Asymptotic expansions and continued fraction approximations for the harmonic number

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Abstract. In this paper, we provide a method to construct a continued fraction approximation based on a given asymptotic expansion. We establish some asymptotic expansions for the harmonic number which employ the nth triangular number. Based on these expansions, we derive the corresponding continued fraction approximations for the harmonic number.

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1 Introduction

The Indian mathematician Ramanujan (see [1, p. 531] and [11, p. 276]) claimed the following asymptotic expansion for the nth harmonic number:

$$
\begin{aligned}
H_n := \sum_{k=1}^{n} \frac{1}{k} \sim \frac{1}{2}\ln(2m) + \gamma + \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \frac{1}{2310m^5} \\
- \frac{191}{360360m^6} + \frac{29}{30030m^7} - \frac{2833}{1166880m^8} + \frac{140051}{17459442m^9} - \cdots \qquad (1.1)
\end{aligned}
$$

as $n \to \infty$, where $m = n(n+1)/2$ is the nth triangular number and γ is the Euler-Mascheroni constant.

Ramanujan's formula (1.1) has been the subject of intense investigations and has motivated a large number of research papers (see, for example, [2, 4–10, 12, 13]).

Villarino [12, Theorem 1.1] first gave a complete proof of expansion (1.1) in terms of the Bernoulli polynomials. Recently, Chen [5] gave a recursive relation for determining the coefficients of Ramanujan's asymptotic expansion (1.1), without the Bernoulli numbers and polynomials

$$
H_n \sim \frac{1}{2}\ln(2m) + \gamma + \sum_{\ell=1}^{\infty} \frac{a_\ell}{m^\ell}, \qquad n \to \infty, \qquad (1.2)
$$

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where the coefficients a_ℓ ($\ell \in \mathbb{N} := \{1, 2, \dots\}$) are given by the recurrence relation

$$a_1 = \frac{1}{12}, \;\; a_\ell = \frac{1}{2^{\ell+1}\ell} \left\{ \frac{1}{2\ell+1} - \sum_{j=1}^{\ell-1} 2^{j+1} a_j \binom{2\ell-j}{2\ell-2j+1} \right\}, \qquad \ell \geq 2. \tag{1.3}$$

Mortici and Villarino [10, Theorem 2] and Chen [2, Theorem 3.3] obtained the following asymptotic expansion:

$$H_n \sim \frac{1}{2} \ln\left(2m + \frac{1}{3}\right) + \gamma + \sum_{j=2}^{\infty} \frac{\rho_j}{(2m + \frac{1}{3})^j}, \qquad n \to \infty. \tag{1.4}$$

Moreover, these authors gave a formula for determining the coefficients ρ_j in (1.4). From a computational viewpoint, (1.4) is an improvement on the formula (1.2).

Chen [2, Theorem 3.1] obtained the following asymptotic expansion:

$$H_n \sim \gamma + \frac{1}{2} \ln\left(2m + \frac{1}{3} + \sum_{\ell=1}^{\infty} \frac{\omega_\ell}{(2m)^\ell}\right), \qquad n \to \infty, \tag{1.5}$$

with the coefficients ω_ℓ ($\ell \in \mathbb{N}$) given by the recursive relation

$$\omega_1 = -\frac{1}{90}, \quad \omega_\ell = b_{2(\ell+1)} - \sum_{j=1}^{\ell-1} \binom{2\ell-j-1}{2\ell-2j} \omega_j, \qquad \ell \geq 2, \tag{1.6}$$

where b_j are given by

$$b_j = \sum_{k_1+2k_2+\cdots+jk_j=j} \frac{(-2)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!} \left(\frac{B_1}{1}\right)^{k_1} \left(\frac{B_2}{2}\right)^{k_2} \cdots \left(\frac{B_j}{j}\right)^{k_j}, \tag{1.7}$$

and B_j are the Bernoulli numbers and the summation is taken over all nonnegative integers k_j satisfying the equation $k_1 + 2k_2 + \cdots + jk_j = j$.

It follows from [3, Corollary 3.1] that

$$H_n - \ln\left(n + \frac{1}{2}\right) - \gamma = \frac{\frac{1}{48}}{m + \frac{17}{80}} + O\left(\frac{1}{n^6}\right), \quad n \to \infty. \tag{1.8}$$

In this paper, we provide a method to construct a continued fraction approximation based on a given asymptotic expansion. We establish some asymptotic expansions for the harmonic number which employ the nth triangular number. Based on these expansions, we derive the corresponding continued fraction approximations for the harmonic number. All results of the present paper are motivated by (1.1), (1.4), (1.5) and (1.8).

The following lemma will be useful in pour present investigation.

Lemma 1.1. *Let $a_1 \neq 0$ and*

$$A(x) \sim \sum_{j=1}^{\infty} \frac{a_j}{x^j}, \qquad x \to \infty$$

be a given asymptotic expansion. Define the function B by

$$A(x) = \frac{a_1}{B(x)}.$$

Then the function $B(x) = a_1/A(x)$ has asymptotic expansion of the following form

$$B(x) \sim x + \sum_{j=0}^{\infty} \frac{b_j}{x^j}, \qquad x \to \infty,$$

where

$$b_0 = -\frac{a_2}{a_1}, \quad b_j = -\frac{1}{a_1} \left(a_{j+2} + \sum_{k=1}^{j} a_{k+1} b_{j-k} \right), \qquad j \geq 1. \tag{1.9}$$

Proof. We can let

$$\frac{a_1}{A(x)} \sim x + \sum_{j=0}^{\infty} \frac{b_j}{x^j}, \qquad x \to \infty, \tag{1.10}$$

where b_j (for $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$) are real numbers to be determined. Write (1.10) as

$$\sum_{j=1}^{\infty} \frac{a_j}{x^j} \left(x + \sum_{k=0}^{\infty} \frac{b_k}{x^k} \right) \sim a_1,$$

$$\sum_{j=0}^{\infty} \frac{a_{j+2}}{x^j} \sim -\sum_{j=0}^{\infty} \frac{a_{j+1}}{x^j} \sum_{k=0}^{\infty} \frac{b_k}{x^k},$$

$$\sum_{j=0}^{\infty} a_{j+2} x^{-j} \sim \sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} (-a_{k+1} b_{j-k}) \right) x^{-j}. \tag{1.11}$$

Equating coefficients of equal powers of x in (1.11), we obtain

$$a_{j+2} = -\sum_{k=0}^{j} a_{k+1} b_{j-k}, \qquad j \geq 0,$$

For $j = 0$ we obtain $b_0 = -a_2/a_1$, and for $j \geq 1$ we have

$$a_{j+2} = -\sum_{k=1}^{j} a_{k+1} b_{j-k} - a_1 b_j, \qquad j \geq 1,$$

which gives the desired formula (1.9). □

Lemma 1.1 provides a method to construct a continued fraction approximation based on a given asymptotic expansion. We state this method as a consequence of Lemma 1.1.

Corollary 1.1. Let $a_1 \neq 0$ and

$$A(x) \sim \sum_{j=1}^{\infty} \frac{a_j}{x^j}, \qquad x \to \infty \tag{1.12}$$

be a given asymptotic expansion. Then the function A has the following continued fraction approximation of the form

$$A(x) \approx \cfrac{a_1}{x + b_0 + \cfrac{b_1}{x + c_0 + \cfrac{c_1}{x + d_0 + \ddots}}}, \qquad x \to \infty, \tag{1.13}$$

where the constants in the right-hand side of (1.13) are given by the following recurrence relations:

$$\begin{cases} b_0 = -\frac{a_2}{a_1}, \quad b_j = -\frac{1}{a_1}\left(a_{j+2} + \sum_{k=1}^{j} a_{k+1} b_{j-k}\right) \\ c_0 = -\frac{b_2}{b_1}, \quad c_j = -\frac{1}{b_1}\left(b_{j+2} + \sum_{k=1}^{j} b_{k+1} c_{j-k}\right) \\ d_0 = -\frac{c_2}{c_1}, \quad d_j = -\frac{1}{c_1}\left(c_{j+2} + \sum_{k=1}^{j} c_{k+1} d_{j-k}\right) \\ \qquad \cdots \quad \cdots \end{cases} \tag{1.14}$$

Remark 1.1. *Clearly,* $a_j \Longrightarrow b_j \Longrightarrow c_j \Longrightarrow d_j \Longrightarrow \ldots$*. Thus, the asymptotic expansion* (1.12) $\Longrightarrow$ *the continued fraction approximation* (1.13)*. Corollary 1.1 transforms the asymptotic expansion* (1.12) *into a corresponding continued fraction of the form* (1.13)*, and provides the system* (1.14) *to determine the constants in the right-hand side of* (1.13)*.*

2 Main results

Theorem 2.1 transforms the asymptotic expansion (1.1) into a corresponding continued fraction of the form (2.1).

Theorem 2.1. *Let* $m = \frac{1}{2}n(n+1)$*. As* $n \to \infty$*, we have*

$$H_n \approx \frac{1}{2}\ln(2m) + \gamma + \cfrac{a_1}{m + b_0 + \cfrac{b_1}{m + c_0 + \cfrac{c_1}{m + d_0 + \ddots}}}, \tag{2.1}$$

where

$$a_1 = \frac{1}{12}, \quad b_0 = \frac{1}{10}, \quad b_1 = -\frac{19}{2100}, \quad c_0 = \frac{91}{190}, \quad c_1 = -\frac{16585}{83391}, \quad d_0 = \frac{2357167}{1638598}, \ \ldots. \tag{2.2}$$

Proof. Denote

$$A(m) = H_n - \frac{1}{2}\ln(2m) - \gamma.$$

It follows from (1.2) that

$$A(m) \sim \sum_{\ell=1}^{\infty} \frac{a_\ell}{m^\ell} = \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \frac{1}{2310m^5} - \frac{191}{360360m^6} + \frac{29}{30030m^7} - \frac{2833}{1166880m^8} + \frac{140051}{17459442m^9} - \cdots \tag{2.3}$$

as $m \to \infty$, where the coefficients a_ℓ ($\ell \in \mathbb{N}$) are given in (1.3). Then, $A(m)$ has the continued fraction approximation of the form

$$A(m) = H_n - \frac{1}{2}\ln(2m) - \gamma \approx \cfrac{a_1}{m + b_0 + \cfrac{b_1}{m + c_0 + \cfrac{c_1}{m + d_0 + \ddots}}}, \qquad m \to \infty, \tag{2.4}$$

where the constants in the right-hand side of (2.4) can be determined using (1.14). Noting that

$$a_1 = \frac{1}{12}, \quad a_2 = -\frac{1}{120}, \quad a_3 = \frac{1}{630}, \quad a_4 = -\frac{1}{1680}, \quad a_5 = \frac{1}{2310}, \quad a_6 = -\frac{191}{360360}, \quad \ldots,$$

we obtain from the first recurrence relation in (1.14) that

$$\begin{aligned}
b_0 &= -\frac{a_2}{a_1} = \frac{1}{10},\\
b_1 &= -\frac{a_3 + a_2 b_0}{a_1} = -\frac{19}{2100},\\
b_2 &= -\frac{a_4 + a_2 b_1 + a_3 b_0}{a_1} = \frac{13}{3000},\\
b_3 &= -\frac{a_5 + a_2 b_2 + a_3 b_1 + a_4 b_0}{a_1} = -\frac{187969}{48510000},\\
b_4 &= -\frac{a_6 + a_2 b_3 + a_3 b_2 + a_4 b_1 + a_5 b_0}{a_1} = \frac{3718037}{700700000}.
\end{aligned}$$

We obtain from the second recurrence relation in (1.14) that

$$\begin{aligned}
c_0 &= -\frac{b_2}{b_1} = \frac{91}{190},\\
c_1 &= -\frac{b_3 + b_2 c_0}{b_1} = -\frac{16585}{83391},\\
c_2 &= -\frac{b_4 + b_2 c_1 + b_3 c_0}{b_1} = \frac{11785835}{41195154}.
\end{aligned}$$

Continuing the above process, we find

$$d_0 = -\frac{c_2}{c_1} = \frac{2357167}{1638598}, \quad \ldots.$$

The proof is complete. $\square$

Remark 2.1. *It is well known that*

$$
\begin{aligned}
H_n - \ln n - \gamma &\sim -\sum_{k=1}^{\infty} \frac{B_k}{kn^k} \\
&= \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6} + \dots, \qquad n \to \infty,
\end{aligned} \tag{2.5}
$$

where B_k are the Bernoulli numbers. Following the same method as was used in the proof of Theorem 2.1, we derive

$$
H_n \approx \ln n + \gamma + \cfrac{\frac{1}{2}}{n + \frac{1}{6} + \cfrac{\frac{1}{36}}{n + \frac{13}{30} + \cfrac{\frac{9}{25}}{n + \frac{17}{630} + \ddots}}}, \qquad n \to \infty. \tag{2.6}
$$

Theorem 2.2. *Let $m = \frac{1}{2}n(n+1)$. The harmonic number has the following asymptotic expansion:*

$$
\begin{aligned}
H_n &\sim \ln\left(n + \frac{1}{2}\right) + \gamma + \sum_{\ell=1}^{\infty} \frac{r_\ell}{m^\ell} \\
&= \ln\left(n + \frac{1}{2}\right) + \gamma + \frac{1}{48m} - \frac{17}{3840m^2} + \frac{407}{322560m^3} - \frac{1943}{3440640m^4} \\
&\quad + \frac{32537}{75694080m^5} - \frac{25019737}{47233105920m^6} + \dots
\end{aligned} \tag{2.7}
$$

as $n \to \infty$, where the coefficients r_ℓ ($\ell \in \mathbb{N}$) are given by the recurrence relation

$$
r_1 = \frac{1}{48}, \quad r_\ell = \frac{1}{2^{\ell+1}\ell} \left\{ \frac{1}{2^{2\ell}(2\ell+1)} - \sum_{j=1}^{\ell-1} 2^{j+1} r_j \binom{2\ell - j}{2\ell - 2j + 1} \right\}, \quad \ell \geq 2. \tag{2.8}
$$

Proof. Denote

$$
I_n = H_n - \ln\left(n + \frac{1}{2}\right) - \gamma \quad \text{and} \quad J_n = \sum_{\ell=1}^{\infty} \frac{r_\ell}{m^\ell}.
$$

Let $I_n \sim J_n$ and

$$
\Delta I_n := I_{n+1} - I_n \sim \Delta J_n := J_{n+1} - J_n
$$

as $n \to \infty$, where r_ℓ ($\ell \in \mathbb{N}$) are real numbers to be determined.

It is easy to see that

$$
\begin{aligned}
\Delta I_n &= \frac{1}{n+1} - \left\{ \ln\left(1 + \frac{1}{2(n+1)}\right) - \ln\left(1 - \frac{1}{2(n+1)}\right) \right\} \\
&= \frac{1}{n+1} - \sum_{k=1}^{\infty} \frac{(-1)^{k-1} + 1}{2^k k (n+1)^k} = -\sum_{\ell=1}^{\infty} \frac{1}{2^{2\ell}(2\ell+1)} (n+1)^{-2\ell-1}.
\end{aligned} \tag{2.9}
$$

and

$$\Delta J_n = \sum_{k=1}^{\infty} \frac{2^k r_k}{(n+1)^{2k}} \left(1 + \frac{1}{n+1}\right)^{-k} - \sum_{k=1}^{\infty} \frac{2^k r_k}{(n+1)^{2k}} \left(1 - \frac{1}{n+1}\right)^{-k}. \tag{2.10}$$

Direct computation yields

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{2^k r_k}{(n+1)^{2k}} \left(1 + \frac{1}{n+1}\right)^{-k} &= \sum_{k=1}^{\infty} \frac{2^k r_k}{(n+1)^{2k}} \sum_{j=0}^{\infty} \binom{-k}{j} \frac{1}{(n+1)^j} \\
&= \sum_{k=1}^{\infty} \frac{2^k r_k}{(n+1)^{2k}} \sum_{j=0}^{\infty} (-1)^j \binom{k+j-1}{j} \frac{1}{(n+1)^j} \\
&= \sum_{k=1}^{\infty} \sum_{j=1}^{k} 2^j r_j (-1)^{k-j} \binom{k-1}{k-j} \frac{1}{(n+1)^{k+j}} \\
&= \sum_{\ell=2}^{\infty} \sum_{j=1}^{\lfloor \frac{\ell}{2} \rfloor} 2^j r_j (-1)^{\ell} \binom{\ell-j-1}{\ell-2j} \frac{1}{(n+1)^{\ell}}
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{2^k r_k}{(n+1)^{2k}} \left(1 - \frac{1}{n+1}\right)^{-k} &= \sum_{k=1}^{\infty} \frac{2^k r_k}{(n+1)^{2k}} \sum_{j=0}^{\infty} \binom{-k}{j} \frac{(-1)^j}{(n+1)^j} \\
&= \sum_{k=1}^{\infty} \frac{2^k r_k}{(n+1)^{2k}} \sum_{j=0}^{\infty} \binom{k+j-1}{j} \frac{1}{(n+1)^j} \\
&= \sum_{k=1}^{\infty} \sum_{j=1}^{k} 2^j r_j \binom{k-1}{k-j} \frac{1}{(n+1)^{k+j}} \\
&= \sum_{\ell=2}^{\infty} \sum_{j=1}^{\lfloor \frac{\ell}{2} \rfloor} 2^j r_j \binom{\ell-j-1}{\ell-2j} \frac{1}{(n+1)^{\ell}}.
\end{aligned} \tag{2.12}$$

Substituting (2.11) and (2.12) into (2.10) yields

$$\Delta J_n = \sum_{\ell=2}^{\infty} \sum_{j=1}^{\lfloor \frac{\ell}{2} \rfloor} \Big((-1)^{\ell} - 1\Big) 2^j r_j \binom{\ell-j-1}{\ell-2j} \frac{1}{(n+1)^{\ell}}. \tag{2.13}$$

Replacement of ℓ by $2\ell+1$ in (2.13) yields

$$\Delta J_n = -\sum_{\ell=1}^{\infty} \sum_{j=1}^{\ell} 2^{j+1} r_j \binom{2\ell-j}{2\ell-2j+1} (n+1)^{-2\ell-1}. \tag{2.14}$$

Equating coefficients of the term $(n+1)^{-2\ell-1}$ on the right-hand sides of (2.9) and (2.14) yields

$$\sum_{j=1}^{\ell} 2^{j+1} r_j \binom{2\ell-j}{2\ell-2j+1} = \frac{1}{2^{2\ell}(2\ell+1)}, \qquad \ell \geq 1. \tag{2.15}$$

For $\ell = 1$ in (2.15) we obtain $r_1 = \frac{1}{48}$, and for $\ell \geq 2$ we have

$$\sum_{j=1}^{\ell-1} 2^{j+1} r_j \binom{2\ell - j}{2\ell - 2j + 1} + 2^{\ell+1} \ell r_\ell = \frac{1}{2^{2\ell}(2\ell + 1)},$$

which gives the desired formula (2.8). □

Remark 2.2. *We here gave the recursive relation* (2.8) *for determining the coefficients r_ℓ in expansion* (2.7)*, without the Bernoulli numbers and polynomials.*

Remark 2.3. *Denote*

$$A^*(m) = H_n - \ln\left(n + \frac{1}{2}\right) - \gamma.$$

It follows from (2.7) *that*

$$A^*(m) \sim \sum_{\ell=1}^{\infty} \frac{r_\ell}{m^\ell} = \frac{1}{48m} - \frac{17}{3840m^2} + \frac{407}{322560m^3} - \frac{1943}{3440640m^4} + \frac{32537}{75694080m^5} - \frac{25019737}{47233105920m^6} + \dots \tag{2.16}$$

as $m \to \infty$, where the coefficients r_ℓ ($\ell \in \mathbb{N}$) are given in (2.8)*. Following the same method as was used in the proof of Theorem 2.1, we derive*

$$H_n \approx \ln\left(n + \frac{1}{2}\right) + \gamma + \cfrac{\lambda_1}{m + u_1 + \cfrac{\lambda_2}{m + \mu_2 + \cfrac{\lambda_3}{m + \mu_3 + \ddots}}}, \tag{2.17}$$

where

$$\lambda_1 = \frac{1}{48}, \quad \mu_1 = \frac{17}{80}, \quad \lambda_2 = -\frac{2071}{134400}, \quad \mu_2 = \frac{117863}{165680},$$
$$\lambda_3 = -\frac{15685119025}{63409182144}, \quad \mu_3 = \frac{2312217133079747}{1351329470432240}, \quad \dots. \tag{2.18}$$

Thus, we develop the approximation formula (1.8) *to produce a continued fraction approximation.*

Theorem 2.3. *Let $m = \frac{1}{2}n(n+1)$. The harmonic number has the following asymptotic expansion:*

$$\begin{aligned} H_n &\sim \frac{1}{2}\ln\left(2m + \frac{1}{3}\right) + \gamma + \sum_{\ell=2}^{\infty} \frac{s_\ell}{m^\ell} \\ &= \frac{1}{2}\ln\left(2m + \frac{1}{3}\right) + \gamma - \frac{1}{720m^2} + \frac{37}{45360m^3} - \frac{181}{362880m^4} + \frac{503}{1197504m^5} \\ &\quad - \frac{1480211}{2802159360m^6} + \frac{2705333}{2802159360m^7} - \frac{793046533}{326651719680m^8} + \frac{470463477509}{58650316268544m^9} - \dots \end{aligned} \tag{2.19}$$

as $n \to \infty$, with the coefficients s_ℓ given by

$$s_\ell = a_\ell - \frac{(-1)^{\ell-1}}{6^\ell 2\ell}, \quad \ell \geq 2, \tag{2.20}$$

where a_ℓ are given in (1.3).

Proof. We find by (1.2) that, as $n \to \infty$,

$$\begin{aligned} H_n - \frac{1}{2}\ln\left(2m + \frac{1}{3}\right) - \gamma &= H_n - \frac{1}{2}\ln(2m) - \gamma - \frac{1}{2}\ln\left(1 + \frac{1}{6m}\right) \\ &\sim \sum_{\ell=1}^{\infty} \frac{a_\ell}{m^\ell} - \frac{1}{2}\sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1}}{\ell(6m)^\ell}. \end{aligned}$$

Noting that $a_1 = \frac{1}{12}$, we obtain, as $n \to \infty$,

$$H_n \sim \frac{1}{2}\ln\left(2m + \frac{1}{3}\right) + \gamma + \sum_{\ell=2}^{\infty}\left\{a_\ell - \frac{(-1)^{\ell-1}}{6^\ell 2\ell}\right\}\frac{1}{m^\ell}.$$

The proof is complete. $\square$

Theorem 2.4. *Let $m = \frac{1}{2}n(n+1)$. As $n \to \infty$, we have*

$$H_n \approx \frac{1}{2}\ln\left(2m + \frac{1}{3}\right) + \gamma + \cfrac{p_1}{m^2 + \frac{37}{63}m + q_1 + \cfrac{p_2}{m + q_2 + \cfrac{p_3}{m + q_3 + \ddots}}}, \tag{2.21}$$

where

$$\begin{aligned} &p_1 = -\frac{1}{720}, \quad q_1 = -\frac{451}{31752}, \quad p_2 = \frac{228764}{2750517}, \quad q_2 = \frac{21448004509}{11990893824}, \\ &p_3 = -\frac{36637398233630775}{36226136230395904}, \quad q_3 = \frac{86442719924955272247584297}{26612223343933404862713600}, \quad \ldots. \end{aligned} \tag{2.22}$$

Proof. Denote

$$F(m) = H_n - \frac{1}{2}\ln\left(2m + \frac{1}{3}\right) - \gamma.$$

It follows from (2.19) that

$$\begin{aligned} F(m) \sim \sum_{\ell=2}^{\infty} \frac{s_\ell}{m^\ell} &= -\frac{1}{720m^2} + \frac{37}{45360m^3} - \frac{181}{362880m^4} + \frac{503}{1197504m^5} - \frac{1480211}{2802159360m^6} \\ &\quad + \frac{2705333}{2802159360m^7} - \frac{793046533}{326651719680m^8} + \frac{470463477509}{58650316268544m^9} - \ldots \end{aligned} \tag{2.23}$$

as $m \to \infty$, where the coefficients s_ℓ ($\ell \in \mathbb{N}$) are given in (2.20).

Define the function $G(m)$ by

$$F(m) = \frac{s_2}{G(m)}.$$

We obtain by (2.23) and Lemma 1.1 that

$$G(m) = \frac{s_2}{F(m)} \sim \frac{s_2}{\sum_{\ell=2}^{\infty} s_\ell m^{-\ell}} = m\left(\frac{s_2}{\sum_{\ell=1}^{\infty} s_{\ell+1} m^{-\ell}}\right) = m^2 + \frac{37}{63}m - \frac{451}{31752} + A^{**}(m),$$

where

$$\begin{aligned} A^{**}(m) \sim \frac{228764}{2750517m} - \frac{21448004509}{144171099072m^2} + \frac{3180925176497}{9082779241536m^3} \\ - \frac{898929405728511653}{856033777956284928m^4} + \frac{2008288563825356198279}{51233621610683652940 8m^5} - \cdots, \end{aligned} \tag{2.24}$$

We then obtain

$$F(m) \sim \frac{-\frac{1}{720}}{m^2 + \frac{37}{63}m - \frac{451}{31752} + A^{**}(m)}. \tag{2.25}$$

Following the same method as was used in the proof of Theorem 2.1, we derive the continued fraction approximation of $A^{**}(m)$ (we here omit the derivation of (2.26))

$$A^{**}(m) \approx \cfrac{p_2}{m + q_2 + \cfrac{p_3}{m + q_3 + \ddots}} \tag{2.26}$$

as $m \to \infty$, where p_j and q_j (for $j \geq 2$) are given in (2.22). Substituting (2.26) into (2.25) yields (2.21). $\square$

Theorem 2.5. *Let $m = \frac{1}{2}n(n+1)$. As $n \to \infty$, we have*

$$H_n \approx \frac{1}{2}\ln\left(2m + \frac{1}{3} + \cfrac{\alpha_1}{m + \beta_1 + \cfrac{\alpha_2}{m + \beta_2 + \cfrac{\alpha_3}{m + \beta_3 + \ddots}}}\right) + \gamma, \tag{2.27}$$

where

$$\begin{aligned} &\alpha_1 = -\frac{1}{180}, \quad \beta_1 = \frac{53}{126}, \quad \alpha_2 = -\frac{26329}{317520}, \quad \beta_2 = \frac{42684239}{36491994}, \\ &\alpha_3 = -\frac{487447163992501}{785108985906960}, \quad \beta_3 = \frac{2049473595024948803087}{847043761130064882714}, \quad \ldots. \end{aligned} \tag{2.28}$$

Proof. Write (1.5) as

$$e^{2(H_n - \gamma)} \sim 2m + \frac{1}{3} + \sum_{\ell=1}^{\infty} \frac{d_\ell}{m^\ell}, \qquad n \to \infty,$$

with the coefficients d_ℓ given by

$$d_1 = -\frac{1}{180}, \quad d_\ell = \frac{\omega_\ell}{2^\ell}, \quad \ell \geq 2, \tag{2.29}$$

where ω_ℓ are given in (1.6). Denote

$$A^{***}(m) = e^{2(H_n-\gamma)} - 2m - \frac{1}{3}.$$

We have, as $m \to \infty$,

$$\begin{aligned} A^{***}(m) &\sim \sum_{\ell=1}^{\infty} \frac{d_\ell}{m^\ell} \\ &= -\frac{1}{180m} + \frac{53}{22680m^2} - \frac{3929}{2721600m^3} + \frac{240673}{179625600m^4} - \frac{488481881}{267478848000m^5} \\ &\quad + \frac{8834570273}{2521943424000m^6} - \frac{652512638837083}{72026704189440000m^7} + \cdots. \end{aligned} \tag{2.30}$$

Following the same method as was used in the proof of Theorem 2.1, we derive

$$A^{***}(m) \approx \cfrac{\alpha_1}{m + \beta_1 + \cfrac{\alpha_2}{m+\beta_2+\cfrac{\alpha_3}{m+\beta_3+\ddots}}} \tag{2.31}$$

as $m \to \infty$, where α_j and β_j are given in (2.28). We here omit the derivation of (2.31). Formula (2.31) can be written as (2.27). $\square$

3 Comparison

Define the sequences $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$, $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ by

$$H_n \approx \frac{1}{2}\ln(2m) + \gamma + \cfrac{\frac{1}{12}}{m + \frac{1}{10} + \cfrac{-\frac{19}{2100}}{m+\frac{91}{190}+\cfrac{-\frac{16585}{83391}}{m+\frac{2357167}{1638598}}}} = u_n, \tag{3.1}$$

$$H_n \approx \ln\left(n+\frac{1}{2}\right) + \gamma + \cfrac{\frac{1}{48}}{m + \frac{17}{80} + \cfrac{-\frac{2071}{134400}}{m+\frac{117863}{165680}+\cfrac{-\frac{15685119025}{63409182144}}{m+\frac{2312217133079747}{1351329470432240}}}} = v_n, \tag{3.2}$$

$$H_n \approx \frac{1}{2}\ln\left(2m + \frac{1}{3} + \cfrac{\alpha_1}{m+\beta_1+\cfrac{\alpha_2}{m+\beta_2+\cfrac{\alpha_3}{m+\beta_3+\ddots}}}\right) + \gamma = x_n, \tag{3.3}$$

$$H_n \approx \frac{1}{2}\ln\left(2m+\frac{1}{3}\right) + \gamma + \cfrac{p_1}{m^2 + \frac{37}{63}m + q_1 + \cfrac{p_2}{m+q_2+\cfrac{p_3}{m+q_3}}} = y_n, \tag{3.4}$$

where α_j and β_j (for $1 \le j \le 3$) are given in (2.28), p_j and q_j (for $1 \le j \le 3$) are given in (2.22).

It is observed from Table 1 that, among approximation formulas (3.1)-(3.4), for $n \in \mathbb{N}$, the formula (3.4) would be the best one.

Table 1. Comparison among approximation formulas (3.1)-(3.4).

n	$H_n - u_n$	$H_n - v_n$	$x_n - H_n$	$H_n - y_n$
1	4.61559×10^{-6}	1.25202×10^{-6}	5.1364×10^{-7}	3.75796×10^{-7}
10	9.65618×10^{-17}	5.65274×10^{-17}	1.62639×10^{-18}	7.04292×10^{-20}
100	1.98169×10^{-30}	1.19140×10^{-30}	3.98848×10^{-34}	2.01636×10^{-37}
1000	2.11271×10^{-44}	1.27055×10^{-44}	4.29495×10^{-50}	2.19258×10^{-55}

In fact, we have (by using the Maple software), as $n \to \infty$,

$$H_n = u_n + O(n^{-14}), \quad H_n = v_n + O(n^{-14}), \quad H_n = x_n + O(n^{-16}), \quad H_n = y_n + O(n^{-18}).$$

4 Conjecture

In view (1.1), (2.7), (2.19) and (2.30), we propose the following conjecture.

Conjecture 4.1. (i) Let a_ℓ ($\ell \in \mathbb{N}$) be given in (1.2). Then we have

$$(-1)^{\ell-1} a_\ell > 0, \qquad \ell \in \mathbb{N} \tag{4.1}$$

and

$$\sum_{\ell=1}^{2p} \frac{a_\ell}{m^\ell} < H_n - \frac{1}{2}\ln(2m) - \gamma < \sum_{\ell=1}^{2p+1} \frac{a_\ell}{m^\ell}, \tag{4.2}$$

where $m = n(n+1)/2$, $n \in \mathbb{N}$ and $p \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

(ii) Let r_ℓ ($\ell \in \mathbb{N}$) be given in (2.8). Then we have

$$(-1)^{\ell-1} r_\ell > 0, \qquad \ell \in \mathbb{N} \tag{4.3}$$

and

$$\sum_{\ell=1}^{2p} \frac{r_\ell}{m^\ell} < H_n - \ln\left(n + \frac{1}{2}\right) - \gamma < \sum_{\ell=1}^{2p+1} \frac{r_\ell}{m^\ell}, \tag{4.4}$$

where $m = n(n+1)/2$, $n \in \mathbb{N}$ and $p \in \mathbb{N}_0$.

(iii) Let s_ℓ ($\ell \geq 2$) be given in (2.20). Then we have

$$(-1)^{\ell-1} s_\ell > 0, \qquad \ell \geq 2 \tag{4.5}$$

and

$$\sum_{\ell=2}^{2q} \frac{s_\ell}{m^\ell} < H_n - \frac{1}{2}\ln\left(2m + \frac{1}{3}\right) - \gamma < \sum_{\ell=2}^{2q+1} \frac{s_\ell}{m^\ell}, \tag{4.6}$$

where $m = n(n+1)/2$, $n \in \mathbb{N}$ and $q \in \mathbb{N}$.

(iv) Let d_ℓ ($\ell \in \mathbb{N}$) be given in (2.29). Then we have

$$(-1)^{\ell} d_\ell > 0, \qquad \ell \geq 1 \tag{4.7}$$

and

$$\frac{1}{2}\ln\left(2m + \frac{1}{3} + \sum_{\ell=1}^{2q-1} \frac{\omega_\ell}{m^\ell}\right) < H_n - \gamma < \frac{1}{2}\ln\left(2m + \frac{1}{3} + \sum_{\ell=1}^{2q} \frac{\omega_\ell}{m^\ell}\right), \tag{4.8}$$

where $m = n(n+1)/2$, $n \in \mathbb{N}$ and $q \in \mathbb{N}$.

References

[1] B.C. Berndt, Ramanujan's Notebooks Part V. Springer, Berlin, 1998.

[2] C.-P. Chen, On the coefficients of asymptotic expansion for the harmonic number by Ramanujan, Ramanujan J. 40 (2016) 279–290.

[3] C.-P. Chen, Inequalities and asymptotics for the Euler–Mascheroni constant based on DeTemple's result, Numer. Algor. 73 (2016) 761–774.

[4] C.-P. Chen, Ramanujan's formula for the harmonic number, Appl. Math. Comput. 317 (2018) 121–128.

[5] C.-P. Chen, On the Ramanujan harmonic number expansion, Results Math. 74 (2019) 1–7.

[6] C.-P. Chen, J.-X. Cheng, Ramanujan's asymptotic expansion for the harmonic numbers, Ramanujan J. 38 (2015) 123–128.

[7] L. Feng, W. Wang, Riordan array approach to the coefficients of Ramanujan's harmonic Number expansion, Results Math. 71 (2017) 1413–1419.

[8] M.D. Hirschhorn, Ramanujan's enigmatic formula for the harmonic series, Ramanujan J. 27 (2012) 343–347.

[9] A. Issaka, An asymptotic series related to Ramanujan's expansion for the nth Harmonic number, Ramanujan J. 39 (2016) 303–313.

[10] C. Mortici, On the Ramanujan-Lodge harmonic number expansion, Appl. Math. Comput. 251 (2015) 423–430 .

[11] S. Ramanujan, Notebook II, Narosa, New Delhi, 1988.

[12] M.B. Villarino, Ramanujan's harmonic number expansion into negative powers of a triangular number, J. Inequal. Pure Appl. Math. 9 (3) (2008) Article 89. `http://www.emis.de/journals/JIPAM/images/245_07_JIPAM/245_07.pdf`.

[13] A. Xu, Ramanujan's harmonic number expansion and two identities for Bernoulli numbers, Results Math. 72 (2017) 1857–1864.