}(x-a)^2u'_-(x)&\le\int_a^b u(t)\,dt-u(x)(b-a)\\
\le\int_a^x(t-x)\left[u'_+(t)-u'_-(x)\right]dt&+\int_x^b(t-x)\left[u'_-(t)-u'_+(x)\right]dt\\
&+\frac{1}{2}(b-x)^2u'_+(x)-\frac{1}{2}(x-a)^2u'_-(x)
\end{aligned}
\tag{3.2}
$$

for $x\in(a,b)$.

Since u is convex, then the lateral derivatives $u'_+(\cdot)$ and $u'_-(\cdot)$ are monotonic nondecreasing and equal except in a countable number of points. Then

$$
\begin{aligned}
\int_a^x(t-x)\left[u'_+(t)-u'_-(x)\right]dt&=\int_a^x(t-x)\left[u'_-(t)-u'_-(x)\right]dt\\
&\le\sup_{t\in(a,x)}\left[u'_-(x)-u'_-(t)\right]\frac{1}{2}(x-a)^2=\frac{1}{2}(x-a)^2\left[u'_-(x)-u'_+(a)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\int_x^b(t-x)\left[u'_-(t)-u'_+(x)\right]dt&=\int_x^b(t-x)\left[u'_+(t)-u'_+(x)\right]dt\\
&\le\sup_{t\in(x,b)}\left[u'_+(t)-u'_+(x)\right]\frac{1}{2}(b-x)^2=\frac{1}{2}(b-x)^2\left[u'_-(b)-u'_+(x)\right]
\end{aligned}
$$

for $x\in(a,b)$.

Therefore

$$
\begin{aligned}
(3.3)\quad & \int_a^x (t-x)\left[u'_+(t)-u'_-(x)\right]dt+\int_x^b (t-x)\left[u'_-(t)-u'_+(x)\right]dt \\
&+\frac{1}{2}(b-x)^2u'_+(x)-\frac{1}{2}(x-a)^2u'_-(x) \\
&\le \frac{1}{2}(x-a)^2\left[u'_-(x)-u'_+(a)\right]+\frac{1}{2}(b-x)^2\left[u'_-(b)-u'_+(x)\right] \\
&+\frac{1}{2}(b-x)^2u'_+(x)-\frac{1}{2}(x-a)^2u'_-(x) \\
&=\frac{1}{2}(b-x)^2u'_-(b)-\frac{1}{2}(x-a)^2u'_+(a)+\frac{1}{2}(x-a)^2u'_-(x) \\
&-\frac{1}{2}(x-a)^2u'_-(x)+\frac{1}{2}(b-x)^2u'_+(x)-\frac{1}{2}(b-x)^2u'_+(x) \\
&=\frac{1}{2}(b-x)^2u'_-(b)-\frac{1}{2}(x-a)^2u'_+(a)
\end{aligned}
$$

for $x\in(a,b)$.

Therefore, by (3.2) and (3.3) we get

$$
\begin{aligned}
(3.4)\quad & \frac{1}{2}(b-x)^2u'_+(x)-\frac{1}{2}(x-a)^2u'_-(x) \\
&\le \int_a^b u(t)\,dt-u(x)(b-a) \\
&\le \int_a^x (t-x)\left[u'_+(t)-u'_-(x)\right]dt+\int_x^b (t-x)\left[u'_-(t)-u'_+(x)\right]dt \\
&+\frac{1}{2}(b-x)^2u'_+(x)-\frac{1}{2}(x-a)^2u'_-(x) \\
&\le \frac{1}{2}(b-x)^2u'_-(b)-\frac{1}{2}(x-a)^2u'_+(a)
\end{aligned}
$$

for $x\in(a,b)$.

If u is differentiable in $x\in(a,b)$, then from (3.4) we get

$$
\begin{aligned}
(3.5)\quad & (b-a)\left(\frac{a+b}{2}-x\right)u'(x)\le \int_a^b u(t)\,dt-u(x)(b-a) \\
&\le \int_a^x (t-x)\left[u'_+(t)-u'(x)\right]dt+\int_x^b (t-x)\left[u'_-(t)-u'(x)\right]dt \\
&+(b-a)\left(\frac{a+b}{2}-x\right)u'(x)\le \frac{1}{2}(b-x)^2u'_-(b)-\frac{1}{2}(x-a)^2u'_+(a)
\end{aligned}
$$

for $x\in(a,b)$.

If in (3.4) we take $x=\frac{a+b}{2}$, then we get

$$
\begin{aligned}
(3.6)\quad 0 &\leq \frac{1}{8}(b-a)^2\left[u_+'\left(\frac{a+b}{2}\right)-u_-'\left(\frac{a+b}{2}\right)\right] \\
&\leq \int_a^b u(t)\,dt-u\left(\frac{a+b}{2}\right)(b-a) \\
&\leq \int_a^{\frac{a+b}{2}}\left(t-\frac{a+b}{2}\right)\left[u_+'(t)-u_-'\left(\frac{a+b}{2}\right)\right]dt \\
&\quad+\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)\left[u_-'(t)-u_+'\left(\frac{a+b}{2}\right)\right]dt \\
&\quad+\frac{1}{8}(b-a)^2\left[u_+'\left(\frac{a+b}{2}\right)-u_-'\left(\frac{a+b}{2}\right)\right]\leq\frac{1}{8}(b-a)^2\left[u_-'(b)-u_+'(a)\right].
\end{aligned}
$$

If u is differentiable in $\frac{a+b}{2}$, then we obtain from (3.6) that

$$
\begin{aligned}
(3.7)\quad 0 &\leq \int_a^b u(t)\,dt-u\left(\frac{a+b}{2}\right)(b-a) \\
&\leq \int_a^{\frac{a+b}{2}}\left(t-\frac{a+b}{2}\right)\left[u_+'(t)-u'\left(\frac{a+b}{2}\right)\right]dt \\
&\quad+\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)\left[u_-'(t)-u'\left(\frac{a+b}{2}\right)\right]dt\leq\frac{1}{8}(b-a)^2\left[u_-'(b)-u_+'(a)\right].
\end{aligned}
$$

If we take in (2.16) $g(t)=-\frac{1}{t}$, $t\in[a,b]\subset(0,\infty)$, then for monotonic nondecreasing functions $f:[a,b]\to\mathbb{R}$ we have

$$
\begin{aligned}
(3.8)\quad 0 &\leq \int_a^b\frac{f(t)}{t}dt+\frac{1}{x}\left[(b-x)f(b)+(x-a)f(a)-\int_a^b f(t)\,dt\right] \\
&\quad-f(b)\ln\left(\frac{b}{x}\right)-f(a)\ln\left(\frac{x}{a}\right)\leq\frac{1}{x}\int_a^b\frac{(t-x)^2}{t}df(t),
\end{aligned}
$$

for $x\in(a,b)$,

For $x=\frac{a+b}{2}$ we get

$$
\begin{aligned}
(3.9)\quad 0 &\leq \int_a^b\frac{f(t)}{t}dt+\frac{2}{a+b}\left[\frac{f(b)+f(a)}{2}(b-a)-\int_a^b f(t)\,dt\right] \\
&\quad-f(b)\ln\left(\frac{2b}{a+b}\right)-f(a)\ln\left(\frac{a+b}{2a}\right)\leq\frac{2}{a+b}\int_a^b\frac{\left(t-\frac{a+b}{2}\right)^2}{t}df(t),
\end{aligned}
$$

while for $x=\sqrt{ab}$ we get

$$
\begin{aligned}
(3.10)\quad 0 &\leq \int_a^b\frac{f(t)}{t}dt+\frac{1}{\sqrt{ab}}\left[\left(b-\sqrt{ab}\right)f(b)+\left(\sqrt{ab}-a\right)f(a)-\int_a^b f(t)\,dt\right] \\
&\quad-\frac{f(b)+f(a)}{2}\ln\left(\frac{b}{a}\right)\leq\frac{1}{\sqrt{ab}}\int_a^b\frac{\left(t-\sqrt{ab}\right)^2}{t}df(t),
\end{aligned}
$$

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