GENERALIZED OSTROWSKI AND TRAPEZOID TYPE RULES FOR APPROXIMATING THE INTEGRAL OF ANALYTIC COMPLEX FUNCTIONS ON PATHS FROM GENERAL DOMAINS

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Abstract. In this paper we establish some generalized Ostrowski and trapezoid type rules for approximating the integral of analytic complex functions on paths from general domains. Error bounds for these expansions in terms of p-norms, Hölder and Lipschitz constants are also provided. Examples for the complex logarithm and the complex exponential are given as well.

1. Introduction

Suppose γ is a *smooth path* parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_{\gamma} f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are *piecewise smooth*. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration*

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by parts formula

$$\int_{\gamma_{u,w}} f(z) g'(z) dz = f(w) g(w) - f(u) g(u) - \int_{\gamma_{u,w}} f'(z) g(z) dz. \tag{1.1}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left| \int_{\gamma} f(z) dz \right| \leq \int_{\gamma} |f(z)| |dz| \leq \|f\|_{\gamma,\infty} \ell(\gamma) \tag{1.2}$$

where $\|f\|_{\gamma,\infty} := \sup_{z \in \gamma} |f(z)|$.

We also define the p-norm with $p \geq 1$ by

$$\|f\|_{\gamma,p} := \left(\int_{\gamma} |f(z)|^p |dz| \right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_{\gamma} |f(z)| |dz|.$$

If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \leq [\ell(\gamma)]^{1/q} \|f\|_{\gamma,p}.$$

In the recent paper [9] we obtained the following identity for the path integral of an analytic function defined on a convex domain:

Theorem 1. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and $x \in D$. Suppose $\gamma \subset D$ is a smooth path parametrized by $z(t)$, $t \in [a, b]$ with $z(a) = u$ and $z(b) = w$ where u, $w \in D$. Then we have the Ostrowski type equality*

$$\begin{aligned} \int_{\gamma} f(z) dz = & \sum_{k=0}^{n} \frac{1}{(k+1)!} f^{(k)}(x) \left[(w - x)^{k+1} + (-1)^k (x - u)^{k+1} \right] \\ & + R_n(x, \gamma), \end{aligned} \tag{1.3}$$

for $n \geq 0$, where the remainder $R_n(x, \gamma)$ is given by

$$\begin{aligned} R_n(x, \gamma) & := \frac{1}{n!} \int_{\gamma} (z - x)^{n+1} \left(\int_0^1 f^{(n+1)} [(1 - s) x + sz] (1 - s)^n ds \right) dz \\ & = \frac{1}{n!} \int_0^1 \left(\int_{\gamma} (z - x)^{n+1} f^{(n+1)} [(1 - s) x + sz] dz \right) (1 - s)^n ds. \end{aligned} \tag{1.4}$$

We obtained amongst other the following simple error bound [9]:

Corollary 1. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and $x \in D$. Suppose $\gamma \subset D$ is a smooth path parametrized by $z(t)$, $t \in [a, b]$ with $z(a) = u$ and $z(b) = w$ where u, $w \in D$. If*

$$\left\| f^{(n+1)} \right\|_{D,\infty} := \sup_{z \in D} \left| f^{(n+1)}(z) \right| < \infty \text{ for some } n \geq 0, \tag{1.5}$$

then we have the representation (1.3) where the remainder $R_n(x, \gamma)$ satisfies the bound

$$|R_n(x, \gamma)| \leq \frac{1}{(n+1)!} \left\| f^{(n+1)} \right\|_{D,\infty} \int_{\gamma} |z - x|^{n+1} |dz|. \tag{1.6}$$

The above results extend the inequalities for real valued functions of a real variable obtained in [6] and [3]. For similar result see [1], [2], [5], [8], [10] and [11].

2. Representation Results

We have the following identity for the integral on a path from a non-necessarily convex domain D as above:

Theorem 2. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the domain D and $x \in D$. Suppose $\gamma \subset D$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = u$, $z(t) = x$ and $z(b) = w$ where $u, w \in D$. Then we have the equality*

$$\int_\gamma f(z)\,dz = \sum_{k=0}^{n-1} \frac{1}{(k+1)!} f^{(k)}(x) \left[(w-x)^{k+1} + (-1)^k (x-u)^{k+1} \right] + O_n(x,\gamma), \tag{2.1}$$

where the remainder $O_n(x,\gamma)$ is given by

$$O_n(x,\gamma) := \frac{(-1)^n}{n!} \int_\gamma K_n(x,z) f^{(n)}(z)\,dz \tag{2.2}$$

and the kernel $K_n : \gamma \times \gamma \to \mathbb{C}$ is defined by

$$K_n(x,z) := \begin{cases} (z-u)^n & \text{if } z \in \gamma_{u,x} \\ (z-w)^n & \text{if } z \in \gamma_{x,w} \end{cases}, \quad x \in \gamma \tag{2.3}$$

and n is a natural number, $n \geq 1$.

Proof. We prove the identity by induction over n. For $n = 1$, we have to prove the equality

$$\int_\gamma f(z)\,dz = (w-u) f(x) - \int_\gamma K_1(x,z) f'(z)\,dz, \tag{2.4}$$

where

$$K_1(x,z) := \begin{cases} z-u & \text{if } z \in \gamma_{u,x} \\ z-w & \text{if } z \in \gamma_{x,w} \end{cases}.$$

Integrating by parts, we have:

$$\begin{aligned} &\int_\gamma K_1(x,z) f'(z)\,dz \\ &= \int_{\gamma_{u,x}} (z-u) f'(z)\,dz + \int_{\gamma_{x,w}} (z-w) f'(z)\,dz \\ &= (z-u) f(z)\big|_u^x - \int_{\gamma_{u,x}} f(z)\,dz + (z-w) f(z)\big|_x^w - \int_{\gamma_{x,w}} f(z)\,dz \\ &= (x-u) f(x) + (w-x) f(x) - \int_\gamma f(z)\,dz \\ &= (w-u) f(x) - \int_\gamma f(z)\,dz \end{aligned}$$

and the identity (2.4) is proved.

Assume that (2.1) holds for "n" and let us prove it for "$n+1$". That is, we have to prove the equality

$$\int_\gamma f(z)\,dz = \sum_{k=0}^{n}\left[\frac{(w-x)^{k+1}+(-1)^k(x-u)^{k+1}}{(k+1)!}\right]f^{(k)}(x) + \frac{(-1)^{n+1}}{(n+1)!}\int_\gamma K_{n+1}(x,z)\,f^{(n+1)}(z)\,dz. \tag{2.5}$$

We have, by using (2.2),

$$\begin{aligned}
&\frac{1}{(n+1)!}\int_\gamma K_{n+1}(x,z)\,f^{(n+1)}(z)\,dz \\
&= \int_{\gamma_{u,x}}\frac{(z-u)^{n+1}}{(n+1)!}f^{(n+1)}(z)\,dz + \int_{\gamma_{x,w}}\frac{(z-w)^{n+1}}{(n+1)!}f^{(n+1)}(z)\,dz
\end{aligned}$$

and integrating by parts gives

$$\begin{aligned}
&\frac{1}{(n+1)!}\int_\gamma K_{n+1}(x,z)\,f^{(n+1)}(z)\,dz \\
&= \frac{(z-u)^{n+1}}{(n+1)!}f^{(n)}(z)\Big|_u^x - \frac{1}{n!}\int_{\gamma_{u,x}}(z-u)^n f^{(n)}(z)\,dz \\
&+ \frac{(z-w)^{n+1}}{(n+1)!}f^{(n)}(z)\Big|_x^w - \frac{1}{n!}\int_{\gamma_{x,w}}(z-w)^n f^{(n)}(z)\,dz \\
&= \frac{(x-u)^{n+1}+(-1)^{n+2}(w-x)^{n+1}}{(n+1)!}f^{(n)}(x) - \frac{1}{n!}\int_\gamma K_n(x,z)\,f^{(n)}(z)\,dz.
\end{aligned}$$

That is

$$\begin{aligned}
\frac{1}{n!}\int_\gamma K_n(x,z)\,f^{(n)}(z)\,dz &= \frac{(x-u)^{n+1}+(-1)^{n+2}(w-x)^{n+1}}{(n+1)!}f^{(n)}(x) \\
&\quad - \frac{1}{(n+1)!}\int_\gamma K_{n+1}(x,z)\,f^{(n+1)}(z)\,dz \\
&= \frac{(x-u)^{n+1}+(-1)^{n}(w-x)^{n+1}}{(n+1)!}f^{(n)}(x) \\
&\quad - \frac{1}{(n+1)!}\int_\gamma K_{n+1}(x,z)\,f^{(n+1)}(z)\,dz.
\end{aligned}$$

Now, using the mathematical induction hypothesis, we get

$$\begin{aligned}\int_\gamma f(z)\,dz &= \sum_{k=0}^{n-1}\left[\frac{(w-x)^{k+1}+(-1)^k(x-u)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\\&+\frac{(w-x)^{n+1}+(-1)^n(x-u)^{n+1}}{(n+1)!}f^{(n)}(x)\\&-\frac{(-1)^n}{(n+1)!}\int_\gamma K_{n+1}(x,z)\,f^{(n+1)}(z)\,dz\\&=\sum_{k=0}^{n}\left[\frac{(w-x)^{k+1}+(-1)^k(x-u)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\\&+\frac{(-1)^{n+1}}{(n+1)!}\int_\gamma K_{n+1}(x,z)\,f^{(n+1)}(z)\,dz.\end{aligned}$$

That is, identity (2.5) and the theorem is thus proved. □

Corollary 2. *With the assumptions of Theorem 2 and for λ_1, λ_2 complex numbers we have the identity*

$$\begin{aligned}\int_\gamma f(z)\,dz &= \sum_{k=0}^{n-1}\frac{1}{(k+1)!}f^{(k)}(x)\left[(w-x)^{k+1}+(-1)^k(x-u)^{k+1}\right]\\&+\lambda_1\frac{(-1)^n}{(n+1)!}(x-u)^{n+1}+\lambda_2\frac{(-1)^{n+1}}{(n+1)!}(x-w)^{n+1}+O_n(x,\gamma,\lambda_1,\lambda_2),\end{aligned}\tag{2.6}$$

where the remainder $O_n(x,\gamma,\lambda_1,\lambda_2)$ is given by

$$\begin{aligned}O_n(x,\gamma,\lambda_1,\lambda_2) &:= \frac{(-1)^n}{n!}\int_{\gamma_{u,x}}(z-u)^n\left[f^{(n)}(z)-\lambda_1\right]dz\\&+\frac{(-1)^n}{n!}\int_{\gamma_{x,w}}(z-w)^n\left[f^{(n)}(z)-\lambda_2\right]dz.\end{aligned}\tag{2.7}$$

In particular, for $\lambda_1=\lambda_2=\lambda$, we have

$$\begin{aligned}\int_\gamma f(z)\,dz &= \sum_{k=0}^{n-1}\frac{1}{(k+1)!}f^{(k)}(x)\left[(w-x)^{k+1}+(-1)^k(x-u)^{k+1}\right]\\&+\lambda\left[\frac{(-1)^n}{(n+1)!}(x-u)^{n+1}+\frac{(-1)^{n+1}}{(n+1)!}(x-w)^{n+1}\right]+O_n(x,\gamma,\lambda),\end{aligned}\tag{2.8}$$

where

$$\begin{aligned}O_n(x,\gamma,\lambda) &:= \frac{(-1)^n}{n!}\int_{\gamma_{u,x}}(z-u)^n\left[f^{(n)}(z)-\lambda\right]dz\\&+\frac{(-1)^n}{n!}\int_{\gamma_{x,w}}(z-w)^n\left[f^{(n)}(z)-\lambda\right]dz.\end{aligned}\tag{2.9}$$

Proof. Observe that

$$\begin{aligned}
&\frac{(-1)^n}{n!}\int_\gamma K_n(x,z)\,f^{(n)}(z)\,dz \\
&=\frac{(-1)^n}{n!}\int_{\gamma_{u,x}}(z-u)^n\left[f^{(n)}(z)-\lambda_1+\lambda_1\right]dz \\
&+\frac{(-1)^n}{n!}\int_{\gamma_{x,w}}(z-w)^n\left[f^{(n)}(z)-\lambda_2+\lambda_2\right]dz \\
&=\frac{(-1)^n}{n!}\int_{\gamma_{u,x}}(z-u)^n\left[f^{(n)}(z)-\lambda_1\right]dz+\lambda_1\frac{(-1)^n}{n!}\int_{\gamma_{u,x}}(z-u)^n\,dz \\
&+\frac{(-1)^n}{n!}\int_{\gamma_{x,w}}(z-w)^n\left[f^{(n)}(z)-\lambda_2\right]dz+\lambda_2\frac{(-1)^n}{n!}\int_{\gamma_{x,w}}(z-w)^n\,dz \\
&=\frac{(-1)^n}{n!}\int_{\gamma_{u,x}}(z-u)^n\left[f^{(n)}(z)-\lambda_1\right]dz+\lambda_1\frac{(-1)^n}{(n+1)!}(x-u)^{n+1} \\
&+\frac{(-1)^n}{n!}\int_{\gamma_{x,w}}(z-w)^n\left[f^{(n)}(z)-\lambda_2\right]dz+\lambda_2\frac{(-1)^{n+1}}{(n+1)!}(x-w)^{n+1}
\end{aligned}$$

and by (2.1) we then get (2.6). □

Corollary 3. *With the assumptions of Theorem 2 and for θ a complex number we have the identity*

$$\int_\gamma f(z)\,dz=\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[\theta(-1)^k f^{(k)}(w)+(1-\theta)f^{(k)}(u)\right](w-u)^{k+1} \\ +T_n(\gamma,\theta) \tag{2.10}$$

where the remainder $T_n(x,\gamma,\theta)$ is given by

$$T_n(\gamma,\theta):=\frac{(-1)^n}{n!}\int_\gamma\left[\theta(z-u)^n+(1-\theta)(z-w)^n\right]f^{(n)}(z)\,dz. \tag{2.11}$$

In particular, for $\theta=\frac{1}{2}$ we have

$$\int_\gamma f(z)\,dz=\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[\frac{(-1)^k f^{(k)}(w)+f^{(k)}(u)}{2}\right](w-u)^{k+1} \\ +T_n(\gamma), \tag{2.12}$$

where

$$T_n(\gamma)=\frac{(-1)^n}{2n!}\int_\gamma\left[(z-u)^n+(z-w)^n\right]f^{(n)}(z)\,dz. \tag{2.13}$$

Proof. From Theorem 2 for $x = w$ we have

$$\int_{\gamma} f(z)\,dz = \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)!} f^{(k)}(w)(w-u)^{k+1} + \frac{(-1)^n}{n!} \int_{\gamma} (z-u)^n f^{(n)}(z)\,dz, \tag{2.14}$$

while for $x = u$ we have

$$\int_{\gamma} f(z)\,dz = \sum_{k=0}^{n-1} \frac{1}{(k+1)!} f^{(k)}(u)(w-u)^{k+1} + \frac{(-1)^n}{n!} \int_{\gamma} (z-w)^n f^{(n)}(z)\,dz. \tag{2.15}$$

If we multiply (2.14) by θ and (2.15) by $1-\theta$ and sum the obtained equalities, then we get

$$\begin{aligned}\int_{\gamma} f(z)\,dz &= \theta \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)!} f^{(k)}(w)(w-u)^{k+1} + \frac{(-1)^n}{n!} \theta \int_{\gamma} (z-u)^n f^{(n)}(z)\,dz \\ &+ (1-\theta) \sum_{k=0}^{n-1} \frac{1}{(k+1)!} f^{(k)}(u)(w-u)^{k+1} + \frac{(-1)^n}{n!}(1-\theta) \int_{\gamma} (z-w)^n f^{(n)}(z)\,dz \\ &= \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[\theta(-1)^k f^{(k)}(w) + (1-\theta) f^{(k)}(u)\right](w-u)^{k+1} \\ &\quad + \frac{(-1)^n}{n!} \int_{\gamma} \left[\theta(z-u)^n + (1-\theta)(z-w)^n\right] f^{(n)}(z)\,dz,\end{aligned}$$

which proves the desired result (2.10). □

For the case of real variable functions see [4] and [7].

Corollary 4. *With the assumptions of Theorem 2 and for θ and λ complex numbers we have the identity*

$$\begin{aligned}\int_{\gamma} f(z)\,dz &= \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[\theta(-1)^k f^{(k)}(w) + (1-\theta) f^{(k)}(u)\right](w-u)^{k+1} \\ &\quad + \frac{(-1)^n}{(n+1)!} \lambda \left[\theta + (-1)^n (1-\theta)\right](w-u)^{n+1} + T_n(\gamma,\theta,\lambda),\end{aligned} \tag{2.16}$$

where the remainder $T_n(\gamma,\theta,\lambda)$ is given by

$$T_n(\gamma,\theta) := \frac{(-1)^n}{n!} \int_{\gamma} \left[\theta(z-u)^n + (1-\theta)(z-w)^n\right]\left[f^{(n)}(z) - \lambda\right] dz. \tag{2.17}$$

In particular, for $\theta = \frac{1}{2}$ we have

$$\begin{aligned}\int_{\gamma} f(z)\,dz &= \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[\frac{(-1)^k f^{(k)}(w) + f^{(k)}(u)}{2}\right](w-u)^{k+1} \\ &\quad + \frac{(-1)^n}{(n+1)!} \left[\frac{1+(-1)^n}{2}\right] \lambda (w-u)^{n+1} + T_n(\gamma,\lambda),\end{aligned} \tag{2.18}$$

where the remainder $T_n(\gamma,\lambda)$ *is given by*

$$T_n(\gamma,\lambda) := \frac{(-1)^n}{2n!}\int_\gamma [(z-u)^n + (z-w)^n]\left[f^{(n)}(z) - \lambda\right] dz. \tag{2.19}$$

3. p-Norm Error Bounds

We have:

Theorem 3. *Let* $f : D \subseteq \mathbb{C} \to \mathbb{C}$ *be an analytic function on the domain* D *and* $x \in D$. *Suppose* $\gamma \subset D$ *is a smooth path parametrized by* $z(t)$, $t \in [a,b]$ *with* $z(a) = u$, $z(t) = x$ *and* $z(b) = w$ *where* $u, w \in D$. *Then we have the equality (2.1) and the remainder* $O_n(x,\gamma)$ *satisfies the bounds*

$$|O_n(x,\gamma)| \le B_n(x,\gamma) \tag{3.1}$$

where

$$B_n(x,\gamma) := \frac{1}{n!}\left[\int_{\gamma_{u,x}} |z-u|^n \left|f^{(n)}(z)\right| |dz| + \int_{\gamma_{x,w}} |z-w|^n \left|f^{(n)}(z)\right| |dz|\right].$$

Moreover, we have

$$B_n(x,\gamma) \le \frac{1}{n!}\begin{cases} \left\|f^{(n)}\right\|_{\gamma_{u,x},\infty} \int_{\gamma_{u,x}} |z-u|^n |dz| \\ \\ \left\|f^{(n)}\right\|_{\gamma_{u,x},p} \left(\int_{\gamma_{u,x}} |z-u|^{qn} |dz|\right)^{1/q}, \\ \text{where } p,\ q > 1,\ \frac{1}{p}+\frac{1}{q} = 1; \\ \\ \left\|f^{(n)}\right\|_{\gamma_{u,x},1} \max_{z\in\gamma_{u,x}} |z-u|^n \end{cases} \tag{3.2}$$

$$+\frac{1}{n!}\begin{cases} \left\|f^{(n)}\right\|_{\gamma_{x,w},\infty} \int_{\gamma_{x,w}} |z-w|^n |dz| \\ \\ \left\|f^{(n)}\right\|_{\gamma_{x,w},p} \left(\int_{\gamma_{x,w}} |z-w|^{qn} |dz|\right)^{1/q}, \\ \text{where } p,\ q > 1,\ \frac{1}{p}+\frac{1}{q} = 1; \\ \\ \left\|f^{(n)}\right\|_{\gamma_{x,w},1} \max_{z\in\gamma_{x,w}} |z-w|^n \end{cases}$$

$$\le \frac{1}{n!}\begin{cases} \left\|f^{(n)}\right\|_{\gamma_{u,w},\infty} \left[\int_{\gamma_{u,x}} |z-u|^n |dz| + \int_{\gamma_{x,w}} |z-w|^n |dz|\right], \\ \\ \left\|f^{(n)}\right\|_{\gamma_{u,w},p} \left(\int_{\gamma_{u,x}} |z-u|^{qn} |dz| + \int_{\gamma_{x,w}} |z-w|^{qn} |dz|\right)^{1/q} \\ \text{where } p,\ q > 1,\ \frac{1}{p}+\frac{1}{q} = 1; \\ \\ \left\|f^{(n)}\right\|_{\gamma_{u,w},1} \max\left\{\max_{z\in\gamma_{u,x}} |z-u|^n, \max_{z\in\gamma_{x,w}} |z-w|^n\right\}. \end{cases}$$

Proof. We have

$$\begin{aligned} |O_n(x,\gamma)| &\le \left|\frac{(-1)^n}{n!}\int_{\gamma_{u,x}} (z-u)^n f^{(n)}(z)\, dz\right| + \left|\frac{(-1)^n}{n!}\int_{\gamma_{x,w}} (z-w)^n f^{(n)}(z)\, dz\right| \\ &\le \frac{1}{n!}\int_{\gamma_{u,x}} |z-u|^n \left|f^{(n)}(z)\right| |dz| + \frac{1}{n!}\int_{\gamma_{x,w}} |z-w|^n \left|f^{(n)}(z)\right| |dz|, \end{aligned}$$

which proves the inequality (3.1).

Using Hölder's integral inequality we have

$$\int_{\gamma_{u,x}} |z-u|^n \left|f^{(n)}(z)\right| |dz| \le \begin{cases} \max_{z\in\gamma_{u,x}} \left|\left|f^{(n)}(z)\right|\right| \int_{\gamma_{u,x}} |z-u|^n |dz| \\ \\ \left(\int_{\gamma_{u,x}} \left|f^{(n)}(z)\right|^p |dz|\right)^{1/p} \left(\int_{\gamma_{u,x}} |z-u|^{qn} |dz|\right)^{1/q}, \\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \int_{\gamma_{u,x}} \left|f^{(n)}(z)\right| |dz| \max_{z\in\gamma_{u,x}} |z-u|^n \end{cases}$$

$$= \begin{cases} \left\|f^{(n)}\right\|_{\gamma_{u,x},\infty} \int_{\gamma_{u,x}} |z-u|^n |dz| \\ \\ \left\|f^{(n)}\right\|_{\gamma_{u,x},p} \left(\int_{\gamma_{u,x}} |z-u|^{qn} |dz|\right)^{1/q}, \\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left\|f^{(n)}\right\|_{\gamma_{u,x},1} \max_{z\in\gamma_{u,x}} |z-u|^n \end{cases}$$

and

$$\int_{\gamma_{x,w}} |z-w|^n \left|f^{(n)}(z)\right| |dz| \le \begin{cases} \left\|f^{(n)}\right\|_{\gamma_{x,w},\infty} \int_{\gamma_{x,w}} |z-w|^n |dz| \\ \\ \left\|f^{(n)}\right\|_{\gamma_{x,w},p} \left(\int_{\gamma_{x,w}} |z-w|^{qn} |dz|\right)^{1/q}, \\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left\|f^{(n)}\right\|_{\gamma_{x,w},1} \max_{z\in\gamma_{x,w}} |z-w|^n. \end{cases}$$

Using the elementary inequality

$$\alpha a+\beta b \le \begin{cases} \max\{\alpha,\beta\}(a+b) \\ \\ (\alpha^p+\beta^p)^{1/p}(a^q+b^q)^{1/q}; \\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \end{cases}$$

we have

$$\begin{aligned} &\left\|f^{(n)}\right\|_{\gamma_{u,x},\infty} \int_{\gamma_{u,x}} |z-u|^n |dz| + \left\|f^{(n)}\right\|_{\gamma_{x,w},\infty} \int_{\gamma_{x,w}} |z-w|^n |dz| \\ &\le \max\left\{\left\|f^{(n)}\right\|_{\gamma_{u,x},\infty}, \left\|f^{(n)}\right\|_{\gamma_{x,w},\infty}\right\} \left[\int_{\gamma_{u,x}} |z-u|^n |dz| + \int_{\gamma_{x,w}} |z-w|^n |dz|\right] \\ &\qquad = \left\|f^{(n)}\right\|_{\gamma_{u,w},\infty} \left[\int_{\gamma_{u,x}} |z-u|^n |dz| + \int_{\gamma_{x,w}} |z-w|^n |dz|\right], \end{aligned}$$

$$\left\|f^{(n)}\right\|_{\gamma_{u,x},p}\left(\int_{\gamma_{u,x}}|z-u|^{qn}|dz|\right)^{1/q}+\left\|f^{(n)}\right\|_{\gamma_{x,w},p}\left(\int_{\gamma_{x,w}}|z-w|^{qn}|dz|\right)^{1/q}$$
$$\leq\left(\left\|f^{(n)}\right\|_{\gamma_{u,x},p}^{p}+\left\|f^{(n)}\right\|_{\gamma_{x,w},p}^{p}\right)^{1/p}\left(\int_{\gamma_{u,x}}|z-u|^{qn}|dz|+\int_{\gamma_{x,w}}|z-w|^{qn}|dz|\right)^{1/q}$$
$$=\left\|f^{(n)}\right\|_{\gamma_{u,w},p}\left(\int_{\gamma_{u,x}}|z-u|^{qn}|dz|+\int_{\gamma_{x,w}}|z-w|^{qn}|dz|\right)^{1/q}$$

and

$$\left\|f^{(n)}\right\|_{\gamma_{u,x},1}\max_{z\in\gamma_{u,x}}|z-u|^{n}+\left\|f^{(n)}\right\|_{\gamma_{x,w},1}\max_{z\in\gamma_{x,w}}|z-w|^{n}$$
$$\leq\max\left\{\max_{z\in\gamma_{u,x}}|z-u|^{n},\max_{z\in\gamma_{x,w}}|z-w|^{n}\right\}\left[\left\|f^{(n)}\right\|_{\gamma_{u,x},1}+\left\|f^{(n)}\right\|_{\gamma_{x,w},1}\right]$$
$$=\left\|f^{(n)}\right\|_{\gamma_{u,w},1}\max\left\{\max_{z\in\gamma_{u,x}}|z-u|^{n},\max_{z\in\gamma_{x,w}}|z-w|^{n}\right\},$$

which proves the inequality (3.2). □

In a similar way we can prove:

Theorem 4. *With the assumptions of Theorem 3 and for θ a complex number we have the identity (2.10) and the remainder $T_n(x,\gamma,\theta)$ satisfies the bounds*

$$|T_n(\gamma,\theta)|\leq C_n(\gamma,\theta), \tag{3.3}$$

where

$$C_n(\gamma,\theta):=\frac{1}{n!}\left[|\theta|\int_{\gamma}|z-u|^{n}\left|f^{(n)}(z)\right||dz|+|1-\theta|\int_{\gamma}|z-w|^{n}\left|f^{(n)}(z)\right||dz|\right]. \tag{3.4}$$

Moreover, we have

$$C_n(\gamma,\theta)\leq\frac{1}{n!}|\theta|\begin{cases}\left\|f^{(n)}\right\|_{\gamma,\infty}\int_{\gamma}|z-u|^{n}|dz|\\[2ex]\left\|f^{(n)}\right\|_{\gamma,p}\left(\int_{\gamma}|z-u|^{qn}|dz|\right)^{1/q}\\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex]\left\|f^{(n)}\right\|_{\gamma,1}\max_{z\in\gamma}|z-u|^{n}\end{cases}+\frac{1}{n!}|1-\theta|\begin{cases}\left\|f^{(n)}\right\|_{\gamma,\infty}\int_{\gamma}|z-w|^{n}|dz|\\[2ex]\left\|f^{(n)}\right\|_{\gamma,p}\left(\int_{\gamma}|z-w|^{qn}|dz|\right)^{1/q}\\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex]\left\|f^{(n)}\right\|_{\gamma,1}\max_{z\in\gamma}|z-w|^{n}\end{cases} \tag{3.5}$$

$$\leq \frac{1}{n!}\max\left\{|\theta|,|1-\theta|\right\} \tag{3.6}$$
$$\times\begin{cases} \left\|f^{(n)}\right\|_{\gamma,\infty}\left[\int_\gamma \left[|z-u|^n+|z-w|^n\right]|dz|\right] \\ \\ \left\|f^{(n)}\right\|_{\gamma,p}\left[\left(\int_\gamma |z-u|^{qn}|dz|\right)^{1/q}+\left(\int_\gamma |z-w|^{qn}|dz|\right)^{1/q}\right] \\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left\|f^{(n)}\right\|_{\gamma,1}\left[\max_{z\in\gamma}|z-u|^n+\max_{z\in\gamma}|z-w|^n\right]. \end{cases}$$

We observe that for $\theta=\frac{1}{2}$ we have the representation (2.12) and the remainder $T_n(\gamma)$ satisfies the inequalities

$$|T_n(\gamma)|\leq \frac{1}{2n!}\int_\gamma \left[|z-u|^n+|z-w|^n\right]\left|f^{(n)}(z)\right||dz| \tag{3.7}$$
$$\leq \frac{1}{2n!}\begin{cases} \left\|f^{(n)}\right\|_{\gamma,\infty}\int_\gamma \left[|z-u|^n+|z-w|^n\right]|dz| \\ \\ \left\|f^{(n)}\right\|_{\gamma,p}\left(\int_\gamma \left[|z-u|^n+|z-w|^n\right]^q|dz|\right)^{1/q} \\ \text{where } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left\|f^{(n)}\right\|_{\gamma,1}\max_{z\in\gamma}\left[|z-u|^n+|z-w|^n\right]. \end{cases}$$

4. Error Bounds for Bounded Derivatives

Suppose $\gamma\subset\mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t\in\gamma$ from $z(a)=u$ to $z(b)=w$. Now, for ϕ, $\Phi\in\mathbb{C}$ and γ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_\gamma(\phi,\Phi):=\left\{f:\gamma\to\mathbb{C}|\operatorname{Re}\left[(\Phi-f(z))\left(\overline{f(z)}-\overline{\phi}\right)\right]\geq 0 \text{ for each } z\in\gamma\right\}$$

and

$$\bar{\Delta}_\gamma(\phi,\Phi):=\left\{f:\gamma\to\mathbb{C}|\ \left|f(z)-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\phi| \text{ for each } z\in\gamma\right\}.$$

The following representation result may be stated.

Proposition 1. *For any ϕ, $\Phi\in\mathbb{C}$, $\phi\neq\Phi$, we have that $\bar{U}_\gamma(\phi,\Phi)$ and $\bar{\Delta}_\gamma(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_\gamma(\phi,\Phi)=\bar{\Delta}_\gamma(\phi,\Phi). \tag{4.1}$$

Proof. We observe that for any $w\in\mathbb{C}$ we have the equivalence

$$\left|w-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\phi|$$

if and only if

$$\operatorname{Re}\left[(\Phi-w)\left(\overline{w}-\overline{\phi}\right)\right]\geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Phi-\phi|^2-\left|w-\frac{\phi+\Phi}{2}\right|^2=\operatorname{Re}\left[(\Phi-w)\left(\overline{w}-\overline{\phi}\right)\right]$$

that holds for any $w\in\mathbb{C}$.

The equality (4.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 5. *For any ϕ, $\Phi\in\mathbb{C}$, $\phi\neq\Phi$, we have that*

$$\bar{U}_{\gamma}(\phi,\Phi)=\{f:\gamma\to\mathbb{C}\mid (\operatorname{Re}\Phi-\operatorname{Re}f(z))(\operatorname{Re}f(z)-\operatorname{Re}\phi)+(\operatorname{Im}\Phi-\operatorname{Im}f(z))(\operatorname{Im}f(z)-\operatorname{Im}\phi)\geq 0 \text{ for each } z\in\gamma\}. \tag{4.2}$$

Now, if we assume that $\operatorname{Re}(\Phi)\geq\operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi)\geq\operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\bar{S}_{\gamma}(\phi,\Phi):=\{f:\gamma\to\mathbb{C}\mid \operatorname{Re}(\Phi)\geq\operatorname{Re}f(z)\geq\operatorname{Re}(\phi) \text{ and } \operatorname{Im}(\Phi)\geq\operatorname{Im}f(z)\geq\operatorname{Im}(\phi) \text{ for each } z\in\gamma\}. \tag{4.3}$$

One can easily observe that $\bar{S}_{\gamma}(\phi,\Phi)$ is closed, convex and

$$\emptyset\neq\bar{S}_{\gamma}(\phi,\Phi)\subseteq\bar{U}_{\gamma}(\phi,\Phi). \tag{4.4}$$

We have the following result:

Theorem 5. *Let $f:D\subseteq\mathbb{C}\to\mathbb{C}$ be an analytic function on the domain D and $x\in D$. Suppose $\gamma\subset D$ is a smooth path parametrized by $z(t)$, $t\in[a,b]$ with $z(a)=u$, $z(t)=x$ and $z(b)=w$ where u, $w\in D$. If $f^{(n)}\in\bar{\Delta}_{\gamma}(\phi_n,\Phi_n)$ for some ϕ_n, $\Phi_n\in\mathbb{C}$, $\phi_n\neq\Phi_n$, then we have the equality*

$$\int_{\gamma}f(z)\,dz=\sum_{k=0}^{n-1}\frac{1}{(k+1)!}f^{(k)}(x)\left[(w-x)^{k+1}+(-1)^{k}(x-u)^{k+1}\right]+\frac{\phi_n+\Phi_n}{2}\frac{(-1)^{n}}{(n+1)!}\left[(x-u)^{n+1}+(-1)^{n}(w-x)^{n+1}\right]+O_n(x,\gamma,\phi_n,\Phi_n) \tag{4.5}$$

and the remainder satisfies the bound

$$|O_n(x,\gamma,\phi_n,\Phi_n)|\leq\frac{1}{2n!}|\Phi_n-\phi_n|\left[\int_{\gamma_{u,x}}|z-u|^{n}|dz|+\int_{\gamma_{x,w}}|z-w|^{n}|dz|\right]. \tag{4.6}$$

Proof. From the equality (2.8) we have for $\lambda=\frac{\phi_n+\Phi_n}{2}$ that

$$\int_{\gamma}f(z)\,dz=\sum_{k=0}^{n-1}\frac{1}{(k+1)!}f^{(k)}(x)\left[(w-x)^{k+1}+(-1)^{k}(x-u)^{k+1}\right]+\frac{\phi_n+\Phi_n}{2}\frac{(-1)^{n}}{(n+1)!}\left[(x-u)^{n+1}+(-1)^{n}(w-x)^{n+1}\right]+O_n(x,\gamma,\phi_n,\Phi_n), \tag{4.7}$$

where

$$O_n(x,\gamma,\phi_n,\Phi_n)=\frac{(-1)^{n}}{n!}\int_{\gamma_{u,x}}(z-u)^{n}\left(f^{(n)}(z)-\frac{\phi_n+\Phi_n}{2}\right)dz+\frac{(-1)^{n}}{n!}\int_{\gamma_{x,w}}(z-w)^{n}\left(f^{(n)}(z)-\frac{\phi_n+\Phi_n}{2}\right)dz. \tag{4.8}$$

Taking the modulus in (4.8) and taking into account that $f^{(n)} \in \bar{\Delta}_{\gamma}(\phi_n, \Phi_n)$, we have

$$\begin{aligned}
|O_n(x,\gamma,\phi_n,\Phi_n)| &\le \frac{1}{n!}\left|\int_{\gamma_{u,x}} (z-u)^n \left(f^{(n)}(z) - \frac{\phi_n+\Phi_n}{2}\right) dz\right| \\
&\quad + \frac{1}{n!}\left|\int_{\gamma_{x,w}} (z-w)^n \left(f^{(n)}(z) - \frac{\phi_n+\Phi_n}{2}\right) dz\right| \\
&\le \frac{1}{n!}\int_{\gamma_{u,x}} \left|(z-u)^n \left(f^{(n)}(z) - \frac{\phi_n+\Phi_n}{2}\right)\right| |dz| \\
&\quad + \frac{1}{n!}\int_{\gamma_{x,w}} \left|(z-w)^n \left(f^{(n)}(z) - \frac{\phi_n+\Phi_n}{2}\right)\right| |dz| \\
&= \frac{1}{n!}\int_{\gamma_{u,x}} |z-u|^n \left|f^{(n)}(z) - \frac{\phi_n+\Phi_n}{2}\right| |dz| \\
&\quad + \frac{1}{n!}\int_{\gamma_{x,w}} |z-w|^n \left|f^{(n)}(z) - \frac{\phi_n+\Phi_n}{2}\right| |dz| \\
&\le \frac{1}{2n!}|\Phi_n-\phi_n|\left[\int_{\gamma_{u,x}} |z-u|^n |dz| + \int_{\gamma_{x,w}} |z-w|^n |dz|\right],
\end{aligned}$$

which proves the desired inequality (4.6). □

We also have:

Theorem 6. *With the assumption of Theorem 5 and for $\theta \in \mathbb{C}$, we have*

$$\begin{aligned}
\int_{\gamma} f(z)\,dz &= \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[\theta(-1)^k f^{(k)}(w) + (1-\theta) f^{(k)}(u)\right](w-u)^{k+1} \\
&\quad + \frac{\phi_n+\Phi_n}{2}\frac{(-1)^n}{(n+1)!}\left[\theta + (-1)^n(1-\theta)\right](w-u)^{n+1} + T_n(\gamma,\theta,\phi_n,\Phi_n)
\end{aligned} \tag{4.9}$$

and the remainder $T_n(\gamma,\theta,\phi_n,\Phi_n)$ satisfies the bound

$$\begin{aligned}
|T_n(\gamma,\theta,\phi_n,\Phi_n)| &\le \frac{1}{2n!}|\Phi_n-\phi_n|\int_{\gamma}|\theta(z-u)^n + (1-\theta)(z-w)^n| |dz| \\
&\le \frac{1}{2n!}|\Phi_n-\phi_n|\left[|\theta|\int_{\gamma}|z-u|^n|dz| + |1-\theta|\int_{\gamma}|z-w|^n|dz|\right] \\
&\le \frac{1}{2n!}|\Phi_n-\phi_n|\max\{|\theta|,|1-\theta|\}\left[\int_{\gamma}\left[|z-u|^n + |z-w|^n\right]|dz|\right].
\end{aligned} \tag{4.10}$$

In particular, for $\theta = \frac{1}{2}$ we get

$$\begin{aligned}
\int_{\gamma} f(z)\,dz &= \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[\frac{(-1)^k f^{(k)}(w) + f^{(k)}(u)}{2}\right](w-u)^{k+1} \\
&\quad + \frac{\phi_n+\Phi_n}{2}\frac{(-1)^n}{(n+1)!}\left[\frac{1+(-1)^n}{2}\right](w-u)^{n+1} + T_n(\gamma,\phi_n,\Phi_n)
\end{aligned} \tag{4.11}$$

and the remainder $T_n(\gamma,\phi_n,\Phi_n)$ *satisfies the bounds*

$$|T_n(\gamma,\phi_n,\Phi_n)| \leq \frac{1}{4n!}|\Phi_n-\phi_n|\int_\gamma |(z-u)^n+(z-w)^n||dz| \\ \leq \frac{1}{4n!}|\Phi_n-\phi_n|\left[\int_\gamma [|z-u|^n+|z-w|^n]|dz|\right].$$

Proof. From the equality (2.16) for $\lambda=\frac{\phi_n+\Phi_n}{2}$ we have

$$\int_\gamma f(z)\,dz=\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[\theta(-1)^k f^{(k)}(w)+(1-\theta)f^{(k)}(u)\right](w-u)^{k+1} \\ +\frac{(-1)^n}{(n+1)!}\frac{\phi_n+\Phi_n}{2}[\theta+(-1)^n(1-\theta)](w-u)^{n+1}+T_n(\gamma,\theta,\phi_n,\Phi_n) \tag{4.12}$$

and the remainder $T_n(\gamma,\theta,\phi_n,\Phi_n)$ is given by

$$T_n(\gamma,\theta,\phi_n,\Phi_n) \\ =\frac{(-1)^n}{n!}\int_\gamma[\theta(z-u)^n+(1-\theta)(z-w)^n]\left(f^{(n)}(z)-\frac{\phi_n+\Phi_n}{2}\right)dz. \tag{4.13}$$

Taking the modulus in (4.13) and taking into account that $f^{(n)}\in\bar{\Delta}_\gamma(\phi_n,\Phi_n)$, we have

$$|T_n(\gamma,\theta,\phi_n,\Phi_n)| \\ \leq\frac{1}{n!}\int_\gamma|\theta(z-u)^n+(1-\theta)(z-w)^n|\left|f^{(n)}(z)-\frac{\phi_n+\Phi_n}{2}\right||dz| \\ \leq\frac{1}{2n!}|\Phi_n-\phi_n|\int_\gamma|\theta(z-u)^n+(1-\theta)(z-w)^n||dz|, \tag{4.14}$$

which proves the first inequality in (4.10). The rest is obvious. □

5. Bounds for Hölder's Continuous Derivatives

A function $g:\gamma\subset D\subseteq\mathbb{C}\to\mathbb{C}\to C$ is *Hölder continuous* on γ with the constant $H>0$ and $r\in(0,1]$ if

$$|f(z)-f(w)|\leq H|z-w|^r$$

for all $z,\,w\in\gamma$.

Theorem 7. *Let* $f:D\subseteq\mathbb{C}\to\mathbb{C}$ *be an analytic function on the domain* D *and* $x\in D$. *Suppose* $\gamma\subset D$ *is a smooth path parametrized by* $z(t)$, $t\in[a,b]$ *with* $z(a)=u$, $z(t)=x$ *and* $z(b)=w$ *where* $u,\,w\in D$. *If* $f^{(n)}$ *is Hölder continuous on* γ *with the constant* $H_n>0$ *and* $r\in(0,1]$

$$\int_\gamma f(z)\,dz=\sum_{k=0}^{n-1}\frac{1}{(k+1)!}f^{(k)}(x)\left[(w-x)^{k+1}+(-1)^k(x-u)^{k+1}\right] \\ +\frac{(-1)^n}{(n+1)!}\left[f^{(n)}(u)(x-u)^{n+1}+f^{(n)}(w)(-1)^n(w-x)^{n+1}\right]+O_n(x,\gamma), \tag{5.1}$$

where the remainder $O_n(x,\gamma)$ *satisfies the bound*

$$|O_n(x,\gamma)| \le \frac{1}{n!} H_n \left[\int_{\gamma_{u,x}} |z-u|^{n+r} |dz| + \int_{\gamma_{x,w}} |z-w|^{n+r} |dz| \right]. \tag{5.2}$$

In particular, if $f^{(n)}$ *is Lipschitzian on* γ *with the constant* $L_n > 0$*, then we have the error bound*

$$|O_n(x,\gamma)| \le \frac{1}{n!} L_n \left[\int_{\gamma_{u,x}} |z-u|^{n+1} |dz| + \int_{\gamma_{x,w}} |z-w|^{n+1} |dz| \right]. \tag{5.3}$$

Proof. From the identity (2.6) we have for $\lambda_1 = f^{(n)}(u)$ and $\lambda_2 = f^{(n)}(w)$

$$\begin{aligned} \int_\gamma f(z)\,dz &= \sum_{k=0}^{n-1} \frac{1}{(k+1)!} f^{(k)}(x) \left[(w-x)^{k+1} + (-1)^k (x-u)^{k+1} \right] \\ &\quad + f^{(n)}(u) \frac{(-1)^n}{(n+1)!} (x-u)^{n+1} + f^{(n)}(w) \frac{(-1)^{n+1}}{(n+1)!} (x-w)^{n+1} + O_n(x,\gamma), \end{aligned} \tag{5.4}$$

with the remainder $O_n(x,\gamma)$ given by

$$\begin{aligned} O_n(x,\gamma) &= \frac{(-1)^n}{n!} \int_{\gamma_{u,x}} (z-u)^n \left[f^{(n)}(z) - f^{(n)}(u) \right] dz \\ &\quad + \frac{(-1)^n}{n!} \int_{\gamma_{x,w}} (z-w)^n \left[f^{(n)}(z) - f^{(n)}(w) \right] dz. \end{aligned} \tag{5.5}$$

Taking the modulus in (5.5) and using the Hölder continuity we have

$$\begin{aligned} |O_n(x,\gamma)| &\le \frac{1}{n!} \int_{\gamma_{u,x}} \left| (z-u)^n \left[f^{(n)}(z) - f^{(n)}(u) \right] \right| |dz| \\ &\quad + \frac{1}{n!} \int_{\gamma_{x,w}} \left| (z-w)^n \left[f^{(n)}(z) - f^{(n)}(w) \right] \right| |dz| \\ &= \frac{1}{n!} \int_{\gamma_{u,x}} |z-u|^n \left| f^{(n)}(z) - f^{(n)}(u) \right| |dz| \\ &\quad + \frac{1}{n!} \int_{\gamma_{x,w}} |z-w|^n \left| f^{(n)}(z) - f^{(n)}(w) \right| |dz| \\ &\le \frac{1}{n!} H_n \left[\int_{\gamma_{u,x}} |z-u|^n |z-u|^r |dz| + \int_{\gamma_{x,w}} |z-w|^n |z-w|^r |dz| \right] \\ &= \frac{1}{n!} H_n \left[\int_{\gamma_{u,x}} |z-u|^{n+r} |dz| + \int_{\gamma_{x,w}} |z-w|^{n+r} |dz| \right], \end{aligned}$$

which proves the desired result (5.2). □

6. Examples for Logarithm and Exponential

Consider the function $f(z) = \frac{1}{z}$, $z \in \mathbb{C} \setminus \{0\}$. Then

$$f^{(k)}(z) = \frac{(-1)^k k!}{z^{k+1}} \text{ for } k \ge 0,\ z \in \mathbb{C} \setminus \{0\}$$

and suppose $\gamma \subset \mathbb{C}_\ell$ is a *smooth path* parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = u$ and $z(b) = w$ where $u,\ w \in \mathbb{C}_\ell$. Then

$$\int_\gamma f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz = \int_{\gamma_{u,w}} \frac{dz}{z} = \operatorname{Log}(w) - \operatorname{Log}(u)$$

for $u,\ w \in \mathbb{C}_\ell$.

Consider the function $f(z) = \operatorname{Log}(z)$ where $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$ and $\operatorname{Arg}(z)$ is such that $-\pi < \operatorname{Arg}(z) \le \pi$. Log is called the *"principal branch"* of the complex logarithmic function. The function f is analytic on all of $\mathbb{C}_\ell := \mathbb{C}\setminus\{x+iy : x \le 0,\ y = 0\}$ and

$$f^{(k)}(z) = \frac{(-1)^{k-1}(k-1)!}{z^k},\ k \ge 1,\ z \in \mathbb{C}_\ell.$$

Suppose $\gamma \subset \mathbb{C}_\ell$ is a *smooth path* parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = u$ and $z(b) = w$ where $u,\ w \in \mathbb{C}_\ell$. Then

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_{\gamma_{u,w}} f(z)\,dz = \int_{\gamma_{u,w}} \operatorname{Log}(z)\,dz = \\ &= z\operatorname{Log}(z)\big|_u^w - \int_{\gamma_{u,w}} (\operatorname{Log}(z))'\,z\,dz \\ &= w\operatorname{Log}(w) - u\operatorname{Log}(u) - \int_{\gamma_{u,w}} dz \\ &= w\operatorname{Log}(w) - u\operatorname{Log}(u) - (w-u),\end{aligned}$$

where $u,\ w \in \mathbb{C}_\ell$.

Consider the function $f(z) = \exp(z)$, $z \in \mathbb{C}$. Then

$$f^{(k)}(z) = \exp(z) \text{ for } k \ge 0,\ z \in \mathbb{C}$$

and suppose $\gamma \subset \mathbb{C}$ is a *smooth path* parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = u$ and $z(b) = w$ where $u,\ w \in \mathbb{C}$. Then

$$\int_\gamma f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz = \int_{\gamma_{u,w}} \exp(z)\,dz = \exp(w) - \exp(u).$$

We have by the equality (2.1) that

$$\begin{aligned}\int_\gamma f(z)\,dz &= f(x)(w-u) \\ &+ \sum_{k=1}^{n-1} \frac{1}{(k+1)!} f^{(k)}(x)\left[(w-x)^{k+1} + (-1)^k (x-u)^{k+1}\right] \\ &+ \frac{(-1)^n}{n!}\left[\int_{\gamma_{u,x}} (z-u)^n\ f^{(n)}(z)\,dz + \int_{\gamma_{x,w}} (z-w)^n f^{(n)}(z)\,dz\right]\end{aligned} \tag{6.1}$$

for $n \ge 2$.

Suppose $\gamma \subset \mathbb{C}_\ell$ is a *smooth path* parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = u$, $z(t) = x$ and $z(b) = w$ where $u,\ x,\ w \in \mathbb{C}_\ell$, then by writing the equality (6.1) for the function $f(z)=\frac{1}{z}$, we get the identity

$$\begin{aligned}(6.2)\quad &\operatorname{Log}(w)-\operatorname{Log}(u)\\ &=\frac{w-u}{x}+\sum_{k=1}^{n-1}\frac{(-1)^{k}}{(k+1)}\left[\left(\frac{w-x}{x}\right)^{k+1}+(-1)^{k}\left(\frac{x-u}{x}\right)^{k+1}\right]\\ &\qquad+\int_{\gamma_{u,x}}\frac{(z-u)^{n}}{z^{n+1}}dz+\int_{\gamma_{x,w}}\frac{(z-w)^{n}}{z^{n+1}}dz,\end{aligned}$$

for $n\geq 2$.

If we write the equality (6.1) for the function $f(z)=\operatorname{Log}(z)$, then we get the identity

$$\begin{aligned}(6.3)\quad &w\operatorname{Log}(w)-u\operatorname{Log}(u)-(w-u)\\ &=\operatorname{Log}(x)(w-u)+x\sum_{k=1}^{n-1}\frac{(-1)^{k-1}}{(k+1)k}\left[\left(\frac{w-x}{x}\right)^{k+1}+(-1)^{k}\left(\frac{x-u}{x}\right)^{k+1}\right]\\ &\qquad-\frac{1}{n}\left[\int_{\gamma_{u,x}}\left(\frac{z-u}{z}\right)^{n}dz+\int_{\gamma_{x,w}}\left(\frac{z-w}{z}\right)^{n}dz\right]\end{aligned}$$

for $n\geq 2$.

Suppose $\gamma\subset\mathbb{C}$ is a *smooth path* parametrized by $z(t)$, $t\in[a,b]$ with $z(a)=u$, $z(t)=x$ and $z(b)=w$ where u, x, $w\in\mathbb{C}$. If we write the equality (6.1) for the function $f(z)=\exp z$, then we get

$$\begin{aligned}(6.4)\quad &\exp(w)-\exp(u)=(w-u)\exp(x)\\ &+\exp(x)\sum_{k=1}^{n-1}\frac{1}{(k+1)!}\left[(w-x)^{k+1}+(-1)^{k}(x-u)^{k+1}\right]\\ &\qquad+\frac{(-1)^{n}}{n!}\left[\int_{\gamma_{u,x}}(z-u)^{n}\exp z\,dz+\int_{\gamma_{x,w}}(z-w)^{n}\exp z\,dz\right]\end{aligned}$$

for $n\geq 2$.

From the identity (2.12) we get

$$\begin{aligned}(6.5)\quad &\int_{\gamma}f(z)\,dz=\frac{f(w)+f(u)}{2}(w-u)\\ &+\sum_{k=1}^{n-1}\frac{1}{(k+1)!}\left[\frac{(-1)^{k}f^{(k)}(w)+f^{(k)}(u)}{2}\right](w-u)^{k+1}\\ &\qquad+\frac{(-1)^{n}}{2n!}\int_{\gamma}\left[(z-u)^{n}+(z-w)^{n}\right]f^{(n)}(z)\,dz\end{aligned}$$

for $n\geq 2$.

Suppose $\gamma\subset\mathbb{C}_{\ell}$ is a *smooth path* parametrized by $z(t)$, $t\in[a,b]$ with $z(a)=u$ and $z(b)=w$ where u, $w\in\mathbb{C}_{\ell}$, then by writing the equality (6.5) for the function

$f(z)=\frac{1}{z}$, we get the identity

$$\begin{aligned}(6.6)\quad \operatorname{Log}(w)-\operatorname{Log}(u) &= \frac{w+u}{2uw}(w-u)\\ &+\sum_{k=1}^{n-1}\frac{1}{(k+1)}\left[\frac{u^{k+1}+(-1)^{k}w^{k+1}}{2u^{k+1}w^{k+1}}\right](w-u)^{k+1}\\ &+\frac{1}{2}\int_{\gamma}\frac{(z-u)^{n}+(z-w)^{n}}{z^{n+1}}dz\end{aligned}$$

for $n\geq 2$.

By writing the equality (6.5) for the function $f(z)=\operatorname{Log}(z)$, we get the identity

$$\begin{aligned}(6.7)\quad w\operatorname{Log}(w)-u\operatorname{Log}(u)-(w-u) &= \frac{\operatorname{Log}(w)+\operatorname{Log}(u)}{2}(w-u)\\ &+\sum_{k=1}^{n-1}\frac{(-1)^{k-1}}{(k+1)k}\left[\frac{(-1)^{k}u^{k}+w^{k}}{2u^{k}w^{k}}\right](w-u)^{k+1}\\ &-\frac{1}{2n}\int_{\gamma}\frac{(z-u)^{n}+(z-w)^{n}}{z^{n}}dz\end{aligned}$$

for $n\geq 2$.

Suppose $\gamma\subset\mathbb{C}$ is a *smooth path* parametrized by $z(t)$, $t\in[a,b]$ with $z(a)=u$ and $z(b)=w$ where u, $w\in\mathbb{C}$. If we write the equality (6.5) for the function $f(z)=\exp z$, then we get

$$\begin{aligned}(6.8)\quad \exp(w)-\exp(u) &= \frac{\exp(w)+\exp(u)}{2}(w-u)\\ &+\sum_{k=1}^{n-1}\frac{1}{(k+1)!}\left[\frac{(-1)^{k}\exp(w)+\exp(u)}{2}\right](w-u)^{k+1}\\ &+\frac{(-1)^{n}}{2n!}\int_{\gamma}\left[(z-u)^{n}+(z-w)^{n}\right]\exp(z)\,dz\end{aligned}$$

for $n\geq 2$.

From the equality (6.2) we get

$$\begin{aligned}(6.9)\quad &\left|\operatorname{Log}(w)-\operatorname{Log}(u)-\frac{w-u}{x}\right.\\ &\left.-\sum_{k=1}^{n-1}\frac{(-1)^{k}}{(k+1)}\left[\left(\frac{w-x}{x}\right)^{k+1}+(-1)^{k}\left(\frac{x-u}{x}\right)^{k+1}\right]\right|\\ &\leq\int_{\gamma_{u,x}}\frac{|z-u|^{n}}{|z|^{n+1}}dz+\int_{\gamma_{x,w}}\frac{|z-w|^{n}}{|z|^{n+1}}dz,\end{aligned}$$

where u, x, $w\in\mathbb{C}_{\ell}$.

If $d_{u,x} := \inf_{z\in\gamma_{u,x}} |z|$ and $d_{x,w} := \inf_{z\in\gamma_{x,w}} |z|$ are positive and finite, then by (6.9) we get

$$\begin{aligned}(6.10)\quad &\Bigg|\mathrm{Log}\,(w) - \mathrm{Log}\,(u) - \frac{w-u}{x} \\ &\quad - \sum_{k=1}^{n-1} \frac{(-1)^k}{(k+1)} \left[\left(\frac{w-x}{x}\right)^{k+1} + (-1)^k \left(\frac{x-u}{x}\right)^{k+1}\right]\Bigg| \\ &\qquad \le \frac{1}{d_{u,x}^{n+1}} \int_{\gamma_{u,x}} |z-u|^n \, dz + \frac{1}{d_{x,w}^{n+1}} \int_{\gamma_{x,w}} |z-w|^n \, dz.\end{aligned}$$

If $d_{u,w} := \inf_{z\in\gamma_{u,w}} |z| \in (0,\infty)$, then we also have

$$\begin{aligned}(6.11)\quad &\Bigg|\mathrm{Log}\,(w) - \mathrm{Log}\,(u) - \frac{w-u}{x} \\ &\quad - \sum_{k=1}^{n-1} \frac{(-1)^k}{(k+1)} \left[\left(\frac{w-x}{x}\right)^{k+1} + (-1)^k \left(\frac{x-u}{x}\right)^{k+1}\right]\Bigg| \\ &\qquad \le \frac{1}{d_{u,w}^{n+1}} \left[\int_{\gamma_{u,x}} |z-u|^n \, dz + \int_{\gamma_{x,w}} |z-w|^n \, dz\right].\end{aligned}$$

From the equality (6.3) we get

$$\begin{aligned}(6.12)\quad &\Bigg|w\,\mathrm{Log}\,(w) - u\,\mathrm{Log}\,(u) - (w-u) \\ &\quad - \mathrm{Log}\,(x)\,(w-u) - x\sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{(k+1)\,k} \left[\left(\frac{w-x}{x}\right)^{k+1} + (-1)^k \left(\frac{x-u}{x}\right)^{k+1}\right]\Bigg| \\ &\qquad \le \frac{1}{n} \left[\int_{\gamma_{u,x}} \left|\frac{z-u}{z}\right|^n dz + \int_{\gamma_{x,w}} \left|\frac{z-w}{z}\right|^n dz\right],\end{aligned}$$

where $u,\ x,\ w \in \mathbb{C}_\ell$.

If $d_{u,x} := \inf_{z\in\gamma_{u,x}} |z|$ and $d_{x,w} := \inf_{z\in\gamma_{x,w}} |z|$ are positive and finite, then by (6.12) we get

$$\begin{aligned}(6.13)\quad &\Bigg|w\,\mathrm{Log}\,(w) - u\,\mathrm{Log}\,(u) - (w-u) \\ &\quad - \mathrm{Log}\,(x)\,(w-u) - x\sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{(k+1)\,k} \left[\left(\frac{w-x}{x}\right)^{k+1} + (-1)^k \left(\frac{x-u}{x}\right)^{k+1}\right]\Bigg| \\ &\qquad \le \frac{1}{n} \left[\frac{1}{d_{u,x}^{n}} \int_{\gamma_{u,x}} |z-u|^n \, dz + \frac{1}{d_{x,w}^{n}} \int_{\gamma_{x,w}} |z-w|^n \, dz\right].\end{aligned}$$

If $d_{u,w} := \inf_{z\in\gamma_{u,w}} |z| \in (0,\infty)$, then we also have

$$\begin{aligned}(6.14)\quad &\Bigg|w\,\mathrm{Log}\,(w) - u\,\mathrm{Log}\,(u) - (w-u) \\ &\quad - \mathrm{Log}\,(x)\,(w-u) - x\sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{(k+1)\,k} \left[\left(\frac{w-x}{x}\right)^{k+1} + (-1)^k \left(\frac{x-u}{x}\right)^{k+1}\right]\Bigg| \\ &\qquad \le \frac{1}{n d_{u,w}^{n}} \left[\int_{\gamma_{u,x}} |z-u|^n \, dz + \int_{\gamma_{x,w}} |z-w|^n \, dz\right].\end{aligned}$$

Similar inequalities may be stated by the use of the equalities (6.6) and (6.7) and we omit the details.

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