AN INTEGRAL REPRESENTATION OF THE REMAINDER IN TAYLOR'S EXPANSION FORMULA FOR ANALYTIC FUNCTION ON GENERAL DOMAINS

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Abstract. In this paper we establish an integral representation of the remainder in Taylor's expansion formula for analytic function defined on non-necessarily convex domains. Error bounds are provided and some examples for the complex logarithm and complex exponential are also given.

1. Introduction

Suppose γ is a *smooth path* parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_{\gamma} f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are *piecewise smooth*. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{1.1}$$

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We recall also the *triangle inequality* for the complex integral, namely

$$\left|\int_{\gamma} f(z)\,dz\right| \leq \int_{\gamma} |f(z)|\,|dz| \leq \|f\|_{\gamma,\infty}\,\ell(\gamma) \tag{1.2}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \geq 1$ by

$$\|f\|_{\gamma,p} := \left(\int_{\gamma} |f(z)|^{p}\,|dz|\right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_{\gamma} |f(z)|\,|dz|.$$

If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \leq [\ell(\gamma)]^{1/q}\,\|f\|_{\gamma,p}.$$

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and y, $x \in D$, then we have the following Taylor's expansion with integral remainder

$$f(y) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x)(y-x)^{k} + \frac{1}{n!}(y-x)^{n+1} \int_{0}^{1} f^{(n+1)}[(1-s)x + sy](1-s)^{n}\,ds \tag{1.3}$$

for $n \geq 0$, see for instance [13].

Consider the function $f(z) = \operatorname{Log}(z)$ where $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$ and $\operatorname{Arg}(z)$ is such that $-\pi < \operatorname{Arg}(z) \leq \pi$. Log is called the *"principal branch"* of the complex logarithmic function. The function f is analytic on all of $\mathbb{C}_{\ell} := \mathbb{C}\setminus\{x + iy : x \leq 0,\ y = 0\}$ and

$$f^{(k)}(z) = \frac{(-1)^{k-1}(k-1)!}{z^{k}},\ k \geq 1,\ z \in \mathbb{C}_{\ell}.$$

Using the representation (1.3) we then have

$$\operatorname{Log}(z) = \operatorname{Log}(x) + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\left(\frac{z-x}{x}\right)^{k} + (-1)^{n}(z-x)^{n+1} \int_{0}^{1} \frac{(1-s)^{n}\,ds}{[(1-s)x + sz]^{n+1}} \tag{1.4}$$

for all z, $x \in \mathbb{C}_{\ell}$ with $(1-s)x + sz \in \mathbb{C}_{\ell}$ for $s \in [0,1]$.

Consider the complex exponential function $f(z) = \exp(z)$, then by (1.3) we get

$$\exp(z) = \sum_{k=0}^{n} \frac{1}{k!}(z-x)^{k}\exp(x) + \frac{1}{n!}(z-x)^{n+1} \int_{0}^{1} (1-s)^{n}\exp[(1-s)x + sz]\,ds \tag{1.5}$$

for all z, $x \in \mathbb{C}$.

For various inequalities related to Taylor's expansions for real functions see [1]-[12].

In this paper we establish a representation of the remainder in Taylor's expansion formula for analytic function defined on non-necessarily convex domains. Error bounds are provided and some examples for the complex logarithm and complex exponential are also given.

2. Integral Remainder Representation

We can extend the Taylor's representation formula (1.3) for non-necessarily convex domains D as follows:

Theorem 1. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the domain D and y, $x \in D$. Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = x$ and $z(b) = y$ then*

$$f(y) = \sum_{k=0}^{n} \frac{(y-x)^k}{k!} f^{(k)}(x) + \frac{1}{n!} \int_{\gamma_{x,y}} (y-z)^n f^{(n+1)}(z)\, dz \tag{2.1}$$

for $n \geq 0$.

Proof. We prove the identity (2.1) by induction over $n \geq 0$.

For $n = 0$ we have

$$f(y) = f(x) + \int_{\gamma_{x,y}} f'(z)\, dz,$$

which is obviously true.

Let assume that (2.1) holds for a natural number $m \geq 1$, namely

$$f(y) = \sum_{k=0}^{m} \frac{(y-x)^k}{k!} f^{(k)}(x) + \frac{1}{m!} \int_{\gamma_{x,y}} (y-z)^m f^{(m+1)}(z)\, dz \tag{2.2}$$

and let us prove it that it also holds for $m+1$, namely

$$f(y) = \sum_{k=0}^{m+1} \frac{(y-x)^k}{k!} f^{(k)}(x) + \frac{1}{(m+1)!} \int_{\gamma_{x,y}} (y-z)^{m+1} f^{(m+2)}(z)\, dz. \tag{2.3}$$

Using integration by parts we have

$$\begin{aligned} &\frac{1}{(m+1)!} \int_{\gamma_{x,y}} (y-z)^{m+1} f^{(m+2)}(z)\, dz \qquad (2.4)\\ &= \frac{1}{(m+1)!} \int_{\gamma_{x,y}} (y-z)^{m+1} \left(f^{(m+1)}(z)\right)' dz \\ &= \frac{1}{(m+1)!} \left[(y-z)^{m+1} f^{(m+1)}(z)\Big|_x^y + (m+1) \int_{\gamma_{x,y}} (y-z)^m f^{(m+1)}(z)\, dz \right] \\ &= \frac{1}{(m+1)!} \left[(m+1) \int_{\gamma_{x,y}} (y-z)^m f^{(m+1)}(z)\, dz - (y-x)^{m+1} f^{(m+1)}(x) \right] \\ &= \frac{1}{m!} \int_{\gamma_{x,y}} (y-z)^m f^{(m+1)}(z)\, dz - \frac{1}{(m+1)!} (y-x)^{m+1} f^{(m+1)}(x). \end{aligned}$$

Using the induction hypothesis (2.2) we have

$$\frac{1}{m!} \int_{\gamma_{x,y}} (y-z)^m f^{(m+1)}(z)\, dz = f(y) - \sum_{k=0}^{m} \frac{(y-x)^k}{k!} f^{(k)}(x)$$

and by (2.4) we then get

$$\begin{aligned}
&\frac{1}{(m+1)!}\int_{\gamma_{x,y}}(y-z)^{m+1}f^{(m+2)}(z)\,dz\\
&=f(y)-\sum_{k=0}^{m}\frac{(y-x)^k}{k!}f^{(k)}(x)-\frac{1}{(m+1)!}(y-x)^{m+1}f^{(m+1)}(x)\\
&=f(y)-\sum_{k=0}^{m+1}\frac{(y-x)^k}{k!}f^{(k)}(x),
\end{aligned}$$

which proves the desired equality (2.3). □

Using the representation (2.1) we then have

$$\operatorname{Log}(y)=\operatorname{Log}(x)+\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\left(\frac{y-x}{x}\right)^k+(-1)^n\int_{\gamma_{x,y}}\frac{(y-z)^n}{z^{n+1}}dz \tag{2.5}$$

for all y, $x\in\mathbb{C}_\ell$ and $\gamma=\gamma_{x,y}\subset\mathbb{C}_\ell$.

Consider the complex exponential function $f(z)=\exp(z)$, then by (1.3) we get

$$\exp(y)=\sum_{k=0}^{n}\frac{1}{k!}(y-x)^k\exp(x)+\frac{1}{n!}\int_{\gamma_{x,y}}(y-z)^n\exp(z)\,dz \tag{2.6}$$

for all y, $x\in\mathbb{C}$ and a smooth path $\gamma=\gamma_{x,y}$ joining the complex numbers y, x.

Corollary 1. *With the assumptions of Theorem 1, then for any $\lambda\in\mathbb{C}$ we have the perturbed identity*

$$\begin{aligned}
f(y)=\sum_{k=0}^{n}\frac{(y-x)^k}{k!}f^{(k)}(x)&+\frac{(y-x)^{n+1}}{(n+1)!}\lambda\\
&+\frac{1}{n!}\int_{\gamma_{x,y}}(y-z)^n\left[f^{(n+1)}(z)-\lambda\right]dz
\end{aligned}\tag{2.7}$$

for $n\geq 0$.

3. Error Bounds

We have the following error bounds for the general perturbed Taylor's expansion (2.7).

Theorem 2. *Let $f:D\subseteq\mathbb{C}\to\mathbb{C}$ be an analytic function on the domain D and y, $x\in D$. Suppose γ is a smooth path parametrized by $z(t)$, $t\in[a,b]$ with $z(a)=x$*

and $z(b) = y$ *and* $\lambda \in \mathbb{C}$, *then*

$$\left| f(y) - \sum_{k=0}^{n} \frac{(y-x)^k}{k!} f^{(k)}(x) - \frac{(y-x)^{n+1}}{(n+1)!}\lambda \right| \le \frac{1}{n!}\int_{\gamma_{x,y}} |y-z|^n \left| f^{(n+1)}(z) - \lambda \right| |dz| \le \frac{1}{n!}\begin{cases} \max_{z\in\gamma_{x,y}} |y-z|^n \left\| f^{(n+1)} - \lambda \right\|_{\gamma_{x,y},1}, \\ \left(\int_{\gamma_{x,y}} |y-z|^{np}|dz|\right)^{1/p} \left\| f^{(n+1)} - \lambda \right\|_{\gamma_{x,y},q} \\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \int_{\gamma_{x,y}} |y-z|^n |dz| \left\| f^{(n+1)} - \lambda \right\|_{\gamma_{x,y},\infty}. \end{cases} \tag{3.1}$$

In particular, for $\lambda = 0$,

$$\left| f(y) - \sum_{k=0}^{n} \frac{(y-x)^k}{k!} f^{(k)}(x) \right| \le \frac{1}{n!}\int_{\gamma_{x,y}} |y-z|^n \left| f^{(n+1)}(z) \right| |dz| \le \frac{1}{n!}\begin{cases} \max_{z\in\gamma_{x,y}} |y-z|^n \left\| f^{(n+1)} \right\|_{\gamma_{x,y},1}, \\ \left(\int_{\gamma_{x,y}} |y-z|^{np}|dz|\right)^{1/p} \left\| f^{(n+1)} \right\|_{\gamma_{x,y},q} \\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \int_{\gamma_{x,y}} |y-z|^n |dz| \left\| f^{(n+1)} \right\|_{\gamma_{x,y},\infty}. \end{cases} \tag{3.2}$$

Proof. Using the representation (2.7) and Hölder's inequality, we have

$$\left| f(y) - \sum_{k=0}^{n} \frac{(y-x)^k}{k!} f^{(k)}(x) - \frac{(y-x)^{n+1}}{(n+1)!}\lambda \right| = \frac{1}{n!}\left| \int_{\gamma_{x,y}} (y-z)^n \left[f^{(n+1)}(z) - \lambda \right] dz \right|$$

$$\le \frac{1}{n!}\int_{\gamma_{x,y}} \left| (y-z)^n \left[f^{(n+1)}(z) - \lambda \right] \right| |dz|$$

$$= \frac{1}{n!}\int_{\gamma_{x,y}} |y-z|^n \left| f^{(n+1)}(z) - \lambda \right| |dz|$$

$$\le \frac{1}{n!}\begin{cases} \max_{z\in\gamma_{x,y}} |y-z|^n \int_{\gamma_{x,y}} \left| f^{(n+1)}(z) - \lambda \right| |dz|, \\ \left(\int_{\gamma_{x,y}} |y-z|^{np}|dz|\right)^{1/p} \left(\int_{\gamma_{x,y}} \left| f^{(n+1)}(z) - \lambda \right|^q |dz|\right)^{1/q} \\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \int_{\gamma_{x,y}} |y-z|^n |dz| \max_{z\in\gamma_{x,y}} \left| f^{(n+1)}(z) - \lambda \right|, \end{cases}$$

which proves the desired result (3.1). □

By using the first inequality in (3.2) we have

$$\left|\operatorname{Log}(y)-\operatorname{Log}(x)-\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\left(\frac{y-x}{x}\right)^{k}\right|\leq\int_{\gamma_{x,y}}\frac{|y-z|^{n}}{|z|^{n+1}}|dz| \tag{3.3}$$

for all y, $x\in\mathbb{C}_{\ell}$ and $\gamma=\gamma_{x,y}\subset\mathbb{C}_{\ell}$.

Now, if we assume that $d_{x,y}=\inf_{z\in\gamma_{x,y}}|z|\in(0,\infty)$, then by (3.3) we get

$$\left|\operatorname{Log}(y)-\operatorname{Log}(x)-\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\left(\frac{y-x}{x}\right)^{k}\right|\leq\frac{1}{d_{x,y}^{n+1}}\int_{\gamma_{x,y}}|y-z|^{n}|dz|. \tag{3.4}$$

By using the first inequality in (3.2) we also have

$$\left|\exp(y)-\sum_{k=0}^{n}\frac{1}{k!}(y-x)^{k}\exp(x)\right|\leq\int_{\gamma_{x,y}}|y-z|^{n}\exp\operatorname{Re}(z)|dz|, \tag{3.5}$$

for all y, $x\in\mathbb{C}$ and a smooth path $\gamma=\gamma_{x,y}$ joining the complex numbers y, x.

If $\operatorname{Re}(z)\leq M$ for $z\in\gamma_{x,y}$, then by (3.5) we get

$$\left|\exp(y)-\sum_{k=0}^{n}\frac{1}{k!}(y-x)^{k}\exp(x)\right|\leq\frac{1}{n!}\exp M\int_{\gamma_{x,y}}|y-z|^{n}\exp\operatorname{Re}(z)|dz|. \tag{3.6}$$

Suppose $\gamma\subset\mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t\in\gamma$ from $z(a)=u$ to $z(b)=w$. Now, for ϕ, $\Phi\in\mathbb{C}$ and γ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{\gamma}(\phi,\Phi):=\left\{f:\gamma\to\mathbb{C}|\operatorname{Re}\left[(\Phi-f(z))\left(\overline{f(z)}-\overline{\phi}\right)\right]\geq0\ \text{ for each }\ z\in\gamma\right\}$$

and

$$\bar{\Delta}_{\gamma}(\phi,\Phi):=\left\{f:\gamma\to\mathbb{C}|\ \left|f(z)-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\phi|\ \text{ for each }\ z\in\gamma\right\}.$$

The following representation result may be stated.

Proposition 1. *For any ϕ, $\Phi\in\mathbb{C}$, $\phi\neq\Phi$, we have that $\bar{U}_{\gamma}(\phi,\Phi)$ and $\bar{\Delta}_{\gamma}(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_{\gamma}(\phi,\Phi)=\bar{\Delta}_{\gamma}(\phi,\Phi). \tag{3.7}$$

Proof. We observe that for any $w\in\mathbb{C}$ we have the equivalence

$$\left|w-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\phi|\ \text{ iff }\ \operatorname{Re}\left[(\Phi-w)\left(\overline{w}-\overline{\phi}\right)\right]\geq0.$$

This follows by the equality

$$\frac{1}{4}|\Phi-\phi|^{2}-\left|w-\frac{\phi+\Phi}{2}\right|^{2}=\operatorname{Re}\left[(\Phi-w)\left(\overline{w}-\overline{\phi}\right)\right]$$

that holds for any $w\in\mathbb{C}$.

The equality (3.7) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 2. *For any* $\phi,\ \Phi\in\mathbb{C},\ \phi\neq\Phi,$*we have that*

$$\bar{U}_{\gamma}(\phi,\Phi)=\{f:\gamma\to\mathbb{C}\mid\ (\operatorname{Re}\Phi-\operatorname{Re}f(z))(\operatorname{Re}f(z)-\operatorname{Re}\phi)+(\operatorname{Im}\Phi-\operatorname{Im}f(z))(\operatorname{Im}f(z)-\operatorname{Im}\phi)\geq 0\ \text{for each}\ z\in\gamma\}. \tag{3.8}$$

Now, if we assume that $\operatorname{Re}(\Phi)\geq\operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi)\geq\operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\bar{S}_{\gamma}(\phi,\Phi):=\{f:\gamma\to\mathbb{C}\mid\ \operatorname{Re}(\Phi)\geq\operatorname{Re}f(z)\geq\operatorname{Re}(\phi)\ \text{and}\ \operatorname{Im}(\Phi)\geq\operatorname{Im}f(z)\geq\operatorname{Im}(\phi)\ \text{for each}\ \ z\in\gamma\}. \tag{3.9}$$

One can easily observe that $\bar{S}_{\gamma}(\phi,\Phi)$ is closed, convex and

$$\emptyset\neq\bar{S}_{\gamma}(\phi,\Phi)\subseteq\bar{U}_{\gamma}(\phi,\Phi). \tag{3.10}$$

Corollary 3. *Let* $f:D\subseteq\mathbb{C}\to\mathbb{C}$ *be an analytic function on the domain* D *and* y, $x\in D$. *Suppose* γ *is a smooth path parametrized by* $z(t)$, $t\in[a,b]$ *with* $z(a)=x$ *and* $z(b)=y$. *If* $\phi,\ \Phi\in\mathbb{C},\ \phi\neq\Phi$ *and* $f^{(n+1)}\in\bar{\Delta}_{\gamma}(\phi,\Phi)$ *for some* $n\geq 0$, *then*

$$\left|f(y)-\sum_{k=0}^{n}\frac{(y-x)^{k}}{k!}f^{(k)}(x)-\frac{(y-x)^{n+1}}{(n+1)!}\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2n!}|\Phi-\phi|\int_{\gamma_{x,y}}|y-z|^{n}|dz|. \tag{3.11}$$

Proof. Since $f^{(n+1)}\in\bar{\Delta}_{\gamma}(\phi,\Phi)$, hence by (3.1) for $\lambda=\frac{\phi+\Phi}{2}$ we get

$$\begin{aligned}\left|f(y)-\sum_{k=0}^{n}\frac{(y-x)^{k}}{k!}f^{(k)}(x)-\frac{(y-x)^{n+1}}{(n+1)!}\frac{\phi+\Phi}{2}\right|&\leq\frac{1}{n!}\int_{\gamma_{x,y}}|y-z|^{n}|dz|\left\|f^{(n+1)}-\frac{\phi+\Phi}{2}\right\|_{\gamma_{x,y},\infty}\\&\leq\frac{1}{2n!}|\Phi-\phi|\int_{\gamma_{x,y}}|y-z|^{n}|dz|,\end{aligned}$$

which proves the desired result (3.11). □

Corollary 4. *Let* $f:D\subseteq\mathbb{C}\to\mathbb{C}$ *be an analytic function on the domain* D *and* y, $x\in D$. *Suppose* γ *is a smooth path parametrized by* $z(t)$, $t\in[a,b]$ *with* $z(a)=x$ *and* $z(b)=y$. *If* $f^{(n+1)}$, *for some* $n\geq 0$, *satisfies the Hölder type condition on* γ

$$\left|f^{(n+1)}(z)-f^{(n+1)}(y)\right|\leq H|z-y|^{r}$$

for $z,\ y\in\gamma$, *where* $H>0$ *and* $r\in(0,1]$ *are given, then*

$$
\left|f(y)-\sum_{k=0}^{n+1}\frac{(y-x)^k}{k!}f^{(k)}(x)\right|\le\frac{1}{n!}H\int_{\gamma_{x,y}}|y-z|^n|z-x|^r|dz| \tag{3.12}
$$
$$
\le\frac{1}{n!}H\begin{cases}\max_{z\in\gamma_{x,y}}|y-z|^n\int_{\gamma_{x,y}}|z-x|^r|dz|,\\ \\ \left(\int_{\gamma_{x,y}}|y-z|^{np}|dz|\right)^{1/p}\left(\int_{\gamma_{x,y}}|z-x|^{qr}|dz|\right)^{1/q}\\ p,\ q>1\ \textit{with}\ \frac{1}{p}+\frac{1}{q}=1,\\ \\ \max_{z\in\gamma_{x,y}}|z-x|^r\int_{\gamma_{x,y}}|y-z|^n|dz|\end{cases}
$$

and

$$
\left|f(y)-\sum_{k=0}^{n}\frac{(y-x)^k}{k!}f^{(k)}(x)-\frac{(y-x)^{n+1}}{(n+1)!}f^{(n+1)}(y)\right|\le\frac{1}{n!}H\int_{\gamma_{x,y}}|y-z|^{n+r}|dz|. \tag{3.13}
$$

In particular, if $f^{(n+1)}$ *is Lipschitzian with the constant* $L>0$, *then*

$$
\left|f(y)-\sum_{k=0}^{n+1}\frac{(y-x)^k}{k!}f^{(k)}(x)\right|\le\frac{1}{n!}L\int_{\gamma_{x,y}}|y-z|^n|z-x||dz| \tag{3.14}
$$
$$
\le\frac{1}{n!}L\begin{cases}\max_{z\in\gamma_{x,y}}|y-z|^n\int_{\gamma_{x,y}}|z-x||dz|,\\ \\ \left(\int_{\gamma_{x,y}}|y-z|^{np}|dz|\right)^{1/p}\left(\int_{\gamma_{x,y}}|z-x|^{q}|dz|\right)^{1/q},\\ p,\ q>1\ \textit{with}\ \frac{1}{p}+\frac{1}{q}=1\\ \\ \max_{z\in\gamma_{x,y}}|z-x|\int_{\gamma_{x,y}}|y-z|^n|dz|\end{cases}
$$

and

$$
\left|f(y)-\sum_{k=0}^{n}\frac{(y-x)^k}{k!}f^{(k)}(x)-\frac{(y-x)^{n+1}}{(n+1)!}f^{(n+1)}(y)\right|\le\frac{1}{n!}L\int_{\gamma_{x,y}}|y-z|^{n+1}|dz|. \tag{3.15}
$$

Proof. From Theorem 2 we have

$$\begin{aligned}
&\left|f(y)-\sum_{k=0}^{n}\frac{(y-x)^{k}}{k!}f^{(k)}(x)-\frac{(y-x)^{n+1}}{(n+1)!}f^{(n+1)}(x)\right| \\
&\leq \frac{1}{n!}\int_{\gamma_{x,y}}|y-z|^{n}\left|f^{(n+1)}(z)-f^{(n+1)}(x)\right||dz| \\
&\leq \frac{1}{n!}H\int_{\gamma_{x,y}}|y-z|^{n}|z-x|^{r}|dz| \\
&\leq \frac{1}{n!}H\begin{cases}
\max_{z\in\gamma_{x,y}}|y-z|^{n}\int_{\gamma_{x,y}}|z-x|^{r}|dz|, \\
\left(\int_{\gamma_{x,y}}|y-z|^{np}|dz|\right)^{1/p}\left(\int_{\gamma_{x,y}}|z-x|^{qr}|dz|\right)^{1/q} \\
p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\
\max_{z\in\gamma_{x,y}}|z-x|^{r}\int_{\gamma_{x,y}}|y-z|^{n}|dz|,
\end{cases}
\end{aligned}$$

which proves the desired inequality (3.12).

From Theorem 2 we also have

$$\begin{aligned}
&\left|f(y)-\sum_{k=0}^{n}\frac{(y-x)^{k}}{k!}f^{(k)}(x)-\frac{(y-x)^{n+1}}{(n+1)!}f^{(n+1)}(y)\right| \\
&\leq \frac{1}{n!}\int_{\gamma_{x,y}}|y-z|^{n}\left|f^{(n+1)}(z)-f^{(n+1)}(y)\right||dz| \\
&\leq \frac{1}{n!}H\int_{\gamma_{x,y}}|y-z|^{n}|z-y|^{r}|dz|=\frac{1}{n!}H\int_{\gamma_{x,y}}|y-z|^{n+r}|dz|,
\end{aligned}$$

which proves (3.13). □

We also have:

Corollary 5. *Let $f:D\subseteq\mathbb{C}\to\mathbb{C}$ be an analytic function on the domain D and y, $x\in D$. Suppose γ is a smooth path parametrized by $z(t)$, $t\in[a,b]$ with $z(a)=x$ and $z(b)=y$ and $x\neq y$, then*

$$\begin{aligned}
&\left|f(y)-\sum_{k=0}^{n}\frac{(y-x)^{k}}{k!}f^{(k)}(x)-\frac{(y-x)^{n}}{(n+1)!}\left[f^{(n)}(y)-f^{(n)}(x)\right]\right| \quad (3.16)\\
&\leq \frac{1}{n!}\int_{\gamma_{x,y}}|y-z|^{n}\left|f^{(n+1)}(z)-\frac{f^{(n)}(y)-f^{(n)}(x)}{y-x}\right||dz| \\
&\leq \frac{1}{n!}\begin{cases}
\max_{z\in\gamma_{x,y}}|y-z|^{n}\left\|f^{(n+1)}-\frac{f^{(n)}(y)-f^{(n)}(x)}{y-x}\right\|_{\gamma_{x,y},1}, \\
\left(\int_{\gamma_{x,y}}|y-z|^{np}|dz|\right)^{1/p}\left\|f^{(n+1)}-\frac{f^{(n)}(y)-f^{(n)}(x)}{y-x}\right\|_{\gamma_{x,y},q} \\
p,\ q>1 \textit{ with } \frac{1}{p}+\frac{1}{q}=1, \\
\int_{\gamma_{x,y}}|y-z|^{n}|dz|\left\|f^{(n+1)}-\frac{f^{(n)}(y)-f^{(n)}(x)}{y-x}\right\|_{\gamma_{x,y},\infty}.
\end{cases}
\end{aligned}$$

We observe that

$$\begin{aligned}
&\int_{\gamma_{x,y}} |y-z|^n \left| f^{(n+1)}(z) - \frac{f^{(n)}(y) - f^{(n)}(x)}{y-x} \right| |dz| \\
&= \int_{\gamma_{x,y}} |y-z|^n \left| f^{(n+1)}(z)(y-x) - \int_{\gamma_{x,y}} f^{(n+1)}(w)\, dw \right| |dz| \\
&= \int_{\gamma_{x,y}} |y-z|^n \left| f^{(n+1)}(z) \int_{\gamma_{x,y}} dw - \int_{\gamma_{x,y}} f^{(n+1)}(w)\, dw \right| |dz| \\
&= \int_{\gamma_{x,y}} |y-z|^n \left| \int_{\gamma_{x,y}} \left(f^{(n+1)}(z) - f^{(n+1)}(w) \right) dw \right| |dz| \\
&= \int_{\gamma_{x,y}} |y-z|^n \\
&\times \left| \int_{\gamma_{x,z}} \left(f^{(n+1)}(z) - f^{(n+1)}(w) \right) dw + \int_{\gamma_{z,y}} \left(f^{(n+1)}(z) - f^{(n+1)}(w) \right) dw \right| |dz| \\
&\leq \int_{\gamma_{x,y}} |y-z|^n \\
&\times \left[\left| \int_{\gamma_{x,z}} \left(f^{(n+1)}(z) - f^{(n+1)}(w) \right) dw \right| + \left| \int_{\gamma_{z,y}} \left(f^{(n+1)}(z) - f^{(n+1)}(w) \right) dw \right| \right] |dz| \\
&\leq \int_{\gamma_{x,y}} |y-z|^n \\
&\times \left[\int_{\gamma_{x,z}} \left| f^{(n+1)}(z) - f^{(n+1)}(w) \right| |dw| + \int_{\gamma_{z,y}} \left| f^{(n+1)}(z) - f^{(n+1)}(w) \right| |dw| \right] |dz| \\
&=: B_n
\end{aligned}$$

Since

$$\begin{aligned}
\int_{\gamma_{x,z}} \left| f^{(n+1)}(z) - f^{(n+1)}(w) \right| |dw| &\leq \left\| f^{(n+2)} \right\|_{\gamma_{x,z},\infty} \int_{\gamma_{x,z}} |z-w|\, |dw| \\
&\leq \left\| f^{(n+2)} \right\|_{\gamma_{x,y},\infty} \int_{\gamma_{x,z}} |z-w|\, |dw|
\end{aligned}$$

and

$$\begin{aligned}
\int_{\gamma_{z,y}} \left| f^{(n+1)}(z) - f^{(n+1)}(w) \right| |dw| &\leq \left\| f^{(n+2)} \right\|_{\gamma_{z,y},\infty} \int_{\gamma_{z,y}} |z-w|\, |dw| \\
&\leq \left\| f^{(n+2)} \right\|_{\gamma_{x,y},\infty} \int_{\gamma_{z,y}} |z-w|\, |dw|.
\end{aligned}$$

Therefore

$$\begin{aligned}
B_n &\leq \int_{\gamma_{x,y}} |y-z|^n \\
&\times \left[\left\| f^{(n+2)} \right\|_{\gamma_{x,z},\infty} \int_{\gamma_{x,z}} |z-w| \, |dw| + \left\| f^{(n+2)} \right\|_{\gamma_{z,y},\infty} \int_{\gamma_{z,y}} |z-w| \, |dw| \right] |dz| \\
&\leq \left\| f^{(n+2)} \right\|_{\gamma_{x,y},\infty} \int_{\gamma_{x,y}} |y-z|^n \left[\int_{\gamma_{x,z}} |z-w| \, |dw| + \int_{\gamma_{z,y}} |z-w| \, |dw| \right] |dz| \\
&= \left\| f^{(n+2)} \right\|_{\gamma_{x,y},\infty} \int_{\gamma_{x,y}} |y-z|^n \left(\int_{\gamma_{x,y}} |z-w| \, |dw| \right) |dz|
\end{aligned}$$

and by (3.16) we get

$$\begin{aligned}
(3.17) \quad & \left| f(y) - \sum_{k=0}^{n} \frac{(y-x)^k}{k!} f^{(k)}(x) - \frac{(y-x)^n}{(n+1)!} \left[f^{(n)}(y) - f^{(n)}(x) \right] \right| \\
&\leq \frac{1}{n!} \int_{\gamma_{x,y}} |y-z|^n \left| f^{(n+1)}(z) - \frac{f^{(n)}(y) - f^{(n)}(x)}{y-x} \right| |dz| \\
&\leq \frac{1}{n!} \left\| f^{(n+2)} \right\|_{\gamma_{x,y},\infty} \int_{\gamma_{x,y}} |y-z|^n \left(\int_{\gamma_{x,y}} |z-w| \, |dw| \right) |dz| .
\end{aligned}$$

We also have:

Corollary 6. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the domain D and y, $x \in D$. Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = x$ and $z(b) = y$, then*

$$\begin{aligned}
(3.18) \quad & \left| f(y) - \sum_{k=0}^{n} \frac{(y-x)^k}{k!} f^{(k)}(x) - \frac{(y-x)^{n+1}}{(n+1)!} \frac{f^{(n+1)}(x) + f^{(n+1)}(y)}{2} \right| \\
&\leq \frac{1}{n!} \int_{\gamma_{x,y}} |y-z|^n \left| f^{(n+1)}(z) - \frac{f^{(n+1)}(x) + f^{(n+1)}(y)}{2} \right| |dz| \\
&\leq \frac{1}{n!} \begin{cases}
\max_{z \in \gamma_{x,y}} |y-z|^n \left\| f^{(n+1)} - \frac{f^{(n+1)}(x)+f^{(n+1)}(y)}{2} \right\|_{\gamma_{x,y},1}, \\[2ex]
\left(\int_{\gamma_{x,y}} |y-z|^{np} |dz| \right)^{1/p} \left\| f^{(n+1)} - \frac{f^{(n+1)}(x)+f^{(n+1)}(y)}{2} \right\|_{\gamma_{x,y},q} \\
p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\[2ex]
\int_{\gamma_{x,y}} |y-z|^n |dz| \left\| f^{(n+1)} - \frac{f^{(n+1)}(x)+f^{(n+1)}(y)}{2} \right\|_{\gamma_{x,y},\infty} .
\end{cases}
\end{aligned}$$

Observe that

$$\begin{aligned}
&\int_{\gamma_{x,y}} |y-z|^n \left| f^{(n+1)}(z) - \frac{f^{(n+1)}(x)+f^{(n+1)}(y)}{2} \right| |dz| \\
&= \int_{\gamma_{x,y}} |y-z|^n \left| \frac{f^{(n+1)}(z) - f^{(n+1)}(x) + f^{(n+1)}(z) - f^{(n+1)}(y)}{2} \right| |dz| \\
&\leq \frac{1}{2} \int_{\gamma_{x,y}} |y-z|^n \left[\left| f^{(n+1)}(z) - f^{(n+1)}(x) \right| + \left| f^{(n+1)}(z) - f^{(n+1)}(y) \right| \right] |dz| \\
&\leq \frac{1}{2} \int_{\gamma_{x,y}} |y-z|^n \left[\left\| f^{(n+2)} \right\|_{\gamma_{x,z},\infty} |z-x| + \left\| f^{(n+2)} \right\|_{\gamma_{z,y},\infty} |y-z| \right] |dz| \\
&\leq \frac{1}{2} \left\| f^{(n+2)} \right\|_{\gamma_{x,y},\infty} \int_{\gamma_{x,y}} |y-z|^n \left[|z-x| + |y-z| \right] |dz| ,
\end{aligned}$$

then by (3.18) we get

$$\begin{aligned}
&\left| f(y) - \sum_{k=0}^{n} \frac{(y-x)^k}{k!} f^{(k)}(x) - \frac{(y-x)^{n+1}}{(n+1)!} \frac{f^{(n+1)}(x)+f^{(n+1)}(y)}{2} \right| \\
&\leq \frac{1}{n!} \int_{\gamma_{x,y}} |y-z|^n \left| f^{(n+1)}(z) - \frac{f^{(n+1)}(x)+f^{(n+1)}(y)}{2} \right| |dz| \\
&\leq \frac{1}{2n!} \left\| f^{(n+2)} \right\|_{\gamma_{x,y},\infty} \int_{\gamma_{x,y}} |y-z|^n \left[|z-x| + |y-z| \right] |dz| .
\end{aligned} \tag{3.19}$$

4. Examples for Circular Paths

Let $[a,b] \subseteq [0,2\pi]$ and the circular path $\gamma_{[a,b],R}$ centered in 0 and with radius $R>0$

$$z(t) = R\exp(it) = R(\cos t + i \sin t), \ t \in [a,b].$$

If $[a,b] = [0,\pi]$ then we get a half circle while for $[a,b] = [0,2\pi]$ we get the full circle.

If $u = R\exp(it)$ and $w = R\exp(is)$ then

$$\begin{aligned}
w - u &= R[\exp(is) - \exp(it)] = R[\cos s + i\sin s - \cos t - i \sin t] \\
&= R[\cos s - \cos t + i(\sin s - \sin t)].
\end{aligned}$$

Since

$$\cos s - \cos t = -2\sin\left(\frac{t+s}{2}\right)\sin\left(\frac{s-t}{2}\right)$$

and

$$\sin s - \sin t = 2\sin\left(\frac{s-t}{2}\right)\cos\left(\frac{t+s}{2}\right),$$

hence

$$\begin{aligned} w-u &= R\left[-2\sin\left(\frac{t+s}{2}\right)\sin\left(\frac{s-t}{2}\right)+2i\sin\left(\frac{s-t}{2}\right)\cos\left(\frac{t+s}{2}\right)\right] \\ &= 2R\sin\left(\frac{s-t}{2}\right)\left[-\sin\left(\frac{t+s}{2}\right)+i\cos\left(\frac{t+s}{2}\right)\right] \\ &= 2Ri\sin\left(\frac{s-t}{2}\right)\left[\cos\left(\frac{t+s}{2}\right)+i\sin\left(\frac{t+s}{2}\right)\right] \\ &= 2Ri\sin\left(\frac{s-t}{2}\right)\exp\left[\left(\frac{t+s}{2}\right)i\right]. \end{aligned}$$

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the domain D and $y, x \in D$. Suppose $\gamma_{[a,b],R} \subset D$ and such that $x = Re^{ia}$, $y = Re^{ib}$ and $z = Re^{it}$ then $dz = Rie^{it}dt$

$$(y-x)^k = 2^k R^k i^k \sin^k\left(\frac{b-a}{2}\right)\exp\left[n\left(\frac{b+a}{2}\right)i\right],$$

$$(y-z)^n = 2^n R^n i^n \sin^n\left(\frac{b-t}{2}\right)\exp\left[n\left(\frac{t+b}{2}\right)i\right]$$

and by (2.1) we get for $n \geq 0$ that

$$\begin{aligned} (4.1)\quad f\left(Re^{ib}\right) &= \sum_{k=0}^{n}\frac{2^k R^k i^k \sin^k\left(\frac{b-a}{2}\right)\exp\left[n\left(\frac{b+a}{2}\right)i\right]}{k!}f^{(k)}\left(Re^{ia}\right) \\ &+\frac{1}{n!}\int_a^b 2^n R^n i^n \sin^n\left(\frac{b-t}{2}\right)\exp\left[n\left(\frac{t+b}{2}\right)i\right]f^{(n+1)}\left(Re^{it}\right)Rie^{it}dt \\ &= \sum_{k=0}^{n}\frac{2^k R^k i^k \sin^k\left(\frac{b-a}{2}\right)\exp\left[n\left(\frac{b+a}{2}\right)i\right]}{k!}f^{(k)}\left(Re^{ia}\right) \\ &+\frac{2^n}{n!}R^{n+1}i^{n+1}\int_a^b \sin^n\left(\frac{b-t}{2}\right)\exp\left\{\left(\left[n\left(\frac{t+b}{2}\right)+t\right]\right)i\right\}f^{(n+1)}\left(Re^{it}\right)dt \end{aligned}$$

If $a = 0$, $R \in D$ and $Re^{ib} \in D$, then from (4.1) we get

$$\begin{aligned} (4.2)\quad f\left(Re^{ib}\right) &= \sum_{k=0}^{n}\frac{2^k R^k i^k \sin^k\left(\frac{b}{2}\right)\exp\left[n\left(\frac{b}{2}\right)i\right]}{k!}f^{(k)}(R) \\ &+\frac{2^n}{n!}R^{n+1}i^{n+1}\int_0^b \sin^n\left(\frac{b-t}{2}\right)\exp\left\{\left(\left[n\left(\frac{t+b}{2}\right)+t\right]\right)i\right\}f^{(n+1)}\left(Re^{it}\right)dt. \end{aligned}$$

We have from (4.2) that

$$\begin{aligned} (4.3)\quad &\left|f\left(Re^{ib}\right)-\sum_{k=0}^{n}\frac{2^k R^k i^k \sin^k\left(\frac{b-a}{2}\right)\exp\left[n\left(\frac{b+a}{2}\right)i\right]}{k!}f^{(k)}\left(Re^{ia}\right)\right| \\ &\leq \frac{2^n}{n!}R^{n+1}\left\|f^{(n+1)}\right\|_{\gamma_{[a,b],R}}\int_a^b \sin^n\left(\frac{b-t}{2}\right)dt \end{aligned}$$

for $n \geq 0$.

Since

$$\int_a^b \sin^n\left(\frac{b-t}{2}\right)dt = \int_a^b \left[\frac{\sin\left(\frac{b-t}{2}\right)}{\frac{b-t}{2}}\right]^n \left(\frac{b-t}{2}\right)^n dt \le \frac{1}{2^n}\frac{(b-a)^{n+1}}{n+1}$$

hence by (4.3) we get for $n \ge 0$ that

$$\left| f\left(Re^{ib}\right) - \sum_{k=0}^{n} \frac{2^k R^k i^k \sin^k\left(\frac{b-a}{2}\right)\exp\left[n\left(\frac{b+a}{2}\right)i\right]}{k!} f^{(k)}\left(Re^{ia}\right) \right| \le \frac{1}{(n+1)!}\left\|f^{(n+1)}\right\|_{\gamma_{[a,b],R}} R^{n+1}(b-a)^{n+1}. \tag{4.4}$$

If $a = 0$, $R \in D$ and $Re^{ib} \in D$, then by (4.4) we get

$$\left| f\left(Re^{ib}\right) - \sum_{k=0}^{n} \frac{2^k R^k i^k \sin^k\left(\frac{b}{2}\right)\exp\left[n\left(\frac{b}{2}\right)i\right]}{k!} f^{(k)}(R) \right| \le \frac{1}{(n+1)!}\left\|f^{(n+1)}\right\|_{\gamma_{[0,b],R}} b^{n+1}R^{n+1}. \tag{4.5}$$

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