Asymptotic expansions and continued fraction approximations related to the constant e

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Abstract. In this paper, we introduce a method to construct a continued fraction approximation based on a given asymptotic expansion. We establish some asymptotic expansions related to the constant e. Based on these expansions, we derive the corresponding continued fraction approximations related to the constant e.

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1 Introduction

The constant e can be defined by the limit

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.$$

With the possible exception of π, e is the most important constant in mathematics since it appears in myriad mathematical contexts involving limits and derivatives. Joost Bürgi seems to have been the first to formulate an approximation to e around 1620, obtaining three-decimal-place accuracy (see [12, p. 31], [19], and [26, pp. 26–27]).

There have been many results in generalizing Carleman-type inequality by estimating $(1+1/n)^n$ (see, for example, [3–5, 9, 11, 13, 21–24, 29, 31, 35–40]). For example, Xie and Zhong [35] proved that, for $x \ge 1$,

$$e\left(1-\frac{7}{14x+12}\right) < \left(1+\frac{1}{x}\right)^x < e\left(1-\frac{6}{12x+11}\right), \tag{1.1}$$

and then applied it to obtain an improvement of Carleman-type inequality. For information about the history of Carleman-type inequalities, see [17, 18, 20, 32].

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Some asymptotic expansions for $(1+1/x)^x$ (as $x \to \infty$) can be found in [2, 3, 6, 8, 16, 19, 23, 30, 34, 38, 39]. For example, Brothers and Knox [2] (see also [6, 19] and [15, p. 14]) derived, without a formula for the general term, the following expansion:

$$\left(1+\frac{1}{x}\right)^x = e\left(1 - \frac{1}{2x} + \frac{11}{24x^2} - \frac{7}{16x^3} + \frac{2447}{5760x^4} - \frac{959}{2304x^5} + \frac{238043}{580608x^6} - \cdots\right) \tag{1.2}$$

for $x < -1$ or $x \geq 1$.

Adding approximation (1.2) and the approximation obtained by replacing x by $-x$ in (1.2), and multiplying the resulting identity by $1/2$, Knox and Brothers [19] (see also [2]) obtained the following better approximation to e than that given by (1.2):

$$\frac{1}{2}\left(\left(1+\frac{1}{x}\right)^x + \left(1-\frac{1}{x}\right)^{-x}\right) = e\left(1 + \frac{11}{24x^2} + \frac{2447}{5760x^4} + \frac{238043}{580608x^6} + \cdots\right). \tag{1.3}$$

With

$$\left(1+\frac{1}{x}\right)^x = e\sum_{j=0}^{\infty} \frac{\omega_j}{x^j}, \qquad x < -1 \quad \text{or} \quad x \geq 1, \tag{1.4}$$

Chen and Choi [6] gave an explicit formula for successively determining the coefficients ω_j in the form

$$\omega_0 = 1, \quad \omega_j = (-1)^j \sum_{k_1+2k_2+\cdots+jk_j=j} \frac{\left(\frac{1}{2}\right)^{k_1}\left(\frac{1}{3}\right)^{k_2}\cdots\left(\frac{1}{j+1}\right)^{k_j}}{k_1!k_2!\cdots k_j!}, \tag{1.5}$$

summed over all nonnegative integers k_j satisfying the equation $k_1 + 2k_2 + \cdots + jk_j = j$. The above result immediately shows that $(-1)^j\omega_j > 0$ so that (1.4) is an alternating series for positive x. The following recurrence relation for $\theta_j = (-1)^j\omega_j$ can be found in [8]:

$$\theta_0 = 1 \quad \text{and} \quad \theta_j = \frac{1}{j}\sum_{k=1}^{j} \frac{k}{k+1}\theta_{j-k} \quad \text{for} \quad j \geq 1. \tag{1.6}$$

The representation using a recursive algorithm for the coefficients $(-1)^j\theta_j = \omega_j$ in (1.6) is more practical for numerical evaluation than the expression in (1.5).

By using (1.4), we find, as $x \to \infty$,

$$\frac{1}{2}\left(\left(1+\frac{1}{x}\right)^x + \left(1-\frac{1}{x}\right)^{-x}\right) = e\sum_{j=0}^{\infty} \frac{(1+(-1)^j)\omega_j}{2x^j} = e\sum_{j=0}^{\infty} \frac{\omega_{2j}}{x^{2j}}, \tag{1.7}$$

where the ω_j (for $j \in \mathbb{N}_0$) are given in (1.5).

The constant e is given in [2] (see also [15, p. 15]) by the unusual limit

$$\lim_{n\to\infty}\left(\frac{(n+1)^{n+1}}{n^n} - \frac{n^n}{(n-1)^{n-1}}\right) = e. \tag{1.8}$$

By using (1.4), we find the following asymptotic expansion:

$$\frac{(n+1)^{n+1}}{n^n} - \frac{n^n}{(n-1)^{n-1}} = (1+n)\left(1+\frac{1}{n}\right)^n + (1-n)\left(1-\frac{1}{n}\right)^{-n} = e\sum_{j=0}^{\infty}\frac{2(\omega_{2j}+\omega_{2j+1})}{n^{2j}} \tag{1.9}$$

as $n\to\infty$, where the ω_j (for $j\in\mathbb{N}_0$) are given in (1.5).

Some continued fraction approximations related to the constant e were presented in [14, 24, 25, 41]. For example, You et al. [41] proved, as $x\to\infty$,

$$\left(1+\frac{1}{x}\right)^x \approx e\left(1+\cfrac{-\frac{1}{2}}{x+\frac{11}{12}+\cfrac{-\frac{5}{144}}{x+\frac{34}{75}+\cfrac{-\frac{481}{10000}}{x+\frac{357866}{757575}+\ddots}}}\right). \tag{1.10}$$

Lu et al. [25] presented, as $x\to\infty$,

$$\left(1+\frac{1}{x}\right)^x \approx \exp\left(1+\cfrac{-\frac{1}{2}}{x+\frac{2}{3}+\cfrac{-\frac{1}{18}}{x+\frac{8}{15}+\cfrac{-\frac{3}{50}}{x+\frac{18}{35}+\ddots}}}\right). \tag{1.11}$$

Lu et al. [24] presented, as $n\to\infty$,

$$\frac{1}{e}\left(1+\frac{1}{n}\right)^n \approx 1+\cfrac{b_1}{n+\cfrac{b_2 n}{n+\cfrac{b_3 n}{n+\cfrac{b_4 n}{n+\cfrac{b_5 n}{n+\cfrac{b_6 n}{n+\ddots}}}}}}, \tag{1.12}$$

where

$$b_1=-\frac{1}{2},\quad b_2=\frac{11}{12},\quad b_3=\frac{5}{132},\quad b_4=\frac{457}{1100},\quad b_5=\frac{5291}{45700},\quad b_6=\frac{19753835}{55393884},\quad \ldots.$$

Fang et al. [14] presented, as $n\to\infty$,

$$\frac{1}{e}\left(1+\frac{1}{n}\right)^n \approx \exp\left(\cfrac{a_1}{n+\cfrac{a_2 n}{n+\cfrac{a_3 n}{n+\cfrac{a_4 n}{n+\cfrac{a_5 n}{n+\ddots}}}}}\right), \tag{1.13}$$

where

$$a_1 = -\frac{1}{2}, \quad a_2 = \frac{2}{3}, \quad a_3 = \frac{1}{12}, \quad a_4 = \frac{9}{20}, \quad a_5 = \frac{2}{15}, \quad a_6 = \frac{8}{21}, \quad \ldots.$$

Very recently, Chen and Wang [10, Corollary 1.1] provided a method to construct a continued fraction approximation based on a given asymptotic expansion. We state this method as follows. Let $a_1 \neq 0$ and

$$f(x) \sim \sum_{j=1}^{\infty} \frac{a_j}{x^j}, \qquad x \to \infty \tag{1.14}$$

be a given asymptotic expansion. Then the function f has the following continued fraction approximation of the form

$$f(x) \approx \cfrac{a_1}{x + b_0 + \cfrac{b_1}{x + c_0 + \cfrac{c_1}{x + d_0 + \ddots}}}, \qquad x \to \infty, \tag{1.15}$$

where the constants in the right-hand side of (1.15) are given by the following recurrence relations:

$$\begin{cases} b_0 = -\frac{a_2}{a_1}, \quad b_j = -\frac{1}{a_1}\left(a_{j+2} + \sum_{k=1}^{j} a_{k+1} b_{j-k}\right) \\ c_0 = -\frac{b_2}{b_1}, \quad c_j = -\frac{1}{b_1}\left(b_{j+2} + \sum_{k=1}^{j} b_{k+1} c_{j-k}\right) \\ d_0 = -\frac{c_2}{c_1}, \quad d_j = -\frac{1}{c_1}\left(c_{j+2} + \sum_{k=1}^{j} c_{k+1} d_{j-k}\right) \\ \qquad \cdots \quad \cdots \end{cases} \tag{1.16}$$

Clearly, $a_j \Longrightarrow b_j \Longrightarrow c_j \Longrightarrow d_j \Longrightarrow \ldots$. Thus, the asymptotic expansion (1.14) $\Longrightarrow$ the continued fraction approximation (1.15).

Based on Ramanujan's asymptotic expansion for the nth harmonic number (see [1, p. 531] and [33, p. 276])

$$\begin{aligned} H_n := \sum_{k=1}^{n} \frac{1}{k} \sim \frac{1}{2}\ln(2m) + \gamma + \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \frac{1}{2310m^5} \\ - \frac{191}{360360m^6} + \frac{29}{30030m^7} - \frac{2833}{1166880m^8} + \frac{140051}{17459442m^9} - \cdots, \quad n \to \infty \end{aligned} \tag{1.17}$$

and using (1.16), Chen and Wang [10, Theorem 2.1] derived the following continued fraction approximation for the nth harmonic number:

$$H_n \approx \frac{1}{2}\ln(2m) + \gamma + \cfrac{\frac{1}{12}}{m + \frac{1}{10} + \cfrac{-\frac{19}{2100}}{m + \frac{91}{190} + \cfrac{-\frac{16585}{83391}}{m + \frac{2357167}{1638598} + \ddots}}}, \qquad n \to \infty, \tag{1.18}$$

where $m = n(n+1)/2$ is the nth triangular number and γ is the Euler-Mascheroni constant.

Remark 1.1. *It is easy to see that*

$$x\ln\left(1+\frac{1}{x}\right)-1=\sum_{j=1}^{\infty}\frac{(-1)^j}{(j+1)x^j},\quad |x|\geq 1 \quad \textit{and} \quad x\neq -1. \tag{1.19}$$

Based on expansion (1.19) *and using* (1.16)*, we derive*

$$x\ln\left(1+\frac{1}{x}\right)-1\approx \cfrac{-\frac{1}{2}}{x+\frac{2}{3}+\cfrac{-\frac{1}{18}}{x+\frac{8}{15}+\cfrac{-\frac{3}{50}}{x+\frac{18}{35}+\ddots}}},\qquad x\to\infty,$$

which can be written as (1.11)*. Thus we give a different derivation of* (1.11) *from that in [25]. Based on expansion* (1.2) *and using* (1.16)*, we can easily derive the continued fraction approximation* (1.10)*.*

All results of the present paper are motivated by the paper [10]. We here present some asymptotic expansions related to the constant e. Based on these expansions, we derive the corresponding continued fraction approximations related to the constant e.

2 Lemmas

The following lemmas will be useful in pour present investigation.

Lemma 2.1 (see [7])**.** *Let*

$$A(x)\sim\sum_{n=1}^{\infty}a_nx^{-n},\qquad x\to\infty$$

be a given asymptotical expansion. Then the composition $\exp(A(x))$ *has asymptotic expansion of the following form*

$$\exp(A(x))\sim\sum_{n=0}^{\infty}b_nx^{-n},\qquad x\to\infty, \tag{2.1}$$

where

$$b_0=1,\quad b_n=\frac{1}{n}\sum_{k=1}^{n}ka_kb_{n-k},\qquad n\geq 1. \tag{2.2}$$

Lemma 2.2 (see [7])**.** *Let* $a_0=1$ *and*

$$g(x)\sim\sum_{n=0}^{\infty}a_nx^{-n},\qquad x\to\infty$$

be a given asymptotic expansion. Then the composition $\ln(g(x))$ *has asymptotic expansion of the following form*

$$\ln(g(x))\sim\sum_{n=1}^{\infty}b_nx^{-n},\qquad x\to\infty,$$

where

$$b_n = a_n - \frac{1}{n}\sum_{k=1}^{n-1} k b_k a_{n-k}, \qquad n \in \mathbb{N}. \tag{2.3}$$

3 Main results

Using the Maple software, we find, as $n \to \infty$,

$$\begin{aligned}\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} \sim e\bigg(1 &+ \frac{1}{24m} - \frac{7}{5760m^2} + \frac{43}{580608m^3} - \frac{7961}{1393459200m^4} + \frac{182521}{367873228800m^5} \\ &- \frac{1115593093}{24103053950976000m^6} + \frac{2620419701}{578473294823424000m^7} \\ &- \frac{333235214791}{7262064747321753600000m^8} + \frac{12937676612987993}{27121197487937713864704000m^9} - \dots\bigg)\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} \sim \exp\bigg(1 &+ \frac{1}{24m} - \frac{1}{480m^2} + \frac{1}{6720m^3} - \frac{1}{80640m^4} + \frac{1}{887040m^5} - \frac{1}{9225216m^6} \\ &+ \frac{1}{92252160m^7} - \frac{1}{896163840m^8} + \frac{1}{8513556480m^9} - \dots\bigg),\end{aligned} \tag{3.2}$$

which are analogues of (1.17).

Even though as many coefficients as we please in the right-hand sides of (3.1) and (3.2) can be obtained by using the Maple software, here we aim at giving a formula for determining these coefficients. And then, based on the obtained expansions (3.1) and (3.2) and using (1.16), we derive the corresponding continued fraction approximations related to the constant e.

Theorem 3.1. *Let $m = \frac{1}{2}n(n+1)$. As $n \to \infty$, we have*

$$\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} \sim e\sum_{\ell=0}^{\infty} \frac{a_\ell}{m^\ell}, \tag{3.3}$$

with the coefficients a_ℓ ($\ell \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \mathbb{N} := \{1, 2, \dots\}$) given by the recurrence relation

$$a_0 = 1, \quad a_\ell = \frac{1}{2^\ell}\left(q_{2\ell} - \sum_{j=0}^{\ell-1} 2^j a_j \binom{2\ell-j-1}{2\ell-2j}\right), \qquad \ell \geq 1, \tag{3.4}$$

where q_ℓ ($\ell \in \mathbb{N}_0$) can be calculated by

$$q_0 = 1, \quad q_\ell = \frac{1}{\ell}\sum_{k=1}^{\ell} \frac{(-1)^k (k-1)}{2(k+1)} q_{\ell-k}, \qquad \ell \geq 1. \tag{3.5}$$

Proof. Denote

$$I_n = \frac{1}{e}\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} \quad \text{and} \quad J_n = \sum_{\ell=0}^{\infty} \frac{a_\ell}{m^\ell}.$$

In view of (3.1), we can let $I_n \sim J_n$ ($n \to \infty$), where a_ℓ ($\ell \in \mathbb{N}_0$) are real numbers to be determined.

Direct computation yields

$$\ln I_n = \left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{n}\right) - 1 = \left(n+\frac{1}{2}\right)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{kn^k} - 1 = \sum_{k=1}^{\infty}\left\{\frac{(-1)^k(k-1)}{2k(k+1)}\right\}\frac{1}{n^k}.$$

We obtain by Lemma 2.1 that

$$I_n = \sum_{\ell=0}^{\infty}\frac{q_\ell}{n^\ell}, \tag{3.6}$$

where

$$q_0 = 1, \quad q_\ell = \frac{1}{\ell}\sum_{k=1}^{\ell}\frac{(-1)^k(k-1)}{2(k+1)}q_{\ell-k}, \qquad \ell \ge 1.$$

Direct computation yields

$$\begin{aligned} J_n = \sum_{k=0}^{\infty}\frac{2^k a_k}{n^{2k}}\left(1+\frac{1}{n}\right)^{-k} &= \sum_{k=0}^{\infty}\frac{2^k a_k}{n^{2k}}\sum_{j=0}^{\infty}\binom{-k}{j}\frac{1}{n^j} \\ &= \sum_{k=0}^{\infty}\frac{2^k a_k}{n^{2k}}\sum_{j=0}^{\infty}(-1)^j\binom{k+j-1}{j}\frac{1}{n^j} \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k}2^j a_j(-1)^{k-j}\binom{k-1}{k-j}\frac{1}{(n+1)^{k+j}} \\ &= \sum_{\ell=0}^{\infty}\sum_{j=0}^{\lfloor\frac{\ell}{2}\rfloor}2^j a_j(-1)^\ell\binom{\ell-j-1}{\ell-2j}\frac{1}{n^\ell}. \end{aligned} \tag{3.7}$$

Equating coefficients of the term $n^{-\ell}$ on the right-hand sides of (3.6) and (3.7) yields

$$\sum_{j=0}^{\lfloor\frac{\ell}{2}\rfloor}2^j a_j(-1)^\ell\binom{\ell-j-1}{\ell-2j} = q_\ell, \qquad \ell \ge 0. \tag{3.8}$$

Replacement of ℓ by 2ℓ in (3.8) yields

$$\sum_{j=0}^{\ell}2^j a_j\binom{2\ell-j-1}{2\ell-2j} = q_{2\ell}, \qquad \ell \ge 0. \tag{3.9}$$

For $\ell = 0$ in (3.9) we obtain $a_0 = q_0 = 1$, and for $\ell \geq 1$ we have

$$\sum_{j=0}^{\ell-1} 2^j a_j \binom{2\ell - j - 1}{2\ell - 2j} + 2^\ell a_\ell = q_{2\ell},$$

which gives the desired formula (3.4). □

Remark 3.1. *Replacement of ℓ by $2\ell + 1$ in (3.8) yields*

$$-\sum_{j=0}^{\ell} 2^j a_j \binom{2\ell - j}{2\ell - 2j + 1} = q_{2\ell+1}, \qquad \ell \geq 0. \tag{3.10}$$

For $\ell = 1$ in (3.10) this yields $a_1 = -\frac{q_3}{2} = \frac{1}{24}$, and for $\ell \geq 2$ we have

$$-\sum_{j=0}^{\ell-1} 2^j a_j \binom{2\ell - j}{2\ell - 2j + 1} - 2^\ell \ell a_\ell = q_{2\ell+1}, \qquad \ell \geq 2.$$

We then obtain the alternative recurrence relation for the coefficients a_ℓ in (3.3) in terms of the odd coefficients q_ℓ:

$$a_0 = 1, \quad a_1 = \frac{1}{24}, \quad a_\ell = -\frac{1}{2^\ell \ell}\left(q_{2\ell+1} + \sum_{j=0}^{\ell-1} 2^j a_j \binom{2\ell - j}{2\ell - 2j + 1}\right), \qquad \ell \geq 2. \tag{3.11}$$

Theorem 3.2. *Let $m = \frac{1}{2}n(n+1)$. As $n \to \infty$, we have*

$$\left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} \approx e \left(1 + \cfrac{a_1}{m + b_0 + \cfrac{b_1}{m + c_0 + \cfrac{c_1}{m + d_0 + \ddots}}}\right), \tag{3.12}$$

where

$$a_1 = \frac{1}{24}, \quad b_0 = \frac{7}{240}, \quad b_1 = -\frac{1121}{1209600}, \quad c_0 = \frac{1409}{22420},$$
$$c_1 = -\frac{651087191}{668814499584}, \quad d_0 = \frac{11867426245291}{189765872688860}, \quad \ldots.$$

Proof. Denote

$$F_1(m) = \frac{1}{e}\left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} - 1.$$

It follows from (3.3) that

$$
\begin{aligned}
F_1(m) &\sim \sum_{\ell=1}^{\infty} \frac{a_\ell}{m^\ell} \\
&= \frac{1}{24m} - \frac{7}{5760m^2} + \frac{43}{580608m^3} - \frac{7961}{1393459200m^4} + \frac{182521}{367873228800m^5} \\
&\quad - \frac{1115593093}{24103053950976000m^6} + \frac{2620419701}{578473294823424000m^7} \\
&\quad - \frac{333235214791}{726206474732175360000m^8} + \frac{12937676612987993}{271211974879377138647040000m^9} - \dots
\end{aligned}
\tag{3.13}
$$

as $m \to \infty$, where the coefficients a_ℓ ($\ell \in \mathbb{N}$) are given in (3.4). Then, $F_1(m)$ has the continued fraction approximation of the form

$$
F_1(m) \approx \cfrac{a_1}{m + b_0 + \cfrac{b_1}{m + c_0 + \cfrac{c_1}{m + d_0 + \ddots}}}, \qquad m \to \infty,
\tag{3.14}
$$

where the constants in the right-hand side of (3.14) can be determined using (1.16). Noting that

$$
\begin{aligned}
&a_1 = \frac{1}{24}, \quad a_2 = -\frac{7}{5760}, \quad a_3 = \frac{43}{580608}, \quad a_4 = -\frac{7961}{1393459200}, \\
&a_5 = \frac{182521}{367873228800}, \quad a_6 = -\frac{1115593093}{24103053950976000}, \quad \dots,
\end{aligned}
$$

we obtain from the first recurrence relation in (1.16) that

$$
\begin{aligned}
b_0 &= -\frac{a_2}{a_1} = \frac{7}{240}, \\
b_1 &= -\frac{a_3 + a_2 b_0}{a_1} = -\frac{1121}{1209600}, \\
b_2 &= -\frac{a_4 + a_2 b_1 + a_3 b_0}{a_1} = \frac{1409}{24192000}, \\
b_3 &= -\frac{a_5 + a_2 b_2 + a_3 b_1 + a_4 b_0}{a_1} = -\frac{73430479}{16094453760000}, \\
b_4 &= -\frac{a_6 + a_2 b_3 + a_3 b_2 + a_4 b_1 + a_5 b_0}{a_1} = \frac{557732611}{1394852659200000}, \quad \dots.
\end{aligned}
$$

We obtain from the second recurrence relation in (1.16) that

$$
\begin{aligned}
c_0 &= -\frac{b_2}{b_1} = \frac{1409}{22420}, \\
c_1 &= -\frac{b_3 + b_2 c_0}{b_1} = -\frac{651087191}{668814499584}, \\
c_2 &= -\frac{b_4 + b_2 c_1 + b_3 c_0}{b_1} = \frac{11867426245291}{194932674048752640}, \quad \dots.
\end{aligned}
$$

Continuing the above process, we find

$$d_0 = -\frac{c_2}{c_1} = \frac{11867426245291}{189765872688860}, \quad \dots .$$

Formula (3.14) can be written as (3.12). The proof is complete. $\square$

Theorem 3.3. *Let $m = \frac{1}{2}n(n+1)$. As $n \to \infty$, we have*

$$\left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} \sim \exp\left(1 + \sum_{\ell=1}^{\infty} \frac{d_\ell}{m^\ell}\right), \tag{3.15}$$

with the coefficients d_ℓ ($\ell \in \mathbb{N}$) given by the recurrence relation

$$d_\ell = a_\ell - \frac{1}{\ell} \sum_{k=1}^{\ell-1} k d_k a_{\ell-k}, \qquad \ell \in \mathbb{N}. \tag{3.16}$$

where the coefficients a_ℓ are given in (3.4).

Proof. By Lemma 2.2, we obtain from (3.3) that

$$\left(n + \frac{1}{2}\right) \ln\left(1 + \frac{1}{n}\right) - 1 \sim \ln\left(\sum_{\ell=0}^{\infty} \frac{a_\ell}{m^\ell}\right) \sim \sum_{\ell=1}^{\infty} \frac{d_\ell}{m^\ell}, \tag{3.17}$$

where

$$d_\ell = a_\ell - \frac{1}{\ell} \sum_{k=1}^{\ell-1} k d_k a_{\ell-k}, \qquad \ell \in \mathbb{N},$$

and a_ℓ are given in (3.4). Formula (3.17) can be written as (3.15). $\square$

Based on the asymptotic expansion (3.15) and using (1.16), we derive another continued fraction approximation for the sequence $\left\{\left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}}\right\}_{n \geq 1}$ asserted by Theorem 3.4. We here omit the calculations of the constants α_j and β_j in the right-hand side of (3.18).

Theorem 3.4. *Let $m = \frac{1}{2}n(n+1)$. As $n \to \infty$, we have*

$$\left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} \approx \exp\left(1 + \cfrac{\alpha_1}{m + \beta_1 + \cfrac{\alpha_2}{m + \beta_2 + \cfrac{\alpha_3}{m + \beta_3 + \ddots}}}\right), \tag{3.18}$$

where

$$\alpha_1 = \frac{1}{24}, \quad \beta_1 = \frac{1}{20}, \quad \alpha_2 = -\frac{3}{2800}, \quad \beta_2 = \frac{11}{180}, \quad \alpha_3 = -\frac{25}{24948}, \quad \beta_3 = \frac{29}{468}, \quad \dots .$$

Remark 3.2. *Write* (1.9) *as*

$$\begin{aligned}
&\frac{(n+1)^{n+1}}{n^n}-\frac{n^n}{(n-1)^{n-1}}=e\sum_{j=0}^{\infty}\frac{b_j}{p^j}\\
&=e\bigg(1+\frac{1}{24p}+\frac{11}{640p^2}+\frac{5525}{580608p^3}+\frac{1212281}{199065600p^4}+\frac{772193}{181665792p^5}\\
&\quad+\frac{6889178449747}{2191186722816000p^6}+\frac{107876982981287}{44497945755648000p^7}+\frac{6225541612992329}{3227584332143001600p^8}+\cdots\bigg),
\end{aligned}\tag{3.19}$$

with the coefficients b_j (for $j\in\mathbb{N}_0$) given by

$$b_j=2(\omega_{2j}+\omega_{2j+1}),\tag{3.20}$$

where the ω_j (for $j\in\mathbb{N}_0$) are given in (1.5)*, and $p=n^2$ is the nth quadrangular number.*

Based on the asymptotic expansion (3.19) *and using* (1.16)*, we derive the following continued fraction approximation:*

$$\frac{(n+1)^{n+1}}{n^n}-\frac{n^n}{(n-1)^{n-1}}\approx e\left(1+\cfrac{\lambda_1}{p+\mu_1+\cfrac{\lambda_2}{p+\mu_2+\cfrac{\lambda_3}{p+\mu_3+\ddots}}}\right)\tag{3.21}$$

as $n\to\infty$, where

$$\lambda_1=\frac{1}{24},\quad \mu_1=-\frac{33}{80},\quad \lambda_2=-\frac{70429}{1209600},\quad \mu_2=-\frac{4054307}{8451480},$$
$$\lambda_3=-\frac{159009926405791}{2639960924477184},\quad \mu_3=-\frac{8570632118726927402873}{17470299766660188768840},\quad \ldots.$$

Remark 3.3. *Based on the asymptotic expansion* (1.7) *and using* (1.16)*, we derive the following continued fraction approximation:*

$$\frac{1}{2}\left(\left(1+\frac{1}{n}\right)^n+\left(1-\frac{1}{n}\right)^{-n}\right)\approx e\left(1+\cfrac{\tau_1}{p+\nu_1+\cfrac{\tau_2}{p+\nu_2+\cfrac{\tau_3}{p+\nu_3+\ddots}}}\right)\tag{3.22}$$

as $n\to\infty$, where

$$\tau_1=\frac{11}{24},\quad \nu_1=-\frac{2447}{2640},\quad \tau_2=-\frac{5179661}{146361600},\quad \nu_2=-\frac{902753063}{2279050840},$$
$$\tau_3=-\frac{61929377534266549}{1298088920616977664},\quad \nu_3=-\frac{75130865553803396336002597}{166802054415628635879142280},\quad \ldots,$$

and $p=n^2$ is the nth quadrangular number.

4 Comparison

Using the Maple software, we find from (1.10), (1.11), (3.21), (3.22), (3.12) and (3.18) that, as $n \to \infty$,

$$\left(1+\frac{1}{n}\right)^n \left(1+\cfrac{-\frac{1}{2}}{n+\frac{11}{12}+\cfrac{-\frac{5}{144}}{n+\frac{34}{75}+\cfrac{-\frac{481}{10000}}{n+\frac{357866}{757575}}}}\right)^{-1} = e + O\left(\frac{1}{n^7}\right), \tag{4.1}$$

$$\frac{\left(1+\frac{1}{n}\right)^n}{\exp\left(\cfrac{-\frac{1}{2}}{n+\frac{2}{3}+\cfrac{-\frac{1}{18}}{n+\frac{8}{15}+\cfrac{-\frac{3}{50}}{n+\frac{18}{35}}}}\right)} = e + O\left(\frac{1}{n^7}\right), \tag{4.2}$$

$$u_n := \left(\frac{(n+1)^{n+1}}{n^n} - \frac{n^n}{(n-1)^{n-1}}\right)\left(1+\cfrac{\lambda_1}{p+\mu_1+\cfrac{\lambda_2}{p+\mu_2+\cfrac{\lambda_3}{p+\mu_3+\ddots}}}\right)^{-1} = e + O\left(\frac{1}{n^{14}}\right), \tag{4.3}$$

$$v_n := \frac{1}{2}\left(\left(1+\frac{1}{n}\right)^n + \left(1-\frac{1}{n}\right)^{-n}\right)\left(1+\cfrac{\tau_1}{p+\nu_1+\cfrac{\tau_2}{p+\nu_2+\cfrac{\tau_3}{p+\nu_3+\ddots}}}\right)^{-1} = e + O\left(\frac{1}{n^{14}}\right), \tag{4.4}$$

$$x_n := \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}\left(1+\cfrac{\frac{1}{24}}{m+\frac{7}{240}+\cfrac{-\frac{1121}{1209600}}{m+\frac{1409}{22420}+\cfrac{-\frac{651087191}{668814499584}}{m+\frac{11867426245291}{189765872688860}}}}\right)^{-1} = e + O\left(\frac{1}{n^{14}}\right), \tag{4.5}$$

$$y_n := \frac{\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}}{\exp\left(\cfrac{\frac{1}{24}}{m+\frac{1}{20}+\cfrac{-\frac{3}{2800}}{m+\frac{11}{180}+\cfrac{-\frac{25}{24948}}{m+\frac{29}{468}}}}\right)} = e + O\left(\frac{1}{n^{14}}\right). \tag{4.6}$$

Clearly, the approximation formulas (4.3)-(4.6) are much stronger than (4.1) and (4.2). The following numerical computations (see Table 1) would show that for $n \geq 2$, the formula (4.5) would be the best one.

Table 1. Comparison among approximation formulas (4.3)-(4.6).

n	$v_n - e$	$u_n - e$	$y_n - e$	$x_n - e$
2	1.85697×10^{-8}	3.59161×10^{-9}	4.80604×10^{-14}	3.98141×10^{-14}
10	1.17920×10^{-18}	2.50982×10^{-19}	7.83678×10^{-23}	6.50190×10^{-23}
100	1.13822×10^{-32}	2.43084×10^{-33}	1.43490×10^{-36}	1.19059×10^{-36}
1000	1.13782×10^{-46}	2.43007×10^{-47}	1.52781×10^{-50}	1.26768×10^{-50}

5 Conjecture

In view (3.1) and (3.2), we propose the following conjecture.

Conjecture 5.1. (i) Let a_ℓ ($\ell \in \mathbb{N}$) be given in (3.4). Then we have

$$(-1)^{\ell-1} a_\ell > 0, \qquad \ell \in \mathbb{N} \tag{5.1}$$

and

$$e\left(1 + \sum_{\ell=1}^{2q} \frac{a_\ell}{m^\ell}\right) < \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} < e\left(1 + \sum_{\ell=1}^{2q+1} \frac{a_\ell}{m^\ell}\right), \tag{5.2}$$

where $m = n(n+1)/2$, $n \in \mathbb{N}$ and $q \in \mathbb{N}_0$.

(ii) Let d_ℓ ($\ell \in \mathbb{N}$) be given in (3.16). Then we have

$$(-1)^{\ell-1} d_\ell > 0, \qquad \ell \in \mathbb{N} \tag{5.3}$$

and

$$\exp\left(1 + \sum_{\ell=1}^{2q} \frac{d_\ell}{m^\ell}\right) < \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} < \exp\left(1 + \sum_{\ell=1}^{2q+1} \frac{d_\ell}{m^\ell}\right), \tag{5.4}$$

where $m = n(n+1)/2$, $n \in \mathbb{N}$ and $q \in \mathbb{N}_0$.

(iii) Let $m = n(n+1)/2$. Then for all $n \in \mathbb{N}$ and $q \in \mathbb{N}_0$,

$$\begin{aligned}\exp\left(1 + \sum_{\ell=1}^{2q} \frac{d_\ell}{m^\ell}\right) < e\left(1 + \sum_{\ell=1}^{2q} \frac{a_\ell}{m^\ell}\right) < \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} \\ < e\left(1 + \sum_{\ell=1}^{2q+1} \frac{a_\ell}{m^\ell}\right) < \exp\left(1 + \sum_{\ell=1}^{2q+1} \frac{d_\ell}{m^\ell}\right).\end{aligned} \tag{5.5}$$

This means that double inequality (5.2) is sharper than (5.4).

6 New derivations of (1.12) and (1.13)

Using a lemma of Mortic [27, 28], Lu et al. [24] proved (1.12), and Fang et al. [14] obtained (1.13). However, these authors did not give a formula for determining the constants in the right-hand sides of (1.12) and (1.13). By using system (6.6) below, we here give new derivations of (1.12) and (1.13). To this end, we first establish the following lemma.

Lemma 6.1. *Let $a_1 \neq 0$ and*

$$A(x) \sim \sum_{j=1}^{\infty} \frac{a_j}{x^j}, \qquad x \to \infty$$

be a given asymptotic expansion. Define the function g by

$$A(x) = \frac{a_1}{x + xB(x)}.$$

Then the function $B(x) = \frac{a_1}{xA(x)} - 1$ has asymptotic expansion of the following form

$$B(x) \sim \sum_{j=1}^{\infty} \frac{b_j}{x^j}, \qquad x \to \infty,$$

where

$$b_1 = -\frac{a_2}{a_1}, \quad b_j = -\frac{1}{a_1}\left(a_{j+1} + \sum_{k=2}^{j} a_k b_{j-k+1}\right), \qquad j \geq 2. \tag{6.1}$$

Proof. We can let

$$\frac{a_1}{A(x)} \sim x + x\sum_{k=1}^{\infty} \frac{b_k}{x^k}, \qquad x \to \infty, \tag{6.2}$$

where b_k (for $k \in \mathbb{N}$) are real numbers to be determined. Write (6.2) as

$$\sum_{j=1}^{\infty} \frac{a_j}{x^j}\left(1 + \sum_{k=1}^{\infty} \frac{b_k}{x^k}\right) \sim \frac{a_1}{x},$$

$$-\sum_{j=2}^{\infty} \frac{a_j}{x^j} \sim \sum_{j=1}^{\infty} \frac{a_j}{x^j} \sum_{k=1}^{\infty} \frac{b_k}{x^k} = \sum_{j=2}^{\infty} \left\{\sum_{k=1}^{j-1} a_k b_{j-k}\right\} \frac{1}{x^j}. \tag{6.3}$$

Equating coefficients of equal powers of x in (6.3), we obtain

$$-a_j = \sum_{k=1}^{j-1} a_k b_{j-k}, \qquad j \geq 2.$$

For $j = 2$ we obtain $b_1 = -a_2/a_1$, and for $j \geq 3$ we have

$$-a_j = a_1 b_{j-1} + \sum_{k=2}^{j-1} a_k b_{j-k}, \qquad j \geq 3,$$

which gives the desired formula (6.1). The proof is complete. $\square$

Lemma 6.1 provides a method to construct a continued fraction approximation based on a given asymptotic expansion. We state this method as a consequence of Lemma 6.1.

Corollary 6.1. Let $a_1 \neq 0$ and

$$A(x) \sim \sum_{j=1}^{\infty} \frac{a_j}{x^j}, \qquad x \to \infty \tag{6.4}$$

be a given asymptotic expansion. Then the function $A(x)$ has the following continued fraction approximation of the form:

$$A(x) \approx \cfrac{a_1}{x + \cfrac{b_1 x}{x + \cfrac{c_1 x}{x + \cfrac{d_1 x}{x + \ddots}}}}, \qquad x \to \infty, \tag{6.5}$$

where the constants in the right-hand side of (6.5) are given by the following recurrence relations:

$$\begin{cases} b_1 = -\frac{a_2}{a_1}, \quad b_j = -\frac{1}{a_1}\left(a_{j+1} + \sum_{k=2}^{j} a_k b_{j-k+1}\right) \\ c_1 = -\frac{b_2}{b_1}, \quad c_j = -\frac{1}{b_1}\left(b_{j+1} + \sum_{k=2}^{j} b_k c_{j-k+1}\right) \\ d_1 = -\frac{c_2}{c_1}, \quad d_j = -\frac{1}{c_1}\left(c_{j+1} + \sum_{k=2}^{j} c_k d_{j-k+1}\right) \\ \qquad \cdots \quad \cdots \end{cases} \tag{6.6}$$

Clearly, $a_j \Longrightarrow b_j \Longrightarrow c_j \Longrightarrow d_j \Longrightarrow \ldots$. Thus, the asymptotic expansion (6.4) $\Longrightarrow$ the continued fraction approximation (6.5).

Based on the asymptotic expansions (1.4) and (1.19), respectively, and using (6.6), we can easily derive the continued fraction approximations (1.12) and (1.13). Here, we only give a derivation of (1.12). The derivation of (1.11) is analogous, and we omit it.

A new derivations of (1.12). Denote

$$A(x) = \frac{1}{e}\left(1 + \frac{1}{x}\right)^x - 1.$$

It follows from (1.4) that

$$A(x) \sim \sum_{j=1}^{\infty} \frac{a_j}{x^\ell} = -\frac{1}{2x} + \frac{11}{24x^2} - \frac{7}{16x^3} + \frac{2447}{5760x^4} - \frac{959}{2304x^5} + \frac{238043}{580608x^6} - \cdots \tag{6.7}$$

as $x \to \infty$, where the coefficients $a_j \equiv \omega_j$ are given in (1.5). Then, $A(x)$ has the continued fraction approximation of the form

$$A(x) \approx \cfrac{a_1}{x + \cfrac{b_1 x}{x + \cfrac{c_1 x}{x + \cfrac{d_1 x}{x + \cfrac{e_1 x}{x + \cfrac{f_1 x}{x + \ddots}}}}}}, \qquad x \to \infty, \tag{6.8}$$

where the constants in the right-hand side of (6.8) can be determined using (6.6). Using (6.6), we now calculate the constants in the right-hand side of (6.8). Noting that

$$a_1 = -\frac{1}{2}, \quad a_2 = \frac{11}{24}, \quad a_3 = -\frac{7}{16}, \quad a_4 = \frac{2447}{5760}, \quad \dots,$$

we obtain from the first recurrence relation in (6.6) that

$$b_1 = -\frac{a_2}{a_1} = \frac{11}{12}, \quad b_2 = -\frac{a_3 + a_2 b_1}{a_1} = -\frac{5}{144}, \quad b_3 = -\frac{a_4 + a_2 b_2 + a_3 b_1}{a_1} = \frac{17}{1080}, \quad \dots.$$

We obtain from the second and third recurrence relations in (6.6) that

$$c_1 = -\frac{b_2}{b_1} = \frac{5}{132}, \quad c_2 = -\frac{b_3 + b_2 c_1}{b_1} = -\frac{457}{29040}, \quad \dots$$

and

$$d_1 = -\frac{c_2}{c_1} = \frac{457}{1100}, \quad \dots.$$

Continuing the above process, we find

$$e_1 = \frac{5291}{45700}, \quad f_1 = \frac{19753835}{55393884}, \quad \dots.$$

We see that formula (6.8) coincides with formula (1.12). □

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