NEW TRAPEZOID TYPE RULES FOR APPROXIMATING THE INTEGRAL OF ANALYTIC COMPLEX FUNCTIONS ON PATHS FROM GENERAL DOMAINS

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Abstract. In this paper we establish some new trapezoid type rules for approximating the integral of analytic complex functions on paths from general domains. Error bounds for these expansions in terms of p-norms, Hölder and Lipschitz constants are also provided. Examples for the complex logarithm and the complex exponential are given as well.

1. Introduction

Suppose γ is a *smooth path* parametrized by $z(t)$, $t\in[a,b]$ and f is a complex function which is continuous on γ. Put $z(a)=u$ and $z(b)=w$ with u, $w\in\mathbb{C}$. We define the integral of f on $\gamma_{u,w}=\gamma$ as

$$\int_{\gamma} f(z)\,dz=\int_{\gamma_{u,w}} f(z)\,dz:=\int_{a}^{b} f(z(t))\,z'(t)\,dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are *piecewise smooth*. Suppose γ is parametrized by $z(t)$, $t\in[a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz:=\int_{\gamma_{u,v}} f(z)\,dz+\int_{\gamma_{v,w}} f(z)\,dz$$

where $v:=z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz|:=\int_{a}^{b} f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma)=\int_{\gamma_{u,w}} |dz|=\int_{a}^{b} |z'(t)|\,dt.$$

Let f and g be holomorphic in G, an open domain and suppose $\gamma\subset G$ is a piecewise smooth path from $z(a)=u$ to $z(b)=w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz=f(w)\,g(w)-f(u)\,g(u)-\int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{1.1}$$

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We recall also the *triangle inequality* for the complex integral, namely

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz| \le \|f\|_{\gamma,\infty}\,\ell(\gamma) \tag{1.2}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma}|f(z)|$.

We also define the p-norm with $p \ge 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p\,|dz|\right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)|\,|dz|.$$

If $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \le [\ell(\gamma)]^{1/q}\,\|f\|_{\gamma,p}.$$

In the recent paper [7] we established the following trapezoid type identity for analytic functions on convex domains:

Theorem 1. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D. Suppose $\gamma \subset D$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = u$ and $z(b) = w$ where $u,\ w \in D$. If $\lambda \in \mathbb{C}$, then we have the weighted trapezoid equality*

$$\begin{aligned}\int_\gamma f(z)\,dz = &\sum_{k=0}^{n} \frac{1}{(k+1)!}\left[\lambda f^{(k)}(u) + (1-\lambda)(-1)^k f^{(k)}(w)\right](w-u)^{k+1} \\ &+ T_n(\lambda,\gamma),\end{aligned} \tag{1.3}$$

where the remainder $T_n(\lambda,\gamma)$ is given by

$$\begin{aligned}T_n(\lambda,\gamma) := &\frac{\lambda}{n!}\int_\gamma (z-u)^{n+1}\left(\int_0^1 f^{(n+1)}[(1-s)u + sz](1-s)^n\,ds\right)dz \\ &+ \frac{(1-\lambda)}{n!}\int_\gamma (z-w)^{n+1}\left(\int_0^1 f^{(n+1)}[(1-s)w + sz](1-s)^n\,ds\right)dz \\ = &\frac{\lambda}{n!}\int_0^1 \left(\int_\gamma (z-u)^{n+1} f^{(n+1)}[(1-s)u + sz]\,dz\right)(1-s)^n\,ds \\ &+ \frac{(1-\lambda)}{n!}\int_0^1 \left(\int_\gamma (z-w)^{n+1} f^{(n+1)}[(1-s)w + sz]\,dz\right)(1-s)^n\,ds.\end{aligned} \tag{1.4}$$

In particular, for $\lambda = \frac{1}{2}$ we have the trapezoid equality

$$\begin{aligned}\int_\gamma f(z)\,dz = &\sum_{k=0}^{n} \frac{1}{(k+1)!}\left[\frac{f^{(k)}(u) + (-1)^k f^{(k)}(w)}{2}\right](w-u)^{k+1} \\ &+ T_n(\gamma),\end{aligned} \tag{1.5}$$

where the remainder $T_n(\gamma)$ is given by

$$
\begin{aligned}
(1.6)\quad T_n(\gamma) &:= \frac{1}{2n!}\int_\gamma (z-u)^{n+1}\left(\int_0^1 f^{(n+1)}[(1-s)u+sz](1-s)^n ds\right)dz \\
&+\frac{1}{2n!}\int_\gamma (z-w)^{n+1}\left(\int_0^1 f^{(n+1)}[(1-s)w+sz](1-s)^n ds\right)dz \\
&=\frac{1}{2n!}\int_0^1\left(\int_\gamma (z-u)^{n+1} f^{(n+1)}[(1-s)u+sz]\,dz\right)(1-s)^n ds \\
&+\frac{1}{2n!}\int_0^1\left(\int_\gamma (z-w)^{n+1} f^{(n+1)}[(1-s)w+sz]\,dz\right)(1-s)^n ds.
\end{aligned}
$$

We also have the error bounds [7]:

Theorem 2. *Let $f: D\subseteq\mathbb{C}\to\mathbb{C}$ be an analytic function on the convex domain D and $x\in D$. Suppose $\gamma\subset D$ is a smooth path parametrized by $z(t)$, $t\in[a,b]$ with $z(a)=u$ and $z(b)=w$ where $u, w\in D$. If $f^{(n+1)}$ satisfies the condition*

$$(1.7)\qquad \left\|f^{(n+1)}\right\|_{D,\infty} := \sup_{z\in D}\left|f^{(n+1)}(z)\right| < \infty$$

for some $n\geq 0$ and $\lambda\in\mathbb{C}$, then we have the representation (1.3) and the remainder $T_n(\lambda,\gamma)$ satisfies the bound

$$
\begin{aligned}
(1.8)\quad &|T_n(\lambda,\gamma)| \\
&\leq \frac{1}{(n+1)!}\left\|f^{(n+1)}\right\|_{D,\infty}\left[|\lambda|\int_\gamma |z-u|^{n+1}|dz| + |1-\lambda|\int_\gamma |z-w|^{n+1}|dz|\right] \\
&\leq \max\{|\lambda|,|1-\lambda|\}\frac{1}{(n+1)!}\left\|f^{(n+1)}\right\|_{D,\infty} \\
&\quad\times\left[\int_\gamma |z-u|^{n+1}|dz| + \int_\gamma |z-w|^{n+1}|dz|\right].
\end{aligned}
$$

In particular, if $\lambda=\frac{1}{2}$, then we have the representation (1.5) and the remainder $T_n(\gamma)$ satisfies the bound

$$(1.9)\quad |T_n(\gamma)| \leq \frac{1}{2(n+1)!}\left\|f^{(n+1)}\right\|_{D,\infty}\left[\int_\gamma |z-u|^{n+1}|dz| + \int_\gamma |z-w|^{n+1}|dz|\right].$$

These results generalize the corresponding results for real functions of a real variable, see [3], [2], [1] and [4]. For other recent results on trapezoid inequality see [5]-[12].

In this paper we establish some new trapezoid type rules for approximating the integral of analytic complex functions on paths from general domains. Error bounds for these expansions in terms of p-norms, Hölder and Lipschitz constants are also provided. Examples for the complex logarithm and the complex exponential are given as well.

2. Trapezoid Type Representation Results

We have the following representation result for functions defined on non-necessarily convex domains D.

Theorem 3. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the domain D and $x \in D$. Suppose $\gamma \subset D$ is a smooth path parametrized by $z(z)$, $t \in [a,b]$ with $z(a) = u$, $z(t) = x$ and $z(b) = w$ where u, $w \in D$. Then we have the equality*

$$\int_\gamma f(z)\,dz = \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[(x-u)^{k+1} f^{(k)}(u) + (-1)^k (w-x)^{k+1} f^{(k)}(w)\right] + \frac{1}{n!}\int_\gamma (x-z)^n f^{(n)}(z)\,dz \tag{2.1}$$

for $n \geq 1$.

Proof. The proof is by mathematical induction over $n \geq 1$. For $n = 1$, we have to prove that

$$\int_\gamma f(z)\,dz = (x-u) f(u) + (w-x) f(w) + \int_\gamma (x-z) f'(z)\,dz, \tag{2.2}$$

which is straightforward as may be seen by the integration by parts formula applied for the integral

$$\int_\gamma (x-z) f'(z)\,dz.$$

Assume that (2.1) holds for "n" and let us prove it for "$n+1$". That is, we wish to show that:

$$\int_\gamma f(z)\,dz = \sum_{k=0}^{n} \frac{1}{(k+1)!}\left[(x-u)^{k+1} f^{(k)}(u) + (-1)^k (w-x)^{k+1} f^{(k)}(w)\right] + \frac{1}{(n+1)!}\int_\gamma (x-z)^{n+1} f^{(n+1)}(z)\,dz. \tag{2.3}$$

Using the integration by parts rule, we have

$$\begin{aligned}
&\frac{1}{(n+1)!}\int_\gamma (x-z)^{n+1} f^{(n+1)}(z)\,dz \\
&= \frac{1}{(n+1)!}\int_\gamma (x-z)^{n+1} \left(f^{(n)}(z)\right)' dz \\
&= \frac{1}{(n+1)!}\left[(x-z)^{n+1} f^{(n)}(z)\Big|_u^w + (n+1)\int_\gamma (x-z)^n f^{(n)}(z)\,dz\right] \\
&= \frac{1}{(n+1)!} \\
&\times \left[(x-w)^{n+1} f^{(n)}(w) - (x-u)^{n+1} f^{(n)}(u) + (n+1)\int_\gamma (x-z)^n f^{(n)}(z)\,dz\right] \\
&= \frac{1}{n!}\int_\gamma (x-z)^n f^{(n)}(z)\,dz \\
&\quad - \frac{1}{(n+1)!}\left[(x-u)^{n+1} f^{(n)}(u) + (-1)^n (w-x)^{n+1} f^{(n)}(w)\right]
\end{aligned} \tag{2.4}$$

which gives that

$$\begin{aligned}(2.5)\quad & \frac{1}{n!}\int_{\gamma}(x-z)^{n} f^{(n)}(z)\,dz \\ &= \frac{1}{(n+1)!}\left[(x-u)^{n+1} f^{(n)}(u)+(-1)^{n}(w-x)^{n+1} f^{(n)}(w)\right] \\ &\qquad + \frac{1}{(n+1)!}\int_{\gamma}(x-z)^{n+1} f^{(n+1)}(z)\,dz.\end{aligned}$$

From the induction hypothesis we have

$$\begin{aligned}(2.6)\quad & \frac{1}{n!}\int_{\gamma}(x-z)^{n} f^{(n)}(z)\,dz \\ &= \int_{\gamma} f(z)\,dz-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-u)^{k+1} f^{(k)}(u)+(-1)^{k}(w-x)^{k+1} f^{(k)}(w)\right].\end{aligned}$$

By making use of (2.5) and (2.6) we get

$$\begin{aligned}& \int_{\gamma} f(z)\,dz-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-u)^{k+1} f^{(k)}(u)+(-1)^{k}(w-x)^{k+1} f^{(k)}(w)\right] \\ &= \frac{1}{(n+1)!}\left[(x-u)^{n+1} f^{(n)}(u)+(-1)^{n}(w-x)^{n+1} f^{(n)}(w)\right] \\ &\qquad + \frac{1}{(n+1)!}\int_{\gamma}(x-z)^{n+1} f^{(n+1)}(z)\,dz,\end{aligned}$$

which is equivalent to (2.3). □

Corollary 1. *With the assumptions of Theorem 3 we have*

$$\begin{aligned}(2.7)\quad \int_{\gamma} f(z)\,dz &= \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-u)^{k+1} f^{(k)}(u)+(-1)^{k}(w-x)^{k+1} f^{(k)}(w)\right] \\ &\quad + \frac{1}{(n+1)!}\left[\lambda_1 (x-u)^{n+1}+\lambda_2 (-1)^{n}(w-x)^{n+1}\right] \\ &\quad + \frac{1}{n!}\int_{\gamma_{u,x}}(x-z)^{n}\left[f^{(n)}(z)-\lambda_1\right]dz+\frac{1}{n!}\int_{\gamma_{x,w}}(x-z)^{n}\left[f^{(n)}(z)-\lambda_2\right]dz\end{aligned}$$

for any $\lambda_1,\ \lambda_2 \in \mathbb{C}$.

In particular, we have the representation

$$\begin{aligned}(2.8)\quad \int_{\gamma} f(z)\,dz &= \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-u)^{k+1} f^{(k)}(u)+(-1)^{k}(w-x)^{k+1} f^{(k)}(w)\right] \\ &\quad + \frac{\lambda}{(n+1)!}\left[(x-u)^{n+1}+(-1)^{n}(w-x)^{n+1}\right] \\ &\qquad + \frac{1}{n!}\int_{\gamma}(x-z)^{n}\left[f^{(n)}(z)-\lambda\right]dz\end{aligned}$$

for any $\lambda \in \mathbb{C}$.

Proof. Observe that

$$\begin{aligned}
&\frac{1}{n!}\int_{\gamma}(x-z)^{n}f^{(n)}(z)\,dz\\
&=\frac{1}{n!}\int_{\gamma_{u,x}}(x-z)^{n}f^{(n)}(z)\,dz+\frac{1}{n!}\int_{\gamma_{x,w}}(x-z)^{n}f^{(n)}(z)\,dz\\
&=\frac{1}{n!}\int_{\gamma_{u,x}}(x-z)^{n}\left[f^{(n)}(z)-\lambda_{1}\right]dz+\lambda_{1}\frac{1}{n!}\int_{\gamma_{u,x}}(x-z)^{n}\,dz\\
&+\frac{1}{n!}\int_{\gamma_{x,w}}(x-z)^{n}\left[f^{(n)}(z)-\lambda_{2}\right]dz+\lambda_{2}\frac{1}{n!}\int_{\gamma_{x,w}}(x-z)^{n}\,dz\\
&=\frac{1}{n!}\int_{\gamma_{u,x}}(x-z)^{n}\left[f^{(n)}(z)-\lambda_{1}\right]dz+\lambda_{1}\frac{(x-u)^{n+1}}{(n+1)!}\\
&\quad+\frac{1}{n!}\int_{\gamma_{x,w}}(x-z)^{n}\left[f^{(n)}(z)-\lambda_{2}\right]dz+\lambda_{2}(-1)^{n}\frac{(w-x)^{n+1}}{(n+1)!},
\end{aligned}$$

and by utilising the representation (2.1), we get the desired result (2.7). □

3. Error Bounds

We have:

Theorem 4. *Let $f:D\subseteq\mathbb{C}\to\mathbb{C}$ be an analytic function on the domain D. Suppose $\gamma\subset D$ is a smooth path parametrized by $z(t)$, $t\in[a,b]$ with $z(a)=u$, $z(t)=x$ and $z(b)=w$ where u, x, $w\in D$. Then we have the inequality*

$$\begin{aligned}
(3.1)\quad &\left|\int_{\gamma}f(z)\,dz\right.\\
&\left.-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-u)^{k+1}f^{(k)}(u)+(-1)^{k}(w-x)^{k+1}f^{(k)}(w)\right]\right|\\
&\leq\frac{1}{n!}\int_{\gamma}|x-z|^{n}\left|f^{(n)}(z)\right||dz|\leq\frac{1}{n!}\times\begin{cases}\left\|f^{(n)}\right\|_{\gamma,1}\max_{z\in\gamma}|x-z|^{n},\\[2ex]\left\|f^{(n)}\right\|_{\gamma,q}\left(\int_{\gamma}|x-z|^{np}|dz|\right)^{1/p}\\ \text{where } p,\ q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1,\\[2ex]\left\|f^{(n)}\right\|_{\gamma,\infty}\int_{\gamma}|x-z|^{n}|dz|,\end{cases}
\end{aligned}$$

for $n\geq1$.

Proof. Follows by the identity (2.1) and by Hölder's integral inequality

$$\int_{\gamma}|x-z|^{n}\left|f^{(n)}(z)\right||dz|\leq\begin{cases}\max_{z\in\gamma}|x-z|^{n}\int_{\gamma}\left|f^{(n)}(z)\right||dz|,\\[2ex]\left(\int_{\gamma}|x-z|^{np}|dz|\right)^{1/p}\left(\int_{\gamma}\left|f^{(n)}(z)\right|^{q}|dz|\right)^{1/q}\\ \text{where } p,\ q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1,\\[2ex]\int_{\gamma}|x-z|^{n}|dz|\max_{z\in\gamma}\left|f^{(n)}(z)\right|.\end{cases}$$

□

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$. Now, for ϕ, $\Phi \in \mathbb{C}$ and γ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_\gamma(\phi, \Phi) := \left\{ f : \gamma \to \mathbb{C} | \operatorname{Re}\left[(\Phi - f(z)) \left(\overline{f(z)} - \overline{\phi} \right) \right] \geq 0 \text{ for each } z \in \gamma \right\}$$

and

$$\bar{\Delta}_\gamma(\phi, \Phi) := \left\{ f : \gamma \to \mathbb{C} | \left| f(z) - \frac{\phi + \Phi}{2} \right| \leq \frac{1}{2} |\Phi - \phi| \text{ for each } z \in \gamma \right\}.$$

The following representation result may be stated.

Proposition 1. *For any ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that $\bar{U}_\gamma(\phi, \Phi)$ and $\bar{\Delta}_\gamma(\phi, \Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_\gamma(\phi, \Phi) = \bar{\Delta}_\gamma(\phi, \Phi). \tag{3.2}$$

Proof. We observe that for any $w \in \mathbb{C}$ we have the equivalence

$$\left| w - \frac{\phi + \Phi}{2} \right| \leq \frac{1}{2} |\Phi - \phi|$$

if and only if

$$\operatorname{Re}\left[(\Phi - w) \left(\overline{w} - \overline{\phi} \right) \right] \geq 0.$$

This follows by the equality

$$\frac{1}{4} |\Phi - \phi|^2 - \left| w - \frac{\phi + \Phi}{2} \right|^2 = \operatorname{Re}\left[(\Phi - w) \left(\overline{w} - \overline{\phi} \right) \right]$$

that holds for any $w \in \mathbb{C}$.

The equality (3.2) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 2. *For any ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that*

$$\begin{aligned} \bar{U}_\gamma(\phi, \Phi) = \{ f : \gamma \to \mathbb{C} \mid & (\operatorname{Re}\Phi - \operatorname{Re} f(z)) (\operatorname{Re} f(z) - \operatorname{Re}\phi) \\ & + (\operatorname{Im}\Phi - \operatorname{Im} f(z)) (\operatorname{Im} f(z) - \operatorname{Im}\phi) \geq 0 \textit{ for each } z \in \gamma \}. \end{aligned} \tag{3.3}$$

Now, if we assume that $\operatorname{Re}(\Phi) \geq \operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi) \geq \operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_\gamma(\phi, \Phi) := \{ f : \gamma \to \mathbb{C} \mid & \operatorname{Re}(\Phi) \geq \operatorname{Re} f(z) \geq \operatorname{Re}(\phi) \\ & \text{and } \operatorname{Im}(\Phi) \geq \operatorname{Im} f(z) \geq \operatorname{Im}(\phi) \text{ for each } z \in \gamma \}. \end{aligned} \tag{3.4}$$

One can easily observe that $\bar{S}_\gamma(\phi, \Phi)$ is closed, convex and

$$\emptyset \neq \bar{S}_\gamma(\phi, \Phi) \subseteq \bar{U}_\gamma(\phi, \Phi). \tag{3.5}$$

We have:

Theorem 5. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the domain D. Suppose $\gamma \subset D$ is a smooth path parametrized by $z(t)$, $t \in [a, b]$ with $z(a) = u$, $z(t) = x$*

and $z(b)=w$ *where* $u,\ x,\ w\in D$. *If* $f^{(n)}\in\bar{\Delta}_{\gamma}(\phi_n,\Phi_n)$ *for some* $\phi_n,\ \Phi_n\in\mathbb{C}$, $\phi_n\neq\Phi_n$, *then we have the inequality*

$$\begin{aligned}(3.6)\quad &\left|\int_{\gamma} f(z)\,dz\right.\\ &-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-u)^{k+1}f^{(k)}(u)+(-1)^{k}(w-x)^{k+1}f^{(k)}(w)\right]\\ &\left.-\frac{1}{(n+1)!}\frac{\phi_n+\Phi_n}{2}\left[(x-u)^{n+1}+(-1)^{n}(w-x)^{n+1}\right]\right|\\ &\leq\frac{1}{2n!}\left|\Phi_n-\phi_n\right|\int_{\gamma}|x-z|^{n}|dz|\end{aligned}$$

for $n\geq 1$.

Proof. By making use of the equality (2.8) and the fact that $f^{(n)}\in\bar{\Delta}_{\gamma}(\phi_n,\Phi_n)$ we have

$$\begin{aligned}&\left|\int_{\gamma} f(z)\,dz-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-u)^{k+1}f^{(k)}(u)+(-1)^{k}(w-x)^{k+1}f^{(k)}(w)\right]\right.\\ &\left.-\frac{1}{(n+1)!}\frac{\phi_n+\Phi_n}{2}\left[(x-u)^{n+1}+(-1)^{n}(w-x)^{n+1}\right]\right|\\ &\leq\frac{1}{n!}\left|\int_{\gamma}(x-z)^{n}\left[f^{(n)}(z)-\frac{\phi_n+\Phi_n}{2}\right]dz\right|\\ &\leq\frac{1}{n!}\int_{\gamma}|x-z|^{n}\left|f^{(n)}(z)-\frac{\phi_n+\Phi_n}{2}\right||dz|\\ &\leq\frac{1}{n!}\int_{\gamma}|x-z|^{n}\left|f^{(n)}(z)-\frac{\phi_n+\Phi_n}{2}\right||dz|\leq\frac{1}{2n!}\left|\Phi_n-\phi_n\right|\int_{\gamma}|x-z|^{n}|dz|,\end{aligned}$$

which proves the desired result (3.6). □

A function $g:\gamma\subset D\subseteq\mathbb{C}\to\mathbb{C}\to C$ is Hölder continuous on γ with the constant $H>0$ and $r\in(0,1]$ if

$$|f(z)-f(w)|\leq H|z-w|^{r}$$

for all $z,\ w\in\gamma$.

Theorem 6. *Let* $f:D\subseteq\mathbb{C}\to\mathbb{C}$ *be an analytic function on the domain* D *and* $x\in D$. *Suppose* $\gamma\subset D$ *is a smooth path parametrized by* $z(t),\ t\in[a,b]$ *with* $z(a)=u,\ z(t)=x$ *and* $z(b)=w$ *where* $u,\ w\in D$. *If* $f^{(n)}$ *is Hölder continuous on*

γ with the constant $H_n > 0$ and $r \in (0,1]$, then

$$\begin{aligned}(3.7)\quad & \Bigg| \int_\gamma f(z)\,dz \\ & - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(x-u)^{k+1} f^{(k)}(u) + (-1)^k (w-x)^{k+1} f^{(k)}(w) \right] \\ & \quad - \frac{f^{(n)}(x)}{(n+1)!} \left[(x-u)^{n+1} + (-1)^n (w-x)^{n+1} \right] \Bigg| \\ & \qquad \le \frac{1}{n!} H_n \int_\gamma |x-z|^{n+r}\,dz \end{aligned}$$

and

$$\begin{aligned}(3.8)\quad & \Bigg| \int_\gamma f(z)\,dz \\ & - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(x-u)^{k+1} f^{(k)}(u) + (-1)^k (w-x)^{k+1} f^{(k)}(w) \right] \\ & - \frac{1}{(n+1)!} \left[f^{(n)}(u)(x-u)^{n+1} + (-1)^n f^{(n)}(w)(w-x)^{n+1} \right] \Bigg| \\ & \le \frac{1}{n!} H_n \left[\int_{\gamma_{u,x}} |x-z|^n |z-u|^r |dz| + \int_{\gamma_{x,w}} |x-z|^n |z-w|^r |dz| \right]. \end{aligned}$$

In particular, if $f^{(n)}$ is Lipschitzian on γ with the constant $L_n > 0$, then

$$\begin{aligned}(3.9)\quad & \Bigg| \int_\gamma f(z)\,dz \\ & - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(x-u)^{k+1} f^{(k)}(u) + (-1)^k (w-x)^{k+1} f^{(k)}(w) \right] \\ & \quad - \frac{f^{(n)}(x)}{(n+1)!} \left[(x-u)^{n+1} + (-1)^n (w-x)^{n+1} \right] \Bigg| \\ & \qquad \le \frac{1}{n!} L_n \int_\gamma |x-z|^{n+1}\,|dz| \end{aligned}$$

and

$$\begin{aligned}(3.10)\quad & \Bigg| \int_\gamma f(z)\,dz \\ & - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(x-u)^{k+1} f^{(k)}(u) + (-1)^k (w-x)^{k+1} f^{(k)}(w) \right] \\ & - \frac{1}{(n+1)!} \left[f^{(n)}(u)(x-u)^{n+1} + (-1)^n f^{(n)}(w)(w-x)^{n+1} \right] \Bigg| \\ & \le \frac{1}{n!} H_n \left[\int_{\gamma_{u,x}} |x-z|^n |z-u|\,|dz| + \int_{\gamma_{x,w}} |x-z|^n |z-w|\,|dz| \right]. \end{aligned}$$

Proof. Using the identity (2.8) we get

$$\int_{\gamma} f(z)\,dz = \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[(x-u)^{k+1} f^{(k)}(u) + (-1)^k (w-x)^{k+1} f^{(k)}(w)\right] + \frac{f^{(n)}(x)}{(n+1)!}\left[(x-u)^{n+1} + (-1)^n (w-x)^{n+1}\right] + \frac{1}{n!}\int_{\gamma} (x-z)^n \left[f^{(n)}(z) - f^{(n)}(x)\right] dz.$$

Since $f^{(n)}$ is Hölder continuous on γ with the constant $H_n > 0$ and $r \in (0,1]$, then

$$\begin{aligned} &\left|\int_{\gamma} f(z)\,dz - \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[(x-u)^{k+1} f^{(k)}(u) + (-1)^k (w-x)^{k+1} f^{(k)}(w)\right] - \frac{f^{(n)}(x)}{(n+1)!}\left[(x-u)^{n+1} + (-1)^n (w-x)^{n+1}\right]\right| \\ &\le \frac{1}{n!}\int_{\gamma} \left|(x-z)^n \left[f^{(n)}(z) - f^{(n)}(x)\right]\right| |dz| \\ &= \frac{1}{n!}\int_{\gamma} |x-z|^n \left|f^{(n)}(z) - f^{(n)}(x)\right| |dz| \\ &\le \frac{1}{n!} H_n \int_{\gamma} |x-z|^{n+r} |dz|, \end{aligned}$$

which proves the desired result (3.7).

Using the identity (2.7) we also have

$$\int_{\gamma} f(z)\,dz = \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[(x-u)^{k+1} f^{(k)}(u) + (-1)^k (w-x)^{k+1} f^{(k)}(w)\right] + \frac{1}{(n+1)!}\left[f^{(n)}(u)(x-u)^{n+1} + (-1)^n f^{(n)}(w)(w-x)^{n+1}\right] + \frac{1}{n!}\int_{\gamma_{u,x}} (x-z)^n \left[f^{(n)}(z) - f^{(n)}(u)\right] dz + \frac{1}{n!}\int_{\gamma_{x,w}} (x-z)^n \left[f^{(n)}(z) - f^{(n)}(w)\right] dz.$$

Since $f^{(n)}$ is Hölder continuous on γ with the constant $H_n > 0$ and $r \in (0,1]$, then

$$\left|\int_{\gamma} f(z)\,dz - \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[(x-u)^{k+1} f^{(k)}(u) + (-1)^k (w-x)^{k+1} f^{(k)}(w)\right] - \frac{1}{(n+1)!}\left[f^{(n)}(u)(x-u)^{n+1} + (-1)^n f^{(n)}(w)(w-x)^{n+1}\right]\right|$$

$$\leq \frac{1}{n!}\int_{\gamma_{u,x}} \left|(x-z)^n \left[f^{(n)}(z)-f^{(n)}(u)\right]\right| |dz|$$
$$+\frac{1}{n!}\int_{\gamma_{x,w}} \left|(x-z)^n \left[f^{(n)}(z)-f^{(n)}(w)\right]\right| |dz|$$
$$=\frac{1}{n!}\int_{\gamma_{u,x}} |x-z|^n \left|f^{(n)}(z)-f^{(n)}(u)\right| |dz|$$
$$+\frac{1}{n!}\int_{\gamma_{x,w}} |x-z|^n \left|f^{(n)}(z)-f^{(n)}(w)\right| |dz|$$
$$\leq \frac{1}{n!}H_n\left[\int_{\gamma_{u,x}} |x-z|^n |z-u|^r |dz| + \int_{\gamma_{x,w}} |x-z|^n |z-w|^r |dz|\right],$$

which proves the desired result (3.9). □

4. Examples for Logarithm and Exponential

Consider the function $f(z)=\frac{1}{z}$, $z\in\mathbb{C}\setminus\{0\}$. Then

$$f^{(k)}(z)=\frac{(-1)^k k!}{z^{k+1}} \text{ for } k\geq 0,\ z\in\mathbb{C}\setminus\{0\}$$

and suppose $\gamma\subset\mathbb{C}_\ell$ is a *smooth path* parametrized by $z(t)$, $t\in[a,b]$ with $z(a)=u$ and $z(b)=w$ where u, $w\in\mathbb{C}_\ell$. Then

$$\int_\gamma f(z)\,dz=\int_{\gamma_{u,w}} f(z)\,dz=\int_{\gamma_{u,w}}\frac{dz}{z}=\operatorname{Log}(w)-\operatorname{Log}(u)$$

for u, $w\in\mathbb{C}_\ell$.

Consider the function $f(z)=\operatorname{Log}(z)$ where $\operatorname{Log}(z)=\ln|z|+i\operatorname{Arg}(z)$ and $\operatorname{Arg}(z)$ is such that $-\pi<\operatorname{Arg}(z)\leq\pi$. Log is called the *"principal branch"* of the complex logarithmic function. The function f is analytic on all of $\mathbb{C}_\ell:=\mathbb{C}\setminus\{x+iy: x\leq 0,\ y=0\}$ and

$$f^{(k)}(z)=\frac{(-1)^{k-1}(k-1)!}{z^k},\ k\geq 1,\ z\in\mathbb{C}_\ell.$$

Suppose $\gamma\subset\mathbb{C}_\ell$ is a *smooth path* parametrized by $z(t)$, $t\in[a,b]$ with $z(a)=u$ and $z(b)=w$ where u, $w\in\mathbb{C}_\ell$. Then

$$\int_\gamma f(z)\,dz=\int_{\gamma_{u,w}} f(z)\,dz=\int_{\gamma_{u,w}}\operatorname{Log}(z)\,dz=$$
$$=z\operatorname{Log}(z)\big|_u^w-\int_{\gamma_{u,w}}(\operatorname{Log}(z))'\,z dz$$
$$=w\operatorname{Log}(w)-u\operatorname{Log}(u)-\int_{\gamma_{u,w}}dz$$
$$=w\operatorname{Log}(w)-u\operatorname{Log}(u)-(w-u),$$

where u, $w\in\mathbb{C}_\ell$.

Consider the function $f(z)=\exp(z)$, $z\in\mathbb{C}$. Then

$$f^{(k)}(z)=\exp(z) \text{ for } k\geq 0,\ z\in\mathbb{C}$$

and suppose $\gamma \subset \mathbb{C}$ is a *smooth path* parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = u$ and $z(b) = w$ where u, $w \in \mathbb{C}$. Then

$$\int_\gamma f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz = \int_{\gamma_{u,w}} \exp(z)\,dz = \exp(w) - \exp(u).$$

We have by the equality (2.1) that

$$\int_\gamma f(z)\,dz = (x-u) f(u) + (w-x) f(w) + \sum_{k=1}^{n-1} \frac{1}{(k+1)\,k}\left[(x-u)^{k+1} f^{(k)}(u) + (-1)^k (w-x)^{k+1} f^{(k)}(w)\right] + \frac{1}{n!}\int_\gamma (x-z)^n f^{(n)}(z)\,dz \tag{4.1}$$

for $n \geq 2$.

Suppose $\gamma \subset \mathbb{C}_\ell$ is a *smooth path* parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = u$, $z(t) = x$ and $z(b) = w$ where u, x, $w \in \mathbb{C}_\ell$, then by writing the equality (4.1) for the function $f(z) = \frac{1}{z}$, we get the identity

$$\operatorname{Log}(w) - \operatorname{Log}(u) = x\left(\frac{w-u}{wu}\right) + \sum_{k=1}^{n-1} \frac{1}{(k+1)}\left[(-1)^k \left(\frac{x-u}{u}\right)^{k+1} + \left(\frac{w-x}{w}\right)^{k+1}\right] + (-1)^n \int_\gamma \frac{(x-z)^n}{z^{n+1}}dz \tag{4.2}$$

for $n \geq 2$.

If we write the equality (4.1) for the function $f(z) = \operatorname{Log}(z)$, then we get the identity

$$w\operatorname{Log}(w) - u\operatorname{Log}(u) - (w-u) = (x-u)\operatorname{Log}(u) + (w-x)\operatorname{Log}(w) - \sum_{k=1}^{n-1} \frac{1}{(k+1)\,k}\left[(-1)^k \frac{(x-u)^{k+1}}{u^k} + \frac{(w-x)^{k+1}}{w^k}\right] + \frac{(-1)^{n-1}}{n}\int_\gamma \left(\frac{x-z}{z}\right)^n dz \tag{4.3}$$

for $n \geq 2$.

Suppose $\gamma \subset \mathbb{C}$ is a *smooth path* parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = u$, $z(t) = x$ and $z(b) = w$ where u, x, $w \in \mathbb{C}$. If we write the equality (4.1) for the function $f(z) = \exp z$, then we get

$$\exp(w) - \exp(u) = (x-u)\exp(u) + (w-x)\exp(w) + \sum_{k=1}^{n-1} \frac{1}{(k+1)\,k}\left[(x-u)^{k+1}\exp(u) + (-1)^k (w-x)^{k+1}\exp(w)\right] + \frac{1}{n!}\int_\gamma (x-z)^n \exp(z)\,dz \tag{4.4}$$

for $n \geq 2$.

Using the identity (4.2) we get

$$\left|\operatorname{Log}(w)-\operatorname{Log}(u)-x\left(\frac{w-u}{wu}\right)\right.$$
$$\left.-\sum_{k=1}^{n-1}\frac{1}{(k+1)}\left[(-1)^k\left(\frac{x-u}{u}\right)^{k+1}+\left(\frac{w-x}{w}\right)^{k+1}\right]\right|$$
$$\leq\int_\gamma\frac{|x-z|^n}{|z|^{n+1}}dz, \tag{4.5}$$

where $\gamma\subset\mathbb{C}_\ell$ is a *smooth path* parametrized by $z(t)$, $t\in[a,b]$ with $z(a)=u$, $z(t)=x$ and $z(b)=w$ where u, x, $w\in\mathbb{C}_\ell$.

If $d_\gamma:=\inf_{z\in\gamma}|z|\in(0,\infty)$, then by (4.5) we get

$$\left|\operatorname{Log}(w)-\operatorname{Log}(u)-x\left(\frac{w-u}{wu}\right)\right.$$
$$\left.-\sum_{k=1}^{n-1}\frac{1}{(k+1)}\left[(-1)^k\left(\frac{x-u}{u}\right)^{k+1}+\left(\frac{w-x}{w}\right)^{k+1}\right]\right|$$
$$\leq\frac{1}{d_\gamma^{n+1}}\int_\gamma|x-z|^n dz. \tag{4.6}$$

From (4.3) we also get

$$|w\operatorname{Log}(w)-u\operatorname{Log}(u)-(w-u)-(x-u)\operatorname{Log}(u)-(w-x)\operatorname{Log}(w)$$
$$\left.+\sum_{k=1}^{n-1}\frac{1}{(k+1)k}\left[(-1)^k\frac{(x-u)^{k+1}}{u^k}+\frac{(w-x)^{k+1}}{w^k}\right]\right|$$
$$\leq\frac{1}{n}\int_\gamma\left|\frac{x-z}{z}\right|^n dz. \tag{4.7}$$

If $d_\gamma:=\inf_{z\in\gamma}|z|\in(0,\infty)$, then by (4.7) we obtain

$$|w\operatorname{Log}(w)-u\operatorname{Log}(u)-(w-u)-(x-u)\operatorname{Log}(u)-(w-x)\operatorname{Log}(w)$$
$$\left.+\sum_{k=1}^{n-1}\frac{1}{(k+1)k}\left[(-1)^k\frac{(x-u)^{k+1}}{u^k}+\frac{(w-x)^{k+1}}{w^k}\right]\right|$$
$$\leq\frac{1}{nd_\gamma^n}\int_\gamma|x-z|^n dz. \tag{4.8}$$

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