APPROXIMATING THE INTEGRAL OF ANALYTIC COMPLEX FUNCTIONS ON PATHS FROM CONVEX DOMAINS IN TERMS OF GENERALIZED OSTROWSKI AND TRAPEZOID TYPE RULES

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Abstract. In this paper we establish some results in approximating the integral of analytic complex functions on paths from convex domains in terms of generalized Ostrowski and Trapezoid type rules. Error bounds for these expansions in terms of p-norms are also provided. Examples for the complex logarithm and the complex exponential are also given.

1. Introduction

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and z, $x \in D$, then we have the following Taylor's expansion with integral remainder

$$f(z) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x)(z-x)^k + \frac{1}{n!}(z-x)^{n+1} \int_0^1 f^{(n+1)}[(1-s)x + sz](1-s)^n ds \tag{1.1}$$

for $n \geq 0$, see for instance [15].

Consider the function $f(z) = \operatorname{Log}(z)$ where $\operatorname{Log}(z) = \ln|z| + i \operatorname{Arg}(z)$ and $\operatorname{Arg}(z)$ is such that $-\pi < \operatorname{Arg}(z) \leq \pi$. Log is called the *"principal branch"* of the complex logarithmic function. The function f is analytic on all of $\mathbb{C}_\ell := \mathbb{C} \setminus \{x + iy : x \leq 0,\ y = 0\}$ and

$$f^{(k)}(z) = \frac{(-1)^{k-1}(k-1)!}{z^k},\ k \geq 1,\ z \in \mathbb{C}_\ell.$$

Using the representation (1.1) we then have

$$\operatorname{Log}(z) = \operatorname{Log}(x) + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \left(\frac{z-x}{x}\right)^k + (-1)^n (z-x)^{n+1} \int_0^1 \frac{(1-s)^n ds}{[(1-s)x + sz]^{n+1}} \tag{1.2}$$

for all z, $x \in \mathbb{C}_\ell$ with $(1-s)x + sz \in \mathbb{C}_\ell$ for $s \in [0,1]$.

1991 *Mathematics Subject Classification.* 30A10, 26D15; 26D10.

Key words and phrases. Taylor's formula, Power series, Logarithmic and exponential functions.

RGMIA Res. Rep. Coll. 22 (2019), Art. 1, 21 pp. Received 08/01/19

Consider the complex exponential function $f(z) = \exp(z)$, then by (1.1) we get

$$\exp(z) = \sum_{k=0}^{n} \frac{1}{k!}(z-x)^k \exp(x) + \frac{1}{n!}(z-x)^{n+1}\int_0^1 (1-s)^n \exp[(1-s)x + sz]\,ds \tag{1.3}$$

for all z, $x \in \mathbb{C}$.

For various inequalities related to Taylor's expansions for real functions see [1]-[14].

Suppose γ is a *smooth path* parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with u, $w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_{\gamma} f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are *piecewise smooth*. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{1.4}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left|\int_{\gamma} f(z)\,dz\right| \le \int_{\gamma} |f(z)|\,|dz| \le \|f\|_{\gamma,\infty}\,\ell(\gamma) \tag{1.5}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \ge 1$ by

$$\|f\|_{\gamma,p} := \left(\int_{\gamma} |f(z)|^p\,|dz|\right)^{1/p}.$$

For $p=1$ we have

$$\|f\|_{\gamma,1} := \int_{\gamma} |f(z)| \, |dz| \, .$$

If p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \leq [\ell(\gamma)]^{1/q} \|f\|_{\gamma,p} \, .$$

In this paper we establish some results in approximating the integral of analytic complex functions on paths from convex domains in terms of generalized Ostrowski and Trapezoid type rules. Error bounds for these expansions in terms of p-norms are also provided. Examples for the complex logarithm and the complex exponential are also given.

2. Ostrowski and Trapezoid Type Equalities

We have:

Theorem 1. *Let $f: D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and $x \in D$. Suppose $\gamma \subset D$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ with $z(a)=u$ and $z(b)=w$ where u, $w \in D$. Then we have the Ostrowski type equality*

$$\int_{\gamma} f(z)\, dz = \sum_{k=0}^{n} \frac{1}{(k+1)!} f^{(k)}(x) \left[(w-x)^{k+1} + (-1)^k (x-u)^{k+1} \right] + R_n(x,\gamma) \tag{2.1}$$

where the remainder $R_n(x,\gamma)$ is given by

$$\begin{aligned} R_n(x,\gamma) &:= \frac{1}{n!} \int_{\gamma} (z-x)^{n+1} \left(\int_0^1 f^{(n+1)}[(1-s)x+sz] (1-s)^n \, ds \right) dz \\ &= \frac{1}{n!} \int_0^1 \left(\int_{\gamma} (z-x)^{n+1} f^{(n+1)}[(1-s)x+sz]\, dz \right) (1-s)^n \, ds. \end{aligned} \tag{2.2}$$

Proof. If we take the integral on the path $\gamma=\gamma_{u,w}$ in the equality (1.1), then we get

$$\begin{aligned} \int_{\gamma} f(z)\, dz &= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x) \int_{\gamma_{u,w}} (z-x)^k \, dz \\ &\quad + \frac{1}{n!} \int_{\gamma_{u,w}} (z-x)^{n+1} \left(\int_0^1 f^{(n+1)}[(1-s)x+sz](1-s)^n \, ds \right) dz \\ &= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x) \frac{(w-x)^{k+1} - (u-x)^{k+1}}{k+1} \\ &\quad + \frac{1}{n!} \int_{\gamma_{u,w}} (z-x)^{n+1} \left(\int_0^1 f^{(n+1)}[(1-s)x+sz](1-s)^n \, ds \right) dz \\ &= \sum_{k=0}^{n} \frac{1}{(k+1)!} f^{(k)}(x) \left[(w-x)^{k+1} + (-1)^{k+2} (x-u)^{k+1} \right] \\ &\quad + \frac{1}{n!} \int_{\gamma_{u,w}} (z-x)^{n+1} \left(\int_0^1 f^{(n+1)}[(1-s)x+sz](1-s)^n \, ds \right) dz, \end{aligned} \tag{2.3}$$

which proves the equality (2.1) with the first representation of the remainder from (2.2).

The second representation in (2.2) follows by Fubini's theorem. □

Corollary 1. *With the assumptions of Theorem 1 we have the mid-point equality*

$$\int_{\gamma} f(z)\,dz=\sum_{k=0}^{n}\frac{1}{2^{k}(k+1)!}f^{(k)}\left(\frac{u+w}{2}\right)\left[\frac{1+(-1)^{k}}{2}\right](w-u)^{k+1}+M_{n}(\gamma)\tag{2.4}$$

where the remainder $R_n(x,\gamma)$ is given by

$$\begin{aligned}M_{n}(\gamma)&:=\frac{1}{n!}\int_{\gamma}\left(z-\frac{u+w}{2}\right)^{n+1}\left(\int_{0}^{1}f^{(n+1)}\left[(1-s)\frac{u+w}{2}+sz\right](1-s)^{n}ds\right)dz\\&=\frac{1}{n!}\int_{0}^{1}\left(\int_{\gamma}\left(z-\frac{u+w}{2}\right)^{n+1}f^{(n+1)}\left[(1-s)\frac{u+w}{2}+sz\right]dz\right)(1-s)^{n}ds.\end{aligned}\tag{2.5}$$

The proof follows from Theorem 1 by taking $x=\frac{u+w}{2}\in D$.

Corollary 2. *With the assumptions of Theorem 1 and if $\lambda\in\mathbb{C}$, then we have the weighted trapezoid equality*

$$\int_{\gamma} f(z)\,dz=\sum_{k=0}^{n}\frac{1}{(k+1)!}\left[\lambda f^{(k)}(u)+(1-\lambda)(-1)^{k}f^{(k)}(w)\right](w-u)^{k+1}+T_{n}(\lambda,\gamma)\tag{2.6}$$

where the remainder $T_n(\lambda,\gamma)$ is given by

$$\begin{aligned}T_{n}(\lambda,\gamma)&:=\frac{\lambda}{n!}\int_{\gamma}(z-u)^{n+1}\left(\int_{0}^{1}f^{(n+1)}[(1-s)u+sz](1-s)^{n}ds\right)dz\\&\quad+\frac{(1-\lambda)}{n!}\int_{\gamma}(z-w)^{n+1}\left(\int_{0}^{1}f^{(n+1)}[(1-s)w+sz](1-s)^{n}ds\right)dz\\&=\frac{\lambda}{n!}\int_{0}^{1}\left(\int_{\gamma}(z-u)^{n+1}f^{(n+1)}[(1-s)u+sz]\,dz\right)(1-s)^{n}ds\\&\quad+\frac{(1-\lambda)}{n!}\int_{0}^{1}\left(\int_{\gamma}(z-w)^{n+1}f^{(n+1)}[(1-s)w+sz]\,dz\right)(1-s)^{n}ds.\end{aligned}\tag{2.7}$$

In particular, for $\lambda=\frac{1}{2}$ we have the trapezoid equality

$$\int_{\gamma} f(z)\,dz=\sum_{k=0}^{n}\frac{1}{(k+1)!}\left[\frac{f^{(k)}(u)+(-1)^{k}f^{(k)}(w)}{2}\right](w-u)^{k+1}+T_{n}(\gamma),\tag{2.8}$$

where the remainder $T_n(\gamma)$ is given by

$$T_n(\gamma) := \frac{1}{2n!}\int_\gamma (z-u)^{n+1}\left(\int_0^1 f^{(n+1)}[(1-s)u+sz](1-s)^n ds\right)dz + \frac{1}{2n!}\int_\gamma (z-w)^{n+1}\left(\int_0^1 f^{(n+1)}[(1-s)w+sz](1-s)^n ds\right)dz = \frac{1}{2n!}\int_0^1\left(\int_\gamma (z-u)^{n+1} f^{(n+1)}[(1-s)u+sz]dz\right)(1-s)^n ds + \frac{1}{2n!}\int_0^1\left(\int_\gamma (z-w)^{n+1} f^{(n+1)}[(1-s)w+sz]dz\right)(1-s)^n ds. \tag{2.9}$$

Proof. We write the equality (2.1) for $x=u$ to get

$$\int_\gamma f(z)\,dz = \sum_{k=0}^{n}\frac{1}{(k+1)!}f^{(k)}(u)(w-u)^{k+1} + R_n(u,\gamma), \tag{2.10}$$

where the remainder $R_n(u,\gamma)$ is given by

$$R_n(u,\gamma) := \frac{1}{n!}\int_\gamma (z-u)^{n+1}\left(\int_0^1 f^{(n+1)}[(1-s)u+sz](1-s)^n ds\right)dz = \frac{1}{n!}\int_0^1\left(\int_\gamma (z-u)^{n+1} f^{(n+1)}[(1-s)u+sz]dz\right)(1-s)^n ds, \tag{2.11}$$

and for $x=w$ to get

$$\int_\gamma f(z)\,dz = \sum_{k=0}^{n}\frac{(-1)^k}{(k+1)!}f^{(k)}(w)(w-u)^{k+1} + R_n(w,\gamma) \tag{2.12}$$

where the remainder $R_n(w,\gamma)$ is given by

$$R_n(w,\gamma) := \frac{1}{n!}\int_\gamma (z-w)^{n+1}\left(\int_0^1 f^{(n+1)}[(1-s)w+sz](1-s)^n ds\right)dz = \frac{1}{n!}\int_0^1\left(\int_\gamma (z-w)^{n+1} f^{(n+1)}[(1-s)w+sz]dz\right)(1-s)^n ds. \tag{2.13}$$

Now, if we multiply the equality (2.10) by λ and the equality (2.13) by $1-\lambda$ and sum, then we obtain the desired result. □

Remark 1. *Let $f: D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and $x \in D$. Suppose $\gamma \subset D$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ with $z(a)=u$ and $z(b)=w$ where u, $w \in D$.*

If we take $n=0$ in Theorem 1, then we obtain the Ostrowski type equality

$$\int_\gamma f(z)\,dz = f(x)(w-u) + R(x,\gamma), \tag{2.14}$$

where the remainder $R(x,\gamma)$ *is given by*

$$R(x,\gamma) := \int_\gamma (z-x)\left(\int_0^1 f'[(1-s)x+sz]\,ds\right)dz = \int_0^1 \left(\int_\gamma (z-x)\, f'[(1-s)x+sz]\,dz\right)ds. \tag{2.15}$$

In particular, for $x=\frac{u+w}{2}$ *we have the midpoint type equality*

$$\int_\gamma f(z)\,dz = f\left(\frac{u+w}{2}\right)(w-u)+R(\gamma), \tag{2.16}$$

where the remainder $R(\gamma)$ *is given by*

$$R(\gamma) := \int_\gamma \left(z-\frac{u+w}{2}\right)\left(\int_0^1 f'\left[(1-s)\frac{u+w}{2}+sz\right]ds\right)dz = \int_0^1 \left(\int_\gamma \left(z-\frac{u+w}{2}\right) f'\left[(1-s)\frac{u+w}{2}+sz\right]dz\right)ds. \tag{2.17}$$

If we take $n=0$ *in Corollary 2, then we obtain the weighted trapezoid equality for* $\lambda\in\mathbb{C}$

$$\int_\gamma f(z)\,dz = [\lambda f(u)+(1-\lambda) f(w)](w-u)+T(\lambda,\gamma), \tag{2.18}$$

where the remainder $T(\lambda,\gamma)$ *is given by*

$$\begin{aligned} T(\lambda,\gamma) &:= \lambda\int_\gamma (z-u)\left(\int_0^1 f'[(1-s)u+sz]\,ds\right)dz \\ &\quad +(1-\lambda)\int_\gamma (z-w)\left(\int_0^1 f'[(1-s)w+sz]\,ds\right)dz \\ &= \lambda\int_0^1 \left(\int_\gamma (z-u)\, f'[(1-s)u+sz]\,dz\right)ds \\ &\quad +(1-\lambda)\int_0^1 \left(\int_\gamma (z-w)\, f'[(1-s)w+sz]\,dz\right)ds. \end{aligned} \tag{2.19}$$

In particular, for $\lambda=\frac{1}{2}$ *we have the trapezoid type equality*

$$\int_\gamma f(z)\,dz = \frac{f(u)+f(w)}{2}(w-u)+T(\gamma), \tag{2.20}$$

where the remainder $T(\gamma)$ *is given by*

$$T(\gamma) := \frac{1}{2}\int_\gamma (z-u)\left(\int_0^1 f'[(1-s)u+sz]\,ds\right)dz + \frac{1}{2}\int_\gamma (z-w)\left(\int_0^1 f'[(1-s)w+sz]\,ds\right)dz = \frac{1}{2}\int_0^1\left(\int_\gamma (z-u)f'[(1-s)u+sz]\,dz\right)ds + \frac{1}{2}\int_0^1\left(\int_\gamma (z-w)f'[(1-s)w+sz]\,dz\right)ds. \tag{2.21}$$

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and $x \in D$. Suppose $\gamma \subset D$ is a *smooth path* parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = u$ and $z(b) = w$ where u, $w \in D$.

For $n = 1$ in (2.1) we get the perturbed Ostrowski's equality

$$\int_\gamma f(z)\,dz = f(x)(w-u) + f'(x)\left(\frac{w+u}{2} - x\right)(w-u) + R_1(x,\gamma), \tag{2.22}$$

where the remainder $R_1(x,\gamma)$ is given by

$$R_1(x,\gamma) := \int_\gamma (z-x)^2\left(\int_0^1 f''[(1-s)x+sz](1-s)\,ds\right)dz = \int_0^1\left(\int_\gamma (z-x)^2 f''[(1-s)x+sz]\,dz\right)(1-s)\,ds. \tag{2.23}$$

In particular, for $x = \frac{w+u}{2}$ we get the mid-point equality

$$\int_\gamma f(z)\,dz = f\left(\frac{w+u}{2}\right)(w-u) + M_1(\gamma), \tag{2.24}$$

where the remainder $M_1(\gamma)$ is given by

$$M_1(\gamma) := \int_\gamma \left(z-\frac{w+u}{2}\right)^2\left(\int_0^1 f''\left[(1-s)\frac{w+u}{2}+sz\right](1-s)\,ds\right)dz = \int_0^1\left(\int_\gamma \left(z-\frac{w+u}{2}\right)^2 f''\left[(1-s)\frac{w+u}{2}+sz\right]dz\right)(1-s)\,ds. \tag{2.25}$$

If we take $n = 1$ in Corollary 2 then we get the perturbed weighted trapezoid equality for $\lambda \in \mathbb{C}$

$$\int_\gamma f(z)\,dz = [\lambda f(u) + (1-\lambda)f(w)](w-u) + \frac{1}{2}[\lambda f'(u) - (1-\lambda)f'(w)](w-u)^2 + T_1(\lambda,\gamma), \tag{2.26}$$

where

$$T_1(\lambda,\gamma) := \lambda\int_\gamma (z-u)^2\left(\int_0^1 f''[(1-s)u+sz](1-s)\,ds\right)dz + (1-\lambda)\int_\gamma (z-w)^2\left(\int_0^1 f''[(1-s)w+sz](1-s)\,ds\right)dz = \lambda\int_0^1\left(\int_\gamma (z-u)^2 f''[(1-s)u+sz]\,dz\right)(1-s)\,ds + (1-\lambda)\int_0^1\left(\int_\gamma (z-w)^2 f''[(1-s)w+sz]\,dz\right)(1-s)\,ds. \tag{2.27}$$

In particular, for $\lambda=\frac{1}{2}$ we have the perturbed trapezoid type equality

$$\int_\gamma f(z)\,dz = \frac{f(u)+f(w)}{2}(w-u)+\frac{1}{4}[f'(u)-f'(w)](w-u)^2 + T_1(\gamma), \tag{2.28}$$

where

$$T_1(\gamma) := \frac{1}{2}\int_\gamma (z-u)^2\left(\int_0^1 f''[(1-s)u+sz](1-s)\,ds\right)dz + \frac{1}{2}\int_\gamma (z-w)^2\left(\int_0^1 f''[(1-s)w+sz](1-s)\,ds\right)dz = \frac{1}{2}\int_0^1\left(\int_\gamma (z-u)^2 f''[(1-s)u+sz]\,dz\right)(1-s)\,ds + \frac{1}{2}\int_0^1\left(\int_\gamma (z-w)^2 f''[(1-s)w+sz]\,dz\right)(1-s)\,ds. \tag{2.29}$$

Consider the function $f(z)=\frac{1}{z}$, $z\in\mathbb{C}\setminus\{0\}$. Then

$$f^{(k)}(z)=\frac{(-1)^k k!}{z^{k+1}} \text{ for } k\geq 0,\ z\in\mathbb{C}\setminus\{0\}$$

and suppose $\gamma\subset\mathbb{C}_\ell$ is a *smooth path* parametrized by $z(t)$, $t\in[a,b]$ with $z(a)=u$ and $z(b)=w$ where u, $w\in\mathbb{C}_\ell$. Then

$$\int_\gamma f(z)\,dz=\int_{\gamma_{u,w}} f(z)\,dz=\int_{\gamma_{u,w}}\frac{dz}{z}=\operatorname{Log}(w)-\operatorname{Log}(u)$$

for u, $w\in\mathbb{C}_\ell$.

Let D be a convex domain included in $\mathbb{C}_\ell$. Assume that $\gamma=\gamma_{u,w}\subset D$ and $x\in D$. Then by Theorem 1 we have

$$\operatorname{Log}(w)-\operatorname{Log}(u)=\sum_{k=0}^{n}\frac{(-1)^k}{(k+1)}\left[\left(\frac{w-x}{x}\right)^{k+1}+(-1)^k\left(\frac{x-u}{x}\right)^{k+1}\right] + R_n(x,\gamma), \tag{2.30}$$

where the remainder $R_n(x,\gamma)$ is given by

$$R_n(x,\gamma) := (n+1)(-1)^{n+1}\int_{\gamma_{u,w}}(z-x)^{n+1}\left(\int_0^1\frac{(1-s)^n\,ds}{[(1-s)x+sz]^{n+2}}\right)dz \tag{2.31}$$
$$= (n+1)(-1)^{n+1}\int_0^1\left(\int_{\gamma_{u,w}}\frac{(z-x)^{n+1}}{[(1-s)x+sz]^{n+2}}dz\right)(1-s)^n\,ds,$$

for $n\geq 0$.

Consider the function $f(z)=\operatorname{Log}(z)$, the *"principal branch"* of the complex logarithmic function. The function f is analytic on all of $\mathbb{C}_\ell := \mathbb{C}\setminus\{x+iy : x\leq 0,\ y=0\}$ and

$$f^{(k)}(z)=\frac{(-1)^{k-1}(k-1)!}{z^k},\ k\geq 1,\ z\in\mathbb{C}_\ell.$$

Suppose $\gamma\subset\mathbb{C}_\ell$ is a *smooth path* parametrized by $z(t)$, $t\in[a,b]$ with $z(a)=u$ and $z(b)=w$ where u, $w\in\mathbb{C}_\ell$. Then

$$\int_\gamma f(z)\,dz=\int_{\gamma_{u,w}}f(z)\,dz=\int_{\gamma_{u,w}}\operatorname{Log}(z)\,dz=$$
$$= z\operatorname{Log}(z)\big|_u^w-\int_{\gamma_{u,w}}(\operatorname{Log}(z))'\,zdz$$
$$= w\operatorname{Log}(w)-u\operatorname{Log}(u)-\int_{\gamma_{u,w}}dz$$
$$= w\operatorname{Log}(w)-u\operatorname{Log}(u)-(w-u),$$

where u, $w\in\mathbb{C}_\ell$.

Let D be a convex domain included in $\mathbb{C}_\ell$. Assume that $\gamma=\gamma_{u,w}\subset D$ and $x\in D$. Then by Theorem 1 we have

$$\int_\gamma f(z)\,dz=f(x)(w-u) \tag{2.32}$$
$$+\sum_{k=1}^n\frac{1}{(k+1)!}f^{(k)}(x)\left[(w-x)^{k+1}+(-1)^k(x-u)^{k+1}\right]+R_n(x,\gamma),$$

which gives

$$w\operatorname{Log}(w)-u\operatorname{Log}(u)-(w-u) \tag{2.33}$$
$$=(w-u)\operatorname{Log}(x)+x\sum_{k=1}^n\frac{(-1)^{k-1}}{(k+1)k}\left[\left(\frac{w-x}{x}\right)^{k+1}+(-1)^k\left(\frac{x-u}{x}\right)^{k+1}\right]$$
$$+R_n(x,\gamma),$$

where

$$R_n(x,\gamma):=(-1)^n\int_\gamma(z-x)^{n+1}\left(\int_0^1\frac{(1-s)^n\,ds}{[(1-s)x+sz]^{n+1}}\right)dz \tag{2.34}$$
$$=(-1)^n\int_0^1\left(\int_\gamma\frac{(z-x)^{n+1}}{[(1-s)x+sz]^{n+1}}dz\right)(1-s)^n\,ds,$$

for $n\geq 1$.

Consider the function $f(z)=\exp(z)$, $z\in\mathbb{C}$. Then

$$f^{(k)}(z)=\exp(z)\ \text{ for } k\geq 0,\ z\in\mathbb{C}$$

and suppose $\gamma\subset\mathbb{C}$ is a *smooth path* parametrized by $z(t)$, $t\in[a,b]$ with $z(a)=u$ and $z(b)=w$ where $u,\ w\in\mathbb{C}$. Then

$$\int_\gamma f(z)\,dz=\int_{\gamma_{u,w}}f(z)\,dz=\int_{\gamma_{u,w}}\exp(z)\,dz=\exp(w)-\exp(u).$$

By Theorem 1 we get

$$\exp(w)-\exp(u)=\exp(x)\sum_{k=0}^{n}\frac{1}{(k+1)!}\left[(w-x)^{k+1}+(-1)^k(x-u)^{k+1}\right]+R_n(x,\gamma), \tag{2.35}$$

where the remainder $R_n(x,\gamma)$ is given by

$$\begin{aligned}R_n(x,\gamma)&:=\frac{1}{n!}\int_\gamma(z-x)^{n+1}\left(\int_0^1\exp[(1-s)x+sz](1-s)^n\,ds\right)dz\\&=\frac{1}{n!}\int_0^1\left(\int_\gamma(z-x)^{n+1}\exp[(1-s)x+sz]\,dz\right)(1-s)^n\,ds\end{aligned} \tag{2.36}$$

for $n\geq 0$.

3. Error Bounds for Ostrowski's Rule

We have the following error bunds:

Theorem 2. *Let $f:D\subseteq\mathbb{C}\to\mathbb{C}$ be an analytic function on the convex domain D and $x\in D$. Suppose $\gamma\subset D$ is a smooth path parametrized by $z(t)$, $t\in[a,b]$ with $z(a)=u$ and $z(b)=w$ where $u,\ w\in D$. Then we have the representation (2.1) where the remainder $R_n(x,\gamma)$, $n\geq 0$ satisfies the bounds*

$$\begin{aligned}|R_n(x,\gamma)|&\leq\frac{1}{n!}\int_\gamma|z-x|^{n+1}\left(\int_0^1\left|f^{(n+1)}[(1-s)x+sz]\right|(1-s)^n\,ds\right)|dz|\\&\leq B_n(x,\gamma)\end{aligned} \tag{3.1}$$

where

$$B_n(x,\gamma):=\frac{1}{n!}\begin{cases}\frac{1}{n+1}\int_\gamma|z-x|^{n+1}\left(\max_{s\in[0,1]}\left|f^{(n+1)}[(1-s)x+sz]\right|\right)|dz|;\\[2ex]\frac{1}{(qn+1)^{1/q}}\int_\gamma|z-x|^{n+1}\left(\int_0^1\left|f^{(n+1)}[(1-s)x+sz]\right|^p ds\right)^{1/p}|dz|\\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1;\\[2ex]\int_\gamma|z-x|^{n+1}\left(\int_0^1\left|f^{(n+1)}[(1-s)x+sz]\right|ds\right)|dz|.\end{cases} \tag{3.2}$$

Moreover, we have

$$B_n(x,\gamma) \le \frac{1}{(n+1)!} \tag{3.3}$$

$$\times \begin{cases} \max_{s\in[0,1],z\in\gamma}\left|f^{(n+1)}[(1-s)x+sz]\right| \int_\gamma |z-x|^{n+1}|dz| \\ \\ \left(\int_\gamma |z-x|^{\alpha(n+1)}|dz|\right)^{1/\alpha}\left[\int_\gamma \left(\max_{s\in[0,1]}\left|f^{(n+1)}[(1-s)x+sz]\right|\right)^{\beta}|dz|\right]^{1/\beta} \\ \alpha,\ \beta>1 \ \textit{with}\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \\ \max_{z\in\gamma}|z-x|^{n+1}\int_\gamma \max_{s\in[0,1]}\left|f^{(n+1)}[(1-s)x+sz]\right||dz|, \end{cases}$$

$$B_n(x,\gamma) \le \frac{1}{n!\,(qn+1)^{1/q}} \tag{3.4}$$

$$\times \begin{cases} \max_{z\in\gamma}\left(\int_0^1\left|f^{(n+1)}[(1-s)x+sz]\right|^p ds\right)^{1/p}\int_\gamma |z-x|^{n+1}|dz| \\ \\ \left(\int_\gamma |z-x|^{\alpha(n+1)}|dz|\right)^{1/\alpha}\left[\left(\int_0^1\left|f^{(n+1)}[(1-s)x+sz]\right|^p ds\right)^{\beta/p}|dz|\right]^{1/\beta} \\ \alpha,\ \beta>1 \ \textit{with}\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \\ \max_{z\in\gamma}|z-x|^{n+1}\int_\gamma\left(\int_0^1\left|f^{(n+1)}[(1-s)x+sz]\right|^p ds\right)^{1/p}|dz| \end{cases}$$

and

$$B_n(x,\gamma) \tag{3.5}$$

$$\le \frac{1}{n!}\begin{cases} \max_{z\in\gamma}\int_0^1\left|f^{(n+1)}[(1-s)x+sz]\right|ds\int_\gamma |z-x|^{n+1}|dz| \\ \\ \left(\int_\gamma |z-x|^{\alpha(n+1)}|dz|\right)^{1/\alpha}\left[\left(\int_0^1\left|f^{(n+1)}[(1-s)x+sz]\right|ds\right)^{\beta}|dz|\right]^{1/\beta} \\ \alpha,\ \beta>1 \ \textit{with}\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \\ \max_{z\in\gamma}|z-x|^{n+1}\int_\gamma\left(\int_0^1\left|f^{(n+1)}[(1-s)x+sz]\right|ds\right)|dz|. \end{cases}$$

Proof. Taking the modulus in the first representation in (2.2) we get

$$\begin{aligned} |R_n(x,\gamma)| &= \frac{1}{n!}\left|\int_\gamma (z-x)^{n+1}\left(\int_0^1 f^{(n+1)}[(1-s)x+sz](1-s)^n ds\right)dz\right| \\ &\le \frac{1}{n!}\int_\gamma\left|(z-x)^{n+1}\left(\int_0^1 f^{(n+1)}[(1-s)x+sz](1-s)^n ds\right)\right||dz| \\ &= \frac{1}{n!}\int_\gamma |z-x|^{n+1}\left|\int_0^1 f^{(n+1)}[(1-s)x+sz](1-s)^n ds\right||dz| \\ &\le \frac{1}{n!}\int_\gamma |z-x|^{n+1}\left(\int_0^1\left|f^{(n+1)}[(1-s)x+sz]\right|(1-s)^n ds\right)|dz| \\ &=: A_n(x,\gamma) \end{aligned}$$

for $x \in D$.

Using Hölder's integral inequality we get

$$\begin{aligned}
&\int_0^1 \left|f^{(n+1)}\left[(1-s)x+sz\right]\right|(1-s)^n\,ds \\
&\le \begin{cases}
\max_{s\in[0,1]}\left|f^{(n+1)}\left[(1-s)x+sz\right]\right|\int_0^1(1-s)^n\,ds; \\
\left(\int_0^1\left|f^{(n+1)}\left[(1-s)x+sz\right]\right|^p ds\right)^{1/p}\left(\int_0^1(1-s)^{qn}\,ds\right)^{1/q} \\
p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\
\int_0^1\left|f^{(n+1)}\left[(1-s)x+sz\right]\right|ds
\end{cases} \\
&= \begin{cases}
\frac{1}{n+1}\max_{s\in[0,1]}\left|f^{(n+1)}\left[(1-s)x+sz\right]\right|; \\
\frac{1}{(qn+1)^{1/q}}\left(\int_0^1\left|f^{(n+1)}\left[(1-s)x+sz\right]\right|^p ds\right)^{1/p} \\
p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\
\int_0^1\left|f^{(n+1)}\left[(1-s)x+sz\right]\right|ds.
\end{cases}
\end{aligned}$$

Therefore

$$A_n(x,\gamma)\le\frac{1}{n!}\begin{cases}
\frac{1}{n+1}\int_\gamma|z-x|^{n+1}\left(\max_{s\in[0,1]}\left|f^{(n+1)}\left[(1-s)x+sz\right]\right|\right)|dz|; \\
\frac{1}{(qn+1)^{1/q}}\int_\gamma|z-x|^{n+1}\left(\int_0^1\left|f^{(n+1)}\left[(1-s)x+sz\right]\right|^p ds\right)^{1/p}|dz| \\
p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\
\int_\gamma|z-x|^{n+1}\left(\int_0^1\left|f^{(n+1)}\left[(1-s)x+sz\right]\right|ds\right)|dz|
\end{cases}$$

for $x\in D$, which proves the second bound in (3.1).

The bounds (3.3)-(3.5) follows by Hölder's integral inequality. □

For a recent survey on Ostrowski type inequalities for functions of a real variable, see [6].

In a similar way we can prove:

Theorem 3. *Let $f:D\subseteq\mathbb{C}\to\mathbb{C}$ be an analytic function on the convex domain D and $x\in D$. Suppose $\gamma\subset D$ is a smooth path parametrized by $z(t)$, $t\in[a,b]$ with $z(a)=u$ and $z(b)=w$ where u, $w\in D$. Then we have the representation (2.1) where the remainder $R_n(x,\gamma)$, $n\ge 0$ satisfies the bounds*

$$(3.6)\qquad |R_n(x,\gamma)|\le\frac{1}{n!}\int_0^1\left(\int_\gamma|z-x|^{n+1}\left|f^{(n+1)}\left[(1-s)x+sz\right]\right||dz|\right)(1-s)^n\,ds \le C_n(x,\gamma)$$

where

$$C_n(x,\gamma) := \begin{cases} \frac{1}{(n+1)!}\max_{s\in[0,1]}\int_\gamma |z-x|^{n+1}\left|f^{(n+1)}[(1-s)x+sz]\right||dz| \\ \\ \frac{1}{n!(nq+1)^{1/q}}\int_0^1\left(\int_\gamma |z-x|^{n+1}\left|f^{(n+1)}[(1-s)x+sz]\right||dz|\right)^{1/p}ds \\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\ \\ \frac{1}{n!}\int_0^1\left(\int_\gamma |z-x|^{n+1}\left|f^{(n+1)}[(1-s)x+sz]\right||dz|\right)ds. \end{cases} \tag{3.7}$$

Moreover, we have

$$C_n(x,\gamma) \le \frac{1}{(n+1)!} \times \begin{cases} \max_{s\in[0,1],z\in\gamma}\left|f^{(n+1)}[(1-s)x+sz]\right|\int_\gamma |z-x|^{n+1}|dz| \\ \\ \left(\int_\gamma |z-x|^{\alpha(n+1)}|dz|\right)^{1/\alpha}\max_{s\in[0,1]}\left(\int_\gamma\left|f^{(n+1)}[(1-s)x+sz]\right|^\beta|dz|\right)^{1/\beta} \\ \alpha,\ \beta>1 \text{ with } \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \\ \max_{z\in\gamma}|z-x|^{n+1}\max_{s\in[0,1]}\int_\gamma\left|f^{(n+1)}[(1-s)x+sz]\right||dz|, \end{cases} \tag{3.8}$$

$$C_n(x,\gamma) \le \frac{1}{n!(nq+1)^{1/q}} \times \begin{cases} \max_{z\in\gamma}|z-x|^{(n+1)/p}\int_0^1\left(\int_\gamma\left|f^{(n+1)}[(1-s)x+sz]\right||dz|\right)^{1/p}ds \\ \\ \left(\int_\gamma |z-x|^{\alpha(n+1)}|dz|\right)^{1/(p\alpha)}\int_0^1\left(\int_\gamma\left|f^{(n+1)}[(1-s)x+sz]\right|^\beta|dz|\right)^{\beta/p}ds \\ \alpha,\ \beta>1 \text{ with } \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \\ \left(\int_\gamma |z-x|^{n+1}|dz|\right)^{1/p}\int_0^1\max_{z\in\gamma}\left|f^{(n+1)}[(1-s)x+sz]\right|^{1/p}ds \end{cases} \tag{3.9}$$

and

$$C_n(x,\gamma) \le \frac{1}{n!} \begin{cases} \max_{z\in\gamma}|z-x|^{n+1}\int_0^1\left(\int_\gamma\left|f^{(n+1)}[(1-s)x+sz]\right||dz|\right)ds \\ \\ \left(\int_\gamma |z-x|^{\alpha(n+1)}|dz|\right)^{1/\alpha}\int_0^1\left(\int_\gamma\left|f^{(n+1)}[(1-s)x+sz]\right|^\beta|dz|\right)^{1/\beta}ds \\ \alpha,\ \beta>1 \text{ with } \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \\ \int_\gamma |z-x|^{n+1}|dz|\int_0^1\max_{z\in\gamma}\left|f^{(n+1)}[(1-s)x+sz]\right|ds. \end{cases} \tag{3.10}$$

The following particular case may be useful for applications:

Corollary 3. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and $x \in D$. Suppose $\gamma \subset D$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = u$ and $z(b) = w$ where u, $w \in D$. If*

$$\left\| f^{(n+1)} \right\|_{D,\infty} := \sup_{z \in D} \left| f^{(n+1)}(z) \right| < \infty \text{ for some } n \geq 0, \tag{3.11}$$

then we have the representation (2.1) where the remainder $R_n(x,\gamma)$ satisfies the bound

$$|R_n(x,\gamma)| \leq \frac{1}{(n+1)!} \left\| f^{(n+1)} \right\|_{D,\infty} \int_\gamma |z - x|^{n+1} |dz|. \tag{3.12}$$

Remark 2. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and $x \in D$. Suppose $\gamma \subset D$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = u$ and $z(b) = w$ where u, $w \in D$.*

For $n = 0$ we get the Ostrowski type inequality

$$\left| \int_\gamma f(z)\, dz - f(x)(w-u) \right| \leq \|f'\|_{D,\infty} \int_\gamma |z-x|\, |dz| \tag{3.13}$$

for $x \in D$, provided $\|f'\|_{D,\infty} < \infty$.

In particular, we have the midpoint type inequality

$$\left| \int_\gamma f(z)\, dz - f\left(\frac{u+w}{2}\right)(w-u) \right| \leq \|f'\|_{D,\infty} \int_\gamma \left| z - \frac{u+w}{2} \right| |dz| \tag{3.14}$$

provided $\|f'\|_{D,\infty} < \infty$.

For $n = 1$ we get the perturbed Ostrowski type inequality

$$\left| \int_\gamma f(z)\, dz - f(x)(w-u) - f'(x)\left(\frac{w+u}{2} - x\right)(w-u) \right| \leq \frac{1}{2} \|f''\|_{D,\infty} \int_\gamma |z-x|^2 |dz| \tag{3.15}$$

for $x \in D$, provided $\|f''\|_{D,\infty} < \infty$.

In particular, we have the midpoint type inequality

$$\left| \int_\gamma f(z)\, dz - f\left(\frac{u+w}{2}\right)(w-u) \right| \leq \frac{1}{2} \|f''\|_{D,\infty} \int_\gamma \left| z - \frac{u+w}{2} \right|^2 |dz| \tag{3.16}$$

provided $\|f''\|_{D,\infty} < \infty$.

We also have:

Theorem 4. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and $x \in D$. Suppose $\gamma \subset D$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = u$ and $z(b) = w$ where u, $w \in D$. If $\left| f^{(n+1)} \right|$ is convex on D, for some $n \geq 0$, then we have the representation (2.1) where the remainder $R_n(x,\gamma)$ satisfies the bounds*

$$|R_n(x,\gamma)| \leq \frac{1}{n!\,(n+2)} \times \left[\left| f^{(n+1)}(x) \right| \int_\gamma |z-x|^{n+1} |dz| + \frac{1}{(n+1)} \int_\gamma |z-x|^{n+1} \left| f^{(n+1)}(z) \right| |dz| \right]. \tag{3.17}$$

Proof. We have by (3.6) and by the convexity of $\left|f^{(n+1)}\right|$ that

$$\begin{aligned}(3.18)\quad & |R_n(x,\gamma)| \\ & \leq \frac{1}{n!}\int_\gamma |z-x|^{n+1}\left(\int_0^1 \left|f^{(n+1)}[(1-s)x+sz]\right|(1-s)^n\,ds\right)|dz| \\ & \leq \frac{1}{n!}\int_\gamma |z-x|^{n+1}\left(\int_0^1 \left[(1-s)\left|f^{(n+1)}(x)\right|+s\left|f^{(n+1)}(z)\right|\right](1-s)^n\,ds\right)|dz| \\ & = \frac{1}{n!}\int_\gamma |z-x|^{n+1} \\ & \times\left[\left|f^{(n+1)}(x)\right|\int_0^1 (1-s)^{n+1}\,ds+\left|f^{(n+1)}(z)\right|\int_0^1 s(1-s)^n\,ds\right]|dz| =: C_n.\end{aligned}$$

Since

$$\int_0^1 (1-s)^{n+1}\,ds=\int_0^1 s^{n+1}ds=\frac{1}{n+2}$$

and

$$\begin{aligned}\int_0^1 s(1-s)^n\,ds &= \int_0^1 (1-s)s^n ds=\int_0^1 \left(s^n-s^{n+1}\right)ds=\frac{1}{n+1}-\frac{1}{n+2} \\ &= \frac{1}{(n+1)(n+2)},\end{aligned}$$

hence

$$\begin{aligned}C_n &= \frac{1}{n!}\int_\gamma |z-x|^{n+1}\left[\left|f^{(n+1)}(x)\right|\frac{1}{n+2}+\left|f^{(n+1)}(z)\right|\frac{1}{(n+1)(n+2)}\right]|dz| \\ &= \frac{1}{n!(n+2)}\int_\gamma |z-x|^{n+1}\left[\left|f^{(n+1)}(x)\right|+\left|f^{(n+1)}(z)\right|\frac{1}{(n+1)}\right]|dz| \\ &= \frac{1}{n!(n+2)} \\ &\times\left[\left|f^{(n+1)}(x)\right|\int_\gamma |z-x|^{n+1}|dz|+\frac{1}{(n+1)}\int_\gamma |z-x|^{n+1}\left|f^{(n+1)}(z)\right||dz|\right]\end{aligned}$$

and by (3.18) we get the desired result (3.18). □

Remark 3. *Let $f: D\subseteq\mathbb{C}\to\mathbb{C}$ be an analytic function on the convex domain D and $x\in D$. Suppose $\gamma\subset D$ is a smooth path parametrized by $z(t)$, $t\in[a,b]$ with $z(a)=u$ and $z(b)=w$ where u, $w\in D$. If $|f'|$ is convex on D, then we have the Ostrowski type inequality*

$$\begin{aligned}(3.19)\quad & \left|\int_\gamma f(z)\,dz-f(x)(w-u)\right| \\ & \leq \frac{1}{2}\left[|f'(x)|\int_\gamma |z-x|\,|dz|+\int_\gamma |z-x|\,|f'(z)|\,|dz|\right]\end{aligned}$$

for any $x\in D$.

In particular, we have the midpoint inequality

$$\left|\int_\gamma f(z)\,dz - f\left(\frac{u+w}{2}\right)(w-u)\right| \le \frac{1}{2}\left[\left|f'\left(\frac{u+w}{2}\right)\right|\int_\gamma \left|z-\frac{u+w}{2}\right||dz| + \int_\gamma \left|z-\frac{u+w}{2}\right||f'(z)||dz|\right]. \tag{3.20}$$

If $|f''|$ is convex on D, then we have the perturbed Ostrowski type inequality

$$\left|\int_\gamma f(z)\,dz - f(x)(w-u) - f'(x)\left(\frac{w+u}{2}-x\right)(w-u)\right| \le \frac{1}{6}\left[|f''(x)|\int_\gamma |z-x|^2|dz| + \frac{1}{2}\int_\gamma |z-x|^2|f''(z)||dz|\right] \tag{3.21}$$

for any $x \in D$.

In particular, we have the midpoint inequality

$$\left|\int_\gamma f(z)\,dz - f\left(\frac{u+w}{2}\right)(w-u)\right| \le \frac{1}{6}\left[\left|f''\left(\frac{w+u}{2}\right)\right|\int_\gamma \left|z-\frac{w+u}{2}\right|^2|dz| + \frac{1}{2}\int_\gamma \left|z-\frac{w+u}{2}\right|^2|f''(z)||dz|\right]. \tag{3.22}$$

4. Error Bounds for Trapezoid Rule

Similar inequalities may be stated for the trapezoid rule, however here we present only the simplest case of bounded derivatives:

Theorem 5. *Let $f: D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and $x \in D$. Suppose $\gamma \subset D$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = u$ and $z(b) = w$ where $u, w \in D$. If $f^{(n+1)}$ satisfies the condition (3.11) for some $n \ge 0$ and $\lambda \in \mathbb{C}$, then we have the representation (2.6) and the remainder $T_n(\lambda,\gamma)$ satisfies the bound*

$$\begin{aligned} |T_n(\lambda,\gamma)| &\le \frac{1}{(n+1)!}\left\|f^{(n+1)}\right\|_{D,\infty}\left[|\lambda|\int_\gamma |z-u|^{n+1}|dz| + |1-\lambda|\int_\gamma |z-w|^{n+1}|dz|\right] \\ &\le \max\{|\lambda|,|1-\lambda|\}\frac{1}{(n+1)!}\left\|f^{(n+1)}\right\|_{D,\infty}\left[\int_\gamma |z-u|^{n+1}|dz| + \int_\gamma |z-w|^{n+1}|dz|\right]. \end{aligned} \tag{4.1}$$

In particular, if $\lambda = \frac{1}{2}$, then we have the bound

$$|T_n(\gamma)| \le \frac{1}{2(n+1)!}\left\|f^{(n+1)}\right\|_{D,\infty}\left[\int_\gamma |z-u|^{n+1}|dz| + \int_\gamma |z-w|^{n+1}|dz|\right]. \tag{4.2}$$

Proof. Using the representation (2.7) we have

$$
\begin{aligned}
(4.3)\quad \left|T_n(\lambda,\gamma)\right| &\le \left|\frac{\lambda}{n!}\int_\gamma (z-u)^{n+1}\left(\int_0^1 f^{(n+1)}\left[(1-s)u+sz\right](1-s)^n ds\right)dz\right| \\
&+\left|\frac{(1-\lambda)}{n!}\int_\gamma (z-w)^{n+1}\left(\int_0^1 f^{(n+1)}\left[(1-s)w+sz\right](1-s)^n ds\right)dz\right| \\
&\le |\lambda|\frac{1}{n!}\int_\gamma |z-u|^{n+1}\left|\int_0^1 f^{(n+1)}\left[(1-s)u+sz\right](1-s)^n ds\right||dz| \\
&+\frac{1}{n!}|1-\lambda|\int_\gamma |z-w|^{n+1}\left|\int_0^1 f^{(n+1)}\left[(1-s)w+sz\right](1-s)^n ds\right||dz|
\end{aligned}
$$

$$
\begin{aligned}
&\le |\lambda|\frac{1}{n!}\int_\gamma |z-u|^{n+1}\left(\int_0^1 f^{(n+1)}\left|\left[(1-s)u+sz\right]\right|(1-s)^n ds\right)|dz| \\
&+\frac{1}{n!}|1-\lambda|\int_\gamma |z-w|^{n+1}\left(\int_0^1 \left|f^{(n+1)}\left[(1-s)w+sz\right]\right|(1-s)^n ds\right)|dz| \\
&\le |\lambda|\frac{1}{n!}\left\|f^{(n+1)}\right\|_{D,\infty}\int_\gamma |z-u|^{n+1}\left(\int_0^1 (1-s)^n ds\right)|dz| \\
&+\frac{1}{n!}|1-\lambda|\left\|f^{(n+1)}\right\|_{D,\infty}\int_\gamma |z-w|^{n+1}\left(\int_0^1 (1-s)^n ds\right)|dz| \\
&= |\lambda|\frac{1}{(n+1)!}\left\|f^{(n+1)}\right\|_{D,\infty}\int_\gamma |z-u|^{n+1}|dz| \\
&\qquad+\frac{1}{(n+1)!}|1-\lambda|\left\|f^{(n+1)}\right\|_{D,\infty}\int_\gamma |z-w|^{n+1}|dz|,
\end{aligned}
$$

which proves the desired result (4.1). □

For some inequalities of trapezoid type for functions of a real variable, see [4].

Remark 4. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and $x \in D$. Suppose $\gamma \subset D$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = u$ and $z(b) = w$ where u, $w \in D$.*

For $n = 0$ we get the generalized trapezoid type inequality

$$
\begin{aligned}
(4.4)\quad &\left|\int_\gamma f(z)\,dz - \left[\lambda f(u) + (1-\lambda) f(w)\right](w-u)\right| \\
&\qquad\le \left\|f'\right\|_{D,\infty}\left[|\lambda|\int_\gamma |z-u||dz| + |1-\lambda|\int_\gamma |z-w||dz|\right]
\end{aligned}
$$

for $\lambda \in \mathbb{C}$, provided $\left\|f'\right\|_{D,\infty} < \infty$.

In particular, we have the trapezoid inequality

$$
\begin{aligned}
(4.5)\quad &\left|\int_\gamma f(z)\,dz - \frac{f(u)+f(w)}{2}(w-u)\right| \\
&\qquad\le \frac{1}{2}\left\|f'\right\|_{D,\infty}\left[\int_\gamma |z-u||dz| + \int_\gamma |z-w||dz|\right]
\end{aligned}
$$

provided $\left\|f'\right\|_{D,\infty} < \infty$.

We also have the perturbed generalized trapezoid inequality

$$
\begin{aligned}
(4.6)\quad & \left|\int_\gamma f(z)\,dz - [\lambda f(u) + (1-\lambda) f(w)](w-u) \right. \\
& \left. -\frac{1}{2}[\lambda f'(u) - (1-\lambda) f'(w)](w-u)^2 \right| \\
& \le \frac{1}{2}\|f''\|_{D,\infty}\left[|\lambda|\int_\gamma |z-u|^2 |dz| + |1-\lambda|\int_\gamma |z-w|^2 |dz|\right] \\
& \le \frac{1}{2}\max\{|\lambda|, |1-\lambda|\}\|f''\|_{D,\infty}\left[\int_\gamma |z-u|^2 |dz| + \int_\gamma |z-w|^2 |dz|\right]
\end{aligned}
$$

for $\lambda \in \mathbb{C}$, *provided* $\|f''\|_{D,\infty} < \infty$.

In particular, we have the perturbed trapezoid inequality

$$
\begin{aligned}
(4.7)\quad & \left|\int_\gamma f(z)\,dz - \frac{f(u)+f(w)}{2}(w-u) \right. \\
& \left. -\frac{1}{4}[f'(u) - f'(w)](w-u)^2 \right| \\
& \le \frac{1}{4}\|f''\|_{D,\infty}\left[\int_\gamma |z-u|^2 |dz| + \int_\gamma |z-w|^2 |dz|\right]
\end{aligned}
$$

provided $\|f''\|_{D,\infty} < \infty$.

We also have:

Theorem 6. *Let* $f : D \subseteq \mathbb{C} \to \mathbb{C}$ *be an analytic function on the convex domain* D *and* $x \in D$. *Suppose* $\gamma \subset D$ *is a smooth path parametrized by* $z(t)$, $t \in [a,b]$ *with* $z(a) = u$ *and* $z(b) = w$ *where* u, $w \in D$. *If* $\left|f^{(n+1)}\right|$ *is convex on* D, *for some* $n \ge 0$, *then we have the representation (2.6) and the remainder* $T_n(\lambda,\gamma)$ *satisfies the bound*

$$
\begin{aligned}
(4.8)\quad & |T_n(\lambda,\gamma)| \le \frac{1}{(n+2)\,n!} \\
& \times \left\{|\lambda|\left[f^{(n+1)}|(u)|\int_\gamma |z-u|^{n+1}|dz| + \frac{1}{(n+1)}\int_\gamma |z-u|^{n+1} f^{(n+1)}|(z)|\,|dz|\right]\right. \\
& \left. + |1-\lambda|\left[f^{(n+1)}|(w)|\int_\gamma |z-w|^{n+1}|dz| + \frac{1}{(n+1)}\int_\gamma |z-w|^{n+1} f^{(n+1)}|(z)|\,|dz|\right]\right\} \\
& \le \frac{1}{(n+2)\,n!}\max\{|\lambda|, |1-\lambda|\} \\
& \times \left[f^{(n+1)}|(u)|\int_\gamma |z-u|^{n+1}|dz| + f^{(n+1)}|(w)|\int_\gamma |z-w|^{n+1}|dz|\right. \\
& \left. + \frac{1}{(n+1)}\int_\gamma \left[|z-u|^{n+1} + |z-w|^{n+1}\right] f^{(n+1)}|(z)|\,|dz|\right]
\end{aligned}
$$

for $\lambda \in \mathbb{C}$.

Proof. Using the representation (2.7) and the convexity of $\left|f^{(n+1)}\right|$ is on D we have

$$\begin{aligned}
(4.9)\quad & |T_n(\lambda,\gamma)| \\
&\leq |\lambda| \frac{1}{n!}\int_\gamma |z-u|^{n+1}\left(\int_0^1 f^{(n+1)}|[(1-s)u+sz]|(1-s)^n ds\right)|dz| \\
&+\frac{1}{n!}|1-\lambda|\int_\gamma |z-w|^{n+1}\left(\int_0^1 \left|f^{(n+1)}[(1-s)w+sz]\right|(1-s)^n ds\right)|dz| \\
&\leq \frac{1}{n!}\left\{|\lambda|\int_\gamma |z-u|^{n+1}\left(\int_0^1\left[(1-s)f^{(n+1)}|(u)|+sf^{(n+1)}|(z)|\right](1-s)^n ds\right)|dz|\right. \\
&\qquad +|1-\lambda| \\
&\left.\times\int_\gamma |z-w|^{n+1}\left(\int_0^1\left[(1-s)\left|f^{(n+1)}(w)\right|+s\left|f^{(n+1)}(z)\right|\right](1-s)^n ds\right)|dz|\right\}
\end{aligned}$$

for $\lambda\in\mathbb{C}$.

Since

$$\begin{aligned}
&\int_\gamma |z-u|^{n+1}\left(\int_0^1\left[(1-s)f^{(n+1)}|(u)|+sf^{(n+1)}|(z)|\right](1-s)^n ds\right)|dz| \\
&\quad = f^{(n+1)}|(u)|\int_\gamma |z-u|^{n+1}|dz|\int_0^1(1-s)^{n+1}ds \\
&\quad + \int_\gamma |z-u|^{n+1} f^{(n+1)}|(z)||dz|\int_0^1 s(1-s)^n ds \\
&\quad = \frac{1}{n+2}f^{(n+1)}|(u)|\int_\gamma |z-u|^{n+1}|dz| \\
&\quad + \frac{1}{(n+1)(n+2)}\int_\gamma |z-u|^{n+1}f^{(n+1)}|(z)||dz| \\
&= \frac{1}{n+2}\left[f^{(n+1)}|(u)|\int_\gamma |z-u|^{n+1}|dz|+\frac{1}{(n+1)}\int_\gamma |z-u|^{n+1}f^{(n+1)}|(z)||dz|\right]
\end{aligned}$$

and, similarly

$$\begin{aligned}
&\int_\gamma |z-w|^{n+1}\left(\int_0^1\left[(1-s)\left|f^{(n+1)}(w)\right|+s\left|f^{(n+1)}(z)\right|\right](1-s)^n ds\right)|dz| \\
&= \frac{1}{n+2}\left[f^{(n+1)}|(w)|\int_\gamma |z-w|^{n+1}|dz|+\frac{1}{(n+1)}\int_\gamma |z-w|^{n+1}f^{(n+1)}|(z)||dz|\right],
\end{aligned}$$

hence by (4.9) we get the desired result (4.8). □

Remark 5. *For $\lambda=\frac{1}{2}$ we have the representation (2.8) and the remainder satisfies the bound*

$$\begin{aligned}
(4.10)\quad & |T_n(\gamma)| \leq \frac{1}{2(n+2)n!} \\
&\times\left\{\left[f^{(n+1)}|(u)|\int_\gamma |z-u|^{n+1}|dz|+f^{(n+1)}|(w)|\int_\gamma |z-w|^{n+1}|dz|\right]\right. \\
&\qquad\left.+\frac{1}{(n+1)}\left[\int_\gamma\left[|z-u|^{n+1}|dz|+|z-w|^{n+1}\right]f^{(n+1)}|(z)||dz|\right]\right\},
\end{aligned}$$

provided $\left|f^{(n+1)}\right|$ *is convex on* D*, for some* $n\geq 0$.

Remark 6. *If* $|f'|$ *is convex on* D*, then we have the inequality*

$$\left|\int_{\gamma} f(z)\,dz - [\lambda f(u) + (1-\lambda) f(w)](w-u)\right| \tag{4.11}$$
$$\leq \frac{1}{2}\left\{|\lambda|\left[|f'(u)|\int_{\gamma}|z-u|\,|dz| + \int_{\gamma}|z-u|\,|f'(z)|\,|dz|\right]\right.$$
$$\left. + |1-\lambda|\left[|f'(w)|\int_{\gamma}|z-w|\,|dz| + \int_{\gamma}|z-w|\,|f'(z)|\,|dz|\right]\right\}$$
$$\leq \frac{1}{2}\max\{|\lambda|,|1-\lambda|\}\left[|f'(u)|\int_{\gamma}|z-u|\,|dz| + |f'(w)|\int_{\gamma}|z-w|\,|dz|\right.$$
$$\left. + \int_{\gamma}[|z-u|+|z-w|]\,|f'(z)|\,|dz|\right]$$

and for $\lambda = \frac{1}{2}$,

$$\left|\int_{\gamma} f(z)\,dz - \frac{f(u)+f(w)}{2}(w-u)\right| \tag{4.12}$$
$$\leq \frac{1}{4}\left[|f'(u)|\int_{\gamma}|z-u|\,|dz| + |f'(w)|\int_{\gamma}|z-w|\,|dz|\right.$$
$$\left. + \int_{\gamma}[|z-u|+|z-w|]\,|f'(z)|\,|dz|\right].$$

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