# AN EXTENSION OF WIRTINGER'S INEQUALITY TO THE COMPLEX INTEGRAL

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Abstract. In this paper we establish a natural extension of the Wirtinger inequality to the case of complex integral of analytic functions. Applications related to the trapezoid inequalities are also provided. Examples for logarithmic and exponential complex functions are given as well.

## 1. Introduction

It is well known that, see for instance [4], or [7], if $u \in C^1([a,b],\mathbb{R})$ satisfies $u(a) = u(b) = 0$, then we have the *Wirtinger inequality*

$$\int_a^b u^2(t)\,dt \leq \frac{(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.1}$$

with the equality holding if and only if $u(t) = K\sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant $K \in \mathbb{R}$.

If $u \in C^1([a,b],\mathbb{R})$ satisfies the condition $u(a) = 0$, then also

$$\int_a^b u^2(t)\,dt \leq \frac{4(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.2}$$

and the equality holds if and only if $u(t) = L\sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant $L \in \mathbb{R}$.

If $h \in C^1([a,b],\mathbb{C})$ is a function with complex values and $h(a) = h(b) = 0$, then $\operatorname{Re} h(a) = \operatorname{Re} h(b) = 0$ and $\operatorname{Im} h(a) = \operatorname{Im} h(b) = 0$ and by writing (1.1) for $\operatorname{Re} h$ and $\operatorname{Im} h$ and adding the obtained inequalities, we get

$$\int_a^b |h(t)|^2\,dt \leq \frac{(b-a)^2}{\pi^2}\int_a^b |h'(t)|^2\,dt \tag{1.3}$$

with the equality holding if and only if

$$h(t) = K\sin\left[\frac{\pi(t-a)}{b-a}\right]$$

for some complex constant $K \in \mathbb{C}$.

Similarly, if $h \in C^1([a,b],\mathbb{C})$ with $h(a) = 0$, then by (1.2) we have

$$\int_a^b |h(t)|^2\,dt \leq \frac{4(b-a)^2}{\pi^2}\int_a^b |h'(t)|^2\,dt \tag{1.4}$$

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and the equality holds if and only if

$$h(t) = L \sin \left[ \frac{\pi (t-a)}{2(b-a)} \right]$$

for some complex constant $L \in \mathbb{R}$.

For some related Wirtinger type integral inequalities see [1], [2], [4] and [6]-[10].

In order to extend this result for the complex integral, we need some preparations as follows.

Suppose $\gamma$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ and $f$ is a complex function which is continuous on $\gamma$. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of $f$ on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt.$$

We observe that that the actual choice of parametrization of $\gamma$ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose $\gamma$ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that $f$ is continuous on $\gamma$ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve $\gamma$ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let $f$ and $g$ be holomorphic in $G$, and open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{1.5}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left| \int_\gamma f(z)\,dz \right| \le \int_\gamma |f(z)|\,|dz| \le \|f\|_{\gamma,\infty}\,\ell(\gamma) \tag{1.6}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the $p$-norm with $p \ge 1$ by

$$\|f\|_{\gamma,p} := \left( \int_\gamma |f(z)|^p\,|dz| \right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)|\,|dz|.$$

If $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \le [\ell(\gamma)]^{1/q} \|f\|_{\gamma,p}.$$

In this paper we establish a natural extension of the Wirtinger inequality to the case of complex integral of analytic functions. Applications related to the trapezoid inequalities are also provided. Examples for logarithmic and exponential complex functions are given as well.

## 2. Wirtinger Type Inequalities

We have the following weighted version of Wirtinger inequality:

**Lemma 1.** *Let $g:[a,b]\to[g(a),g(b)]$ be a continuous strictly increasing function that is of class $C^1$ on $(a,b)$.*

(i) *If $h\in C^1([a,b],\mathbb{C})$ is a function with complex values and $h(a)=h(b)=0$, then*

$$\int_a^b |h(t)|^2 g'(t)\,dt \le \frac{[g(b)-g(a)]^2}{\pi^2}\int_a^b \frac{|h'(t)|^2}{g'(t)}dt. \tag{2.1}$$

*The equality holds in (2.1) iff*

$$h(t)=K\sin\left[\frac{\pi(g(t)-g(a))}{g(b)-g(a)}\right],\ K\in\mathbb{C}.$$

(ii) *If $h\in C^1([a,b],\mathbb{C})$ is a function with complex values and $h(a)=0$, then*

$$\int_a^b |h(t)|^2 g'(t)\,dt \le \frac{4[g(b)-g(a)]^2}{\pi^2}\int_a^b \frac{|h'(t)|^2}{g'(t)}dt. \tag{2.2}$$

*The equality holds in (2.2) iff*

$$h(t)=K\sin\left[\frac{\pi(g(t)-g(a))}{2(g(b)-g(a))}\right],\ K\in\mathbb{C}.$$

*Proof.* (i) We write the inequality (1.3) for the function $h=h\circ g^{-1}$ on the interval $[g(a),g(b)]$ to get

$$\int_{g(a)}^{g(b)} \left|\left(h\circ g^{-1}\right)(z)\right|^2 dz \le \frac{(g(b)-g(a))^2}{\pi^2}\int_{g(a)}^{g(b)} \left|\left(h\circ g^{-1}\right)'(z)\right|^2 dz. \tag{2.3}$$

If $h:[c,d]\to\mathbb{C}$ is absolutely continuous on $[c,d]$, then $h\circ g^{-1}:[g(c),g(d)]\to\mathbb{C}$ is absolutely continuous on $[g(c),g(d)]$ and using the chain rule and the derivative of inverse functions we have

$$\left(h\circ g^{-1}\right)'(z)=\left(h'\circ g^{-1}\right)(z)\left(g^{-1}\right)'(z)=\frac{\left(h'\circ g^{-1}\right)(z)}{\left(g'\circ g^{-1}\right)(z)} \tag{2.4}$$

for almost every (a.e.) $z\in[g(c),g(d)]$.

Using the inequality (2.3) we then get

$$\int_{g(a)}^{g(b)} \left|\left(h\circ g^{-1}\right)(z)\right|^2 dz \le \frac{(g(b)-g(a))^2}{\pi^2}\int_{g(a)}^{g(b)} \left|\frac{\left(h'\circ g^{-1}\right)(z)}{\left(g'\circ g^{-1}\right)(z)}\right|^2 dz, \tag{2.5}$$

provided $\left(h\circ g^{-1}\right)(g(a))=h(a)=0$ and $\left(h\circ g^{-1}\right)(g(b))=h(b)=0$.

Observe also that, by the change of variable $t = g^{-1}(z)$, $z \in [g(a), g(b)]$, we have $z = g(t)$ that gives $dz = g'(t)\,dt$ and

$$\int_{g(a)}^{g(b)} \left|\left(h \circ g^{-1}\right)(z)\right|^2 dz = \int_a^b |h(t)|^2 g'(t)\,dt. \tag{2.6}$$

We also have

$$\int_{g(a)}^{g(b)} \left|\frac{\left(h' \circ g^{-1}\right)(z)}{\left(g' \circ g^{-1}\right)(z)}\right|^2 dz = \int_a^b \left|\frac{h'(t)}{g'(t)}\right|^2 g'(t)\,dt = \int_a^b \frac{|h'(t)|^2}{g'(t)} dt.$$

By making use of (2.5) we get (2.1).

The equality holds in (2.5) provided

$$\left(h \circ g^{-1}\right)(z) = K \sin\left[\frac{\pi (z - g(a))}{g(b) - g(a)}\right],\ K \in \mathbb{C}$$

for $z \in [g(a), g(b)]$. If we take $t \in [a, b]$ and $z = g(t)$, we then get

$$h(t) = K \sin\left[\frac{\pi (g(t) - g(a))}{g(b) - g(a)}\right],\ K \in \mathbb{C}.$$

(ii) Follows in a similar way by (1.4). □

If $w : [a, b] \to \mathbb{R}$ is continuous and positive on the interval $[a, b]$, then the function $W : [a, b] \to [0, \infty)$, $W(x) := \int_a^x w(s)\,ds$ is strictly increasing and differentiable on $(a, b)$. We have $W'(x) = w(x)$ for any $x \in (a, b)$.

**Corollary 1.** *Assume that $w : [a, b] \to (0, \infty)$ is continuous on $[a, b]$ and $h \in C^1([a, b], \mathbb{C})$ is a function with complex values and $h(a) = h(b) = 0$, then*

$$\int_a^b |h(t)|^2 w(t)\,dt \le \frac{1}{\pi^2}\left(\int_a^b w(s)\,ds\right)^2 \int_a^b \frac{|h'(t)|^2}{w(t)} dt. \tag{2.7}$$

*The equality holds in (2.7) iff*

$$h(t) = K \sin\left[\frac{\pi \int_a^t w(s)\,ds}{\int_a^b w(s)\,ds}\right],\ K \in \mathbb{C}.$$

*If $h(a) = 0$, then*

$$\int_a^b |h(t)|^2 w(t)\,dt \le \frac{4}{\pi^2}\left(\int_a^b w(s)\,ds\right)^2 \int_a^b \frac{|h'(t)|^2}{w(t)} dt \tag{2.8}$$

*with equality iff*

$$h(t) = K \sin\left[\frac{\pi \int_a^t w(s)\,ds}{2\int_a^b w(s)\,ds}\right],\ K \in \mathbb{C}.$$

We have the following Wirtinger type inequality for complex functions:

**Theorem 1.** *Let $f$ be analytic in $G$, a domain of complex numbers and suppose $\gamma \subset G$ is a smooth path parametrized by $z(t)$, $t \in [a, b]$ from $z(a) = u$ to $z(b) = w$ and $z'(t) \neq 0$ for $t \in (a, b)$.*

(i) *If* $f(u)=f(w)=0$, *then*

$$\int_{\gamma}|f(z)|^{2}|dz|\leq\frac{1}{\pi^{2}}\ell^{2}(\gamma)\int_{\gamma}|f'(z)|^{2}|dz|. \tag{2.9}$$

*The equality holds in (2.9) iff*

$$f(v)=K\sin\left[\frac{\pi\ell(\gamma_{u,v})}{\ell(\gamma)}\right],\ K\in\mathbb{C} \tag{2.10}$$

*where* $v=z(t)$, $t\in[a,b]$ *and* $\ell(\gamma_{u,v})=\int_{a}^{t}|z'(s)|\,ds$.

(ii) *If* $f(u)=0$, *then*

$$\int_{\gamma}|f(z)|^{2}|dz|\leq\frac{4}{\pi^{2}}\ell^{2}(\gamma)\int_{\gamma}|f'(z)|^{2}|dz|. \tag{2.11}$$

*The equality holds in (2.11) iff*

$$f(v)=K\sin\left[\frac{\pi\ell(\gamma_{u,v})}{2\ell(\gamma)}\right],\ K\in\mathbb{C} \tag{2.12}$$

*where* $v=z(t)$, $t\in[a,b]$.

*Proof.* (i) Consider the function $h(t)=f(z(t))$ and $w(t)=|z'(t)|$, $t\in[a,b]$. Then $h'(t)=(f(z(t)))'=f'(z(t))z'(t)$ for $t\in(a,b)$. Also $h(a)=f(z(a))=f(u)=0$ and $h(b)=f(z(b))=f(w)=0$. By utilising the inequality (2.7) for these choices, we get

$$\begin{aligned}\int_{a}^{b}|f(z(t))|^{2}|z'(t)|\,dt&\leq\frac{1}{\pi^{2}}\left(\int_{a}^{b}|z'(s)|\,ds\right)^{2}\int_{a}^{b}\frac{|f'(z(t))z'(t)|^{2}}{|z'(t)|}dt\\&=\frac{1}{\pi^{2}}\left(\int_{a}^{b}|z'(s)|\,ds\right)^{2}\int_{a}^{b}\frac{|f'(z(t))|^{2}|z'(t)|^{2}}{|z'(t)|}dt\\&=\frac{1}{\pi^{2}}\left(\int_{a}^{b}|z'(s)|\,ds\right)^{2}\int_{a}^{b}|f'(z(t))|^{2}|z'(t)|\,dt,\end{aligned}$$

which is equivalent to (2.9).

The equality (2.10) follows by the corresponding equality in Corollary 1.

(ii) Follows by the corresponding result from Corollary 1. □

## 3. Some Trapezoid Type Inequalities

We have:

**Proposition 1.** *Let* $g$ *be analytic in* $G$, *a domain of complex numbers and suppose* $\gamma\subset G$ *is a smooth path parametrized by* $z(t)$, $t\in[a,b]$ *from* $z(a)=u$ *to* $z(b)=w$, $w\neq u$ *and* $z'(t)\neq0$ *for* $t\in(a,b)$. *Then*

$$\begin{aligned}&\left|\frac{1}{w-u}\int_{\gamma}g(z)\,dz-\frac{g(u)+g(w)}{2}\right|\\&\qquad\leq\frac{1}{\pi}\frac{\ell(\gamma)}{|w-u|}\left(\frac{1}{\ell(\gamma)}\int_{\gamma}\left|g'(z)-\frac{g(w)-g(u)}{w-u}\right|^{2}|dz|\right)^{1/2}.\end{aligned} \tag{3.1}$$

*Proof.* Consider the function $f:G\to\mathbb{C}$ defined by

$$f(z):=g(z)-\frac{g(u)(w-z)+g(w)(z-u)}{w-u},\ z\in\gamma.$$

Observe that $f(u)=f(w)=0$ and by (2.9) we get

$$\int_\gamma\left|g(z)-\frac{g(u)(w-z)+g(w)(z-u)}{w-u}\right|^2|dz| \leq\frac{1}{\pi^2}\ell^2(\gamma)\int_\gamma\left|g'(z)-\frac{g(w)-g(u)}{w-u}\right|^2|dz|. \tag{3.2}$$

Using Cauchy-Bunyakovsky-Schwarz integral inequality we also have

$$\begin{aligned}&\left|\int_\gamma\left[g(z)-\frac{g(u)(w-z)+g(w)(z-u)}{w-u}\right]dz\right|^2\\ &\leq\int_\gamma|dz|\int_\gamma\left|g(z)-\frac{g(u)(w-z)+g(w)(z-u)}{w-u}\right|^2|dz|\\ &=\ell(\gamma)\int_\gamma\left|g(z)-\frac{g(u)(w-z)+g(w)(z-u)}{w-u}\right|^2|dz|\end{aligned}$$

and since

$$\begin{aligned}&\int_\gamma\left[g(z)-\frac{g(u)(w-z)+g(w)(z-u)}{w-u}\right]dz\\ &=\int_\gamma g(z)\,dz-\frac{g(u)\int_\gamma(w-z)\,dz+g(w)\int_\gamma(z-u)\,dz}{w-u}\\ &=\int_\gamma g(z)\,dz-\frac{g(u)+g(w)}{2}(w-u),\end{aligned}$$

hence

$$\begin{aligned}&\left|\int_\gamma g(z)\,dz-\frac{g(u)+g(w)}{2}(w-u)\right|^2\\ &\leq\ell(\gamma)\int_\gamma\left|g(z)-\frac{g(u)(w-z)+g(w)(z-u)}{w-u}\right|^2|dz|\end{aligned}$$

and by (3.2) we get

$$\begin{aligned}&\left|\int_\gamma g(z)\,dz-\frac{g(u)+g(w)}{2}(w-u)\right|^2\\ &\leq\ell(\gamma)\int_\gamma\left|g(z)-\frac{g(u)(w-z)+g(w)(z-u)}{w-u}\right|^2|dz|\\ &\leq\frac{1}{\pi^2}\ell^3(\gamma)\int_\gamma\left|g'(z)-\frac{g(w)-g(u)}{w-u}\right|^2|dz|,\end{aligned}$$

which implies the desired result (3.1). □

We also have:

**Proposition 2.** *Let $g$ be analytic in $G$, a domain of complex numbers and suppose $\gamma \subset G$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ from $z(a) = u$ to $z(b) = w$, $w \neq u$ and $z'(t) \neq 0$ for $t \in (a,b)$. If $u + w - z \in G$ for $z \in \gamma$, then*

$$\left| \frac{1}{w-u} \int_\gamma \widehat{g(z)} dz - \frac{g(u)+g(w)}{2} \right| \leq \frac{1}{2\pi} \frac{\ell(\gamma)}{|w-u|} \left( \frac{1}{\ell(\gamma)} \int_\gamma |g'(z) - g'(u+w-z)|^2 |dz| \right)^{1/2}, \tag{3.3}$$

*where*

$$\widehat{g(z)} := \frac{g(z) + g(u+w-z)}{2}, \; z \in \gamma.$$

*Proof.* Consider the function $f : G \to \mathbb{C}$ defined by

$$f(z) := \frac{g(z) + g(u+w-z)}{2} - \frac{g(u)+g(w)}{2}, \; z \in \gamma.$$

Observe that $f(u) = f(w) = 0$ and by (2.9) we get

$$\int_\gamma \left| \frac{g(z) + g(u+w-z)}{2} - \frac{g(u)+g(w)}{2} \right|^2 |dz| \leq \frac{1}{4\pi^2} \ell^2(\gamma) \int_\gamma |g'(z) - g'(u+w-z)|^2 |dz|. \tag{3.4}$$

By Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$\left| \int_\gamma \left[ \frac{g(z) + g(u+w-z)}{2} - \frac{g(u)+g(w)}{2} \right] dz \right|^2 \leq \int_\gamma |dz| \int_\gamma \left| \frac{g(z) + g(u+w-z)}{2} - \frac{g(u)+g(w)}{2} \right|^2 |dz|,$$

which gives

$$\left| \int_\gamma \frac{g(z) + g(u+w-z)}{2} dz - \frac{g(u)+g(w)}{2} (w-u) \right|^2 \leq \ell(\gamma) \int_\gamma \left| \frac{g(z) + g(u+w-z)}{2} - \frac{g(u)+g(w)}{2} \right|^2 |dz|.$$

By (3.4) we then get

$$\left| \int_\gamma \frac{g(z) + g(u+w-z)}{2} dz - \frac{g(u)+g(w)}{2} (w-u) \right|^2 \leq \frac{1}{4\pi^2} \ell^3(\gamma) \int_\gamma |g'(z) - g'(u+w-z)|^2 |dz|,$$

which is equivalent to (3.3). □

**Corollary 2.** *With the assumption of Proposition 2 and if*

$$|g'(z) - g'(u+w-z)| \leq |2z - u - w| L$$

*for some $L > 0$ and for any $z \in \gamma$, then*

$$\left| \frac{1}{w-u} \int_{\gamma} \widehat{g(z)} dz - \frac{g(u)+g(w)}{2} \right| \leq \frac{1}{\pi} \frac{\ell(\gamma) L}{|w-u|} \left( \frac{1}{\ell(\gamma)} \int_{\gamma} \left| z - \frac{u+w}{2} \right|^2 |dz| \right)^{1/2}. \tag{3.5}$$

**Remark 1.** *If $\|g''\|_{\infty,G} := \sup_{z\in G} |g''(z)| < \infty$, then we can take above $L = \|g''\|_{\infty,G}$.*

## 4. Some Examples for Logarithmic and Exponential Functions

Consider the function $g(z) = \frac{1}{z}$, $z \in \mathbb{C}\setminus\{0\}$. Then

$$g^{(k)}(z) = \frac{(-1)^k k!}{z^{k+1}} \text{ for } k \geq 0,\ z \in \mathbb{C}\setminus\{0\}$$

and suppose $\gamma \subset \mathbb{C}_\ell := \mathbb{C}\setminus\{x+iy : x \leq 0,\ y = 0\}$ is a *smooth path* parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = u$ and $z(b) = w$ where $u, w \in \mathbb{C}_\ell$ and $z'(t) \neq 0$ for $t \in (a,b)$. Then

$$\int_{\gamma} g(z)\, dz = \int_{\gamma_{u,w}} g(z)\, dz = \int_{\gamma_{u,w}} \frac{dz}{z} = \operatorname{Log}(w) - \operatorname{Log}(u)$$

for $u, w \in \mathbb{C}_\ell$.

By making use of the inequality (3.1) we get

$$\left| \frac{\operatorname{Log}(w) - \operatorname{Log}(u)}{w-u} - \frac{u+w}{2uw} \right| \leq \frac{1}{\pi} \frac{\ell(\gamma)}{|w-u|\,|wu|} \left( \frac{1}{\ell(\gamma)} \int_{\gamma} \left| \frac{z^2 - uw}{z^2} \right|^2 |dz| \right)^{1/2}. \tag{4.1}$$

Observe also that

$$\int_{\gamma_{u,w}} \frac{dz}{u+w-z} = -\operatorname{Log}(u+w-z)\Big|_u^w = -\operatorname{Log}(u) + \operatorname{Log}(w),$$

therefore, by the inequality (3.3) we get

$$\left| \frac{\operatorname{Log}(w) - \operatorname{Log}(u)}{w-u} - \frac{u+w}{2uw} \right| \leq \frac{1}{\pi} \frac{|w+u|\,\ell(\gamma)}{|w-u|} \left( \frac{1}{\ell(\gamma)} \int_{\gamma} \left| \frac{z - \frac{u+w}{2}}{z^2 (u+w-z)^2} \right|^2 |dz| \right)^{1/2}. \tag{4.2}$$

Consider the function $g(z) = \exp z$, $z \in \mathbb{C}$. Suppose $\gamma$ is a *smooth path* parametrized by $z(t)$, $t \in [a,b]$ with $z(a) = u$ and $z(b) = w$ where $u, w \in \mathbb{C}$ and $z'(t) \neq 0$ for $t \in (a,b)$.

By making use of the inequality (3.1) we get

$$\left|\frac{\exp w-\exp u}{w-u}-\frac{\exp u+\exp w}{2}\right| \leq \frac{1}{\pi}\frac{\ell(\gamma)}{|w-u|}\left(\frac{1}{\ell(\gamma)}\int_{\gamma}\left|\exp z-\frac{\exp w-\exp u}{w-u}\right|^{2}|dz|\right)^{1/2}, \tag{4.3}$$

while from (3.3) we get

$$\left|\frac{\exp w-\exp u}{w-u}-\frac{\exp u+\exp w}{2}\right| \leq \frac{1}{2\pi}\frac{\ell(\gamma)}{|w-u|}\left(\frac{1}{\ell(\gamma)}\int_{\gamma}\left|\exp z-\exp(u+w-z)\right|^{2}|dz|\right)^{1/2}. \tag{4.4}$$

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