AN EXTENSION OF OPIAL'S INEQUALITY TO THE COMPLEX INTEGRAL

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Abstract. In this paper we establish an extension of Opial inequality to the case of complex integral of analityc functions.

1. Introduction

We recall the following Opial type inequalities:

Theorem 1. *Assume that $u:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is an absolutely continuous function on the interval $[a,b]$ and such that $u'\in L_2[a,b]$.*

(i) *If $u(a)=u(b)=0$, then*

$$\int_a^b |u(t)\,u'(t)|\,dt\le\frac{1}{4}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.1}$$

with equality if and only if

$$u(t)=\begin{cases} c(t-a) & \text{if } a\le t\le\frac{a+b}{2},\\ c(b-t) & \text{if } \frac{a+b}{2}<t\le b,\end{cases}$$

where c is an arbitrary constant;

(ii) *If $u(a)=0$, then*

$$\int_a^b |u(t)\,u'(t)|\,dt\le\frac{1}{2}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.2}$$

with equality if and only if $u(t)=c(t-a)$ for some constant c.

The inequality (1.1) was obtained by Olech in [9] in which he gave a simplified proof of an inequality originally due in a slightly less general form to Zdzislaw Opial [10].

Embedded in Olech's proof is the half-interval form of Opial's inequality, also discovered by Beesack [1], which is satisfied by those u vanishing only at a.

For various proofs of the above inequalities, see [5]-[8] and [12].

In order to extend this result for the complex integral, we need some preparations as follows.

Suppose γ is a *smooth path* parametrized by $z(t)$, $t\in[a,b]$ and f is a complex function which is continuous on γ. Put $z(a)=u$ and $z(b)=w$ with $u,w\in\mathbb{C}$. We define the integral of f on $\gamma_{u,w}=\gamma$ as

$$\int_\gamma f(z)\,dz=\int_{\gamma_{u,w}} f(z)\,dz:=\int_a^b f(z(t))\,z'(t)\,dt.$$

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We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are *piecewise smooth*. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{1.3}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz| \le \|f\|_{\gamma,\infty}\,\ell(\gamma) \tag{1.4}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \ge 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p\,|dz|\right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)|\,|dz|.$$

If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \le [\ell(\gamma)]^{1/q}\,\|f\|_{\gamma,p}.$$

2. Some Preliminary Results

We have the following refinement and generalization for complex valued function of the Opial inequality:

Theorem 2. *Assume that* $f : [a,b] \to \mathbb{C}$ *are absolutely continuous on* $[a,b]$ *and* $f' \in L_2[a,b]$.

(i) *If either* $f(a)=0$ *or* $f(b)=0$, *then*

$$\int_a^b |f'(t) f(t)| \, dt \le \left(\int_a^b (t-a) |f'(t)|^2 \, dt \right)^{1/2} \left(\int_a^b (b-t) |f'(t)|^2 \, dt \right)^{1/2} \le \frac{1}{2}(b-a) \int_a^b |f'(t)|^2 \, dt. \tag{2.1}$$

(ii) *If* $f(a)=f(b)=0$, *then*

$$\int_a^b |f'(t) f(t)| \, dt \le \left[\int_a^b K(t) |f'(t)|^2 \, dt \right]^{1/2} \left[\int_a^b \left| \frac{a+b}{2} - t \right| |f'(t)|^2 \, dt \right]^{1/2} \le \frac{1}{4}(b-a) \int_a^b |f'(t)|^2 \, dt, \tag{2.2}$$

where

$$K(t) := \begin{cases} t-a \text{ if } a \le t \le \frac{a+b}{2}, \\ b-t \text{ if } \frac{a+b}{2} < t \le b. \end{cases}$$

Proof. (i) Since $f(a)=0$, then $f(t)=\int_a^t f'(s)\,ds$ for $t \in [a,b]$. We have

$$\int_a^b |f'(t) f(t)| \, dt = \int_a^b |f'(t)| |f(t)| \, dt = \int_a^b (t-a)^{1/2} |f'(t)| (t-a)^{-1/2} |f(t)| \, dt = \int_a^b (t-a)^{1/2} |f'(t)| (t-a)^{-1/2} \left| \int_a^t f'(s) \, ds \right| dt =: A. \tag{2.3}$$

Using Cauchy-Bunyakovsky-Schwarz (CBS) inequality, we have

$$A \le \left(\int_a^b \left[(t-a)^{1/2} |f'(t)| \right]^2 dt \right)^{1/2} \times \left(\int_a^b \left[(t-a)^{-1/2} \left| \int_a^t f'(s) \, ds \right| \right]^2 dt \right)^{1/2} = \left(\int_a^b (t-a) |f'(t)|^2 \, dt \right)^{1/2} \left(\int_a^b (t-a)^{-1} \left| \int_a^t f'(s) \, ds \right|^2 dt \right)^{1/2} =: B. \tag{2.4}$$

By (CBS) inequality we also have

$$(t-a)^{-1} \left| \int_a^t f'(s) \, ds \right|^2 \le \int_a^t |f'(s)|^2 \, ds,$$

which gives

$$B \le \left(\int_a^b (t-a) |f'(t)|^2 \, dt \right)^{1/2} \left(\int_a^b \left(\int_a^t |f'(s)|^2 \, ds \right) dt \right)^{1/2}. \tag{2.5}$$

Using integration by parts, we have

$$\int_a^b \left(\int_a^t |f'(s)|^2 \, ds \right) dt = b \int_a^b |f'(s)|^2 \, ds - \int_a^b t\, |f'(t)|^2 \, dt = \int_a^b (b-t)\, |f'(t)|^2$$

and by (2.4) we get the first inequality in (2.1).

The last part follows by the elementary inequality

$$\sqrt{\alpha\beta} \leq \frac{1}{2}(\alpha+\beta), \ \alpha, \beta \geq 0. \tag{2.6}$$

(ii) If we write the inequality (2.1) on the interval $\left[a, \frac{a+b}{2}\right]$, we have

$$\begin{aligned} &\int_a^{\frac{a+b}{2}} |f'(t)\, f(t)|\, dt \\ &\leq \left(\int_a^{\frac{a+b}{2}} (t-a)\, |f'(t)|^2 \, dt \right)^{1/2} \left(\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t \right) |f'(t)|^2 \, dt \right)^{1/2} \end{aligned} \tag{2.7}$$

and if we write the inequality (2.1) on the interval $\left[\frac{a+b}{2}, b\right]$, we have

$$\begin{aligned} &\int_{\frac{a+b}{2}}^b |f'(t)\, f(t)|\, dt \\ &\leq \left(\int_{\frac{a+b}{2}}^b (b-t)\, |f'(t)|^2 \, dt \right)^{1/2} \left(\int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2} \right) |f'(t)|^2 \, dt \right)^{1/2}. \end{aligned} \tag{2.8}$$

If we add the inequalities (2.7) and (2.8) we get

$$\begin{aligned} &\int_a^b |f'(t)\, f(t)|\, dt \\ &\leq \left(\int_a^{\frac{a+b}{2}} (t-a)\, |f'(t)|^2 \, dt \right)^{1/2} \left(\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t \right) |f'(t)|^2 \, dt \right)^{1/2} \\ &+ \left(\int_{\frac{a+b}{2}}^b (b-t)\, |f'(t)|^2 \, dt \right)^{1/2} \left(\int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2} \right) |f'(t)|^2 \, dt \right)^{1/2} \\ &\leq \left[\int_a^{\frac{a+b}{2}} (t-a)\, |f'(t)|^2 \, dt + \int_{\frac{a+b}{2}}^b (b-t)\, |f'(t)|^2 \, dt \right]^{1/2} \\ &\times \left[\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t \right) |f'(t)|^2 \, dt + \int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2} \right) |f'(t)|^2 \, dt \right]^{1/2} \\ &= \left[\int_a^b K(t)\, |f'(t)|^2 \, dt \right]^{1/2} \left[\int_a^b \left| \frac{a+b}{2} - t \right| |f'(t)|^2 \, dt \right]^{1/2}, \end{aligned}$$

where for the last inequality we used the elementary (CBS) inequality

$$\alpha\beta + \gamma\delta \leq \left(\alpha^2+\gamma^2\right)^{1/2} \left(\beta^2+\delta^2\right)^{1/2}, \ \alpha, \ \beta, \ \gamma, \ \delta \geq 0.$$

The last part follows by (2.6), namely

$$\left[\int_a^b K(t)\left|f'(t)\right|^2 dt\right]^{1/2}\left[\int_a^b \left|\frac{a+b}{2}-t\right|\left|f'(t)\right|^2 dt\right]^{1/2}$$
$$\leq \frac{1}{2}\left[\int_a^b K(t)\left|f'(t)\right|^2 dt+\int_a^b \left|\frac{a+b}{2}-t\right|\left|f'(t)\right|^2 dt\right]$$
$$=\frac{1}{2}\int_a^b \left[K(t)+\left|\frac{a+b}{2}-t\right|\right]\left|f'(t)\right|^2 dt=\frac{1}{4}\int_a^b \left|f'(t)\right|^2 dt,$$

since

$$K(t)+\left|\frac{a+b}{2}-t\right|=\frac{1}{2}(b-a) \text{ for } t\in[a,b].$$

□

3. Weighted Inequalities

We also have the following composite inequality:

Theorem 3. *Let $g:[a,b]\to[g(a),g(b)]$ be a continuous strictly increasing function that is of class C^1 on (a,b). Assume that $f:[a,b]\subset\mathbb{R}\to\mathbb{C}$ is an absolutely continuous complex valued function on the interval $[a,b]$ and such that $\frac{f'}{[g']^{1/2}}\in L_2[a,b]$.*

(i) *If $f(a)=0$ or then $f(b)=0$, then*

$$\int_a^b |f(t)f'(t)|\,dt \tag{3.1}$$
$$\leq\left(\int_a^b (g(t)-g(a))\frac{\left|f'(t)\right|^2}{g'(t)}dt\right)^{1/2}\left(\int_a^b (g(b)-g(t))\frac{\left|f'(t)\right|^2}{g'(t)}dt\right)^{1/2}$$
$$\leq\frac{1}{2}[g(b)-g(a)]\int_a^b\frac{\left|f'(t)\right|^2}{g'(t)}dt.$$

(ii) *If $f(a)=f(b)=0$, then*

$$\int_a^b |f(t)f'(t)|\,dt \tag{3.2}$$
$$\leq\left[\int_a^b\left(\frac{1}{2}(g(b)-g(a))-\left|\frac{g(a)+g(b)}{2}-g(t)\right|\right)\frac{\left|f'(t)\right|^2}{g'(t)}dt\right]^{1/2}$$
$$\times\left[\int_a^b\left|\frac{g(a)+g(b)}{2}-g(t)\right|\frac{\left|f'(t)\right|^2}{g'(t)}dt\right]^{1/2}$$
$$\leq\frac{1}{4}[g(b)-g(a)]\int_a^b\frac{\left|f'(t)\right|^2}{g'(t)}dt.$$

Proof. (i) Consider the function $u:=f\circ g^{-1}:[g(a),g(b)]\to\mathbb{R}$. The function u is absolutely continuous on $[g(a),g(b)]$, $u(g(a))=f\circ g^{-1}(g(a))=f(a)=0$ or $u(g(b))=f\circ g^{-1}(g(b))=f(b)=0$.

Using the chain rule and the derivative of inverse functions we have

$$(f\circ g^{-1})'(z)=(f'\circ g^{-1})(z)(g^{-1})'(z)=\frac{(f'\circ g^{-1})(z)}{(g'\circ g^{-1})(z)} \tag{3.3}$$

for almost every (a.e.) $z\in[g(a),g(b)]$.

If we apply the inequality (2.1) for the function $u=f\circ g^{-1}$ on the interval $[g(a),g(b)]$, then we get

$$\begin{aligned}\int_{g(a)}^{g(b)}&\left|f\circ g^{-1}(z)\frac{(f'\circ g^{-1})(z)}{(g'\circ g^{-1})(z)}\right|dz \\ &\le\left(\int_{g(a)}^{g(b)}(z-g(a))\left|\frac{(f'\circ g^{-1})(z)}{(g'\circ g^{-1})(z)}\right|^2dz\right)^{1/2} \\ &\times\left(\int_{g(a)}^{g(b)}(g(b)-z)\left|\frac{(f'\circ g^{-1})(z)}{(g'\circ g^{-1})(z)}\right|^2dz\right)^{1/2} \\ &\le\frac{1}{2}[g(b)-g(a)]\int_{g(a)}^{g(b)}\left|\frac{(f'\circ g^{-1})(z)}{(g'\circ g^{-1})(z)}\right|^2dz.\end{aligned} \tag{3.4}$$

If we make the change of variable $t=g^{-1}(z)$, $z\in[g(a),g(b)]$, then $z=g(t)$, $dz=g'(t)\,dt$,

$$\begin{aligned}\int_{g(a)}^{g(b)}\left|f\circ g^{-1}(z)\frac{(f'\circ g^{-1})(z)}{(g'\circ g^{-1})(z)}\right|dz&=\int_a^b\left|f(t)\frac{f'(t)}{g'(t)}\right|g'(t)\,dt \\ &=\int_a^b|f(t)f'(t)|\,dt,\end{aligned}$$

$$\begin{aligned}\int_{g(a)}^{g(b)}(z-g(a))\left|\frac{(f'\circ g^{-1})(z)}{(g'\circ g^{-1})(z)}\right|^2dz&=\int_a^b(g(t)-g(a))\left|\frac{f'(t)}{g'(t)}\right|^2g'(t)\,dt \\ &=\int_a^b(g(t)-g(a))\frac{|f'(t)|^2}{g'(t)}dt\end{aligned}$$

$$\begin{aligned}\int_{g(a)}^{g(b)}(g(b)-z)\left|\frac{(f'\circ g^{-1})(z)}{(g'\circ g^{-1})(z)}\right|^2dz&=\int_{g(a)}^{g(b)}(g(b)-g(t))\left|\frac{f'(t)}{g'(t)}\right|^2g'(t)\,dt \\ &=\int_a^b(g(b)-g(t))\frac{|f'(t)|^2}{g'(t)}dt\end{aligned}$$

and

$$\int_{g(a)}^{g(b)}\left|\frac{(f'\circ g^{-1})(z)}{(g'\circ g^{-1})(z)}\right|^2dz=\int_a^b\left|\frac{f'(t)}{g'(t)}\right|^2g'(t)\,dt=\int_a^b\frac{|f'(t)|^2}{g'(t)}dt.$$

By utilising (3.4), we then get the desired inequality (3.1).

(ii) By using the inequality (2.2) for the function $u = f \circ g^{-1}$ on the interval $[g(a), g(b)]$, then we get

$$\begin{aligned}
(3.5)\quad & \int_{g(a)}^{g(b)} \left| f \circ g^{-1}(z) \frac{\left(f' \circ g^{-1}\right)(z)}{\left(g' \circ g^{-1}\right)(z)} \right| dz \\
& \leq \left[\int_{g(a)}^{g(b)} \left(\frac{1}{2}(g(b) - g(a)) - \left| \frac{g(a) + g(b)}{2} - z \right| \right) \left| \frac{\left(f' \circ g^{-1}\right)(z)}{\left(g' \circ g^{-1}\right)(z)} \right|^2 dz \right]^{1/2} \\
& \quad \times \left[\int_a^b \left| \frac{g(a) + g(b)}{2} - z \right| \left| \frac{\left(f' \circ g^{-1}\right)(z)}{\left(g' \circ g^{-1}\right)(z)} \right|^2 dz \right]^{1/2} \\
& \leq \frac{1}{4}[g(b) - g(a)] \int_{g(a)}^{g(b)} \left| \frac{\left(f' \circ g^{-1}\right)(z)}{\left(g' \circ g^{-1}\right)(z)} \right|^2 dz.
\end{aligned}$$

If we make the change of variable $t = g^{-1}(z)$, $z \in [g(a), g(b)]$, then by (3.5) we get the desired result (3.2). □

If $w : [a, b] \to \mathbb{R}$ is continuous and positive on the interval $[a, b]$, then the function $W : [a, b] \to [0, \infty)$, $W(x) := \int_a^x w(s)\, ds$ is strictly increasing and differentiable on (a, b). We have $W'(x) = w(x)$ for any $x \in (a, b)$.

Corollary 1. *Assume that $w : [a, b] \to (0, \infty)$ is continuous on $[a, b]$ and that $f : [a, b] \subset \mathbb{R} \to \mathbb{C}$ is an absolutely continuous complex valued function on the interval $[a, b]$ and such that $\frac{f'}{w^{1/2}} \in L_2[a, b]$.*

(i) *If $f(a) = 0$ or $f(b) = 0$, then*

$$\begin{aligned}
(3.6)\quad & \int_a^b |f(t) f'(t)|\, dt \\
& \leq \left(\int_a^b \left(\int_a^t w(s)\, ds \right) \frac{|f'(t)|^2}{w(t)} dt \right)^{1/2} \left(\int_a^b \left(\int_t^b w(s)\, ds \right) \frac{|f'(t)|^2}{w(t)} dt \right)^{1/2} \\
& \leq \frac{1}{2} \int_a^b w(s)\, ds \int_a^b \frac{|f'(t)|^2}{w(t)} dt.
\end{aligned}$$

(ii) *If $f(a) = f(b) = 0$, then*

$$\begin{aligned}
(3.7)\quad & \int_a^b |f(t) f'(t)|\, dt \\
& \leq \frac{1}{2} \left[\int_a^b \left(\int_a^b w(s)\, ds - \left| \int_t^b w(s)\, ds - \int_a^t w(s)\, ds \right| \right) \frac{|f'(t)|^2}{w(t)} dt \right]^{1/2} \\
& \quad \times \left[\int_a^b \left| \int_t^b w(s)\, ds - \int_a^t w(s)\, ds \right| \frac{|f'(t)|^2}{w(t)} dt \right]^{1/2} \\
& \leq \frac{1}{4} \int_a^b w(s)\, ds \int_a^b \frac{|f'(t)|^2}{w(t)} dt.
\end{aligned}$$

4. Opial Type Inequalities for Complex Integral

We have the following Wirtinger type inequality for complex functions:

Theorem 4. *Let f be analytic in G, a domain of complex numbers and suppose $\gamma \subset G$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ from $z(a) = u$ to $z(b) = w$ and $z'(t) \neq 0$ for $t \in (a,b)$.*

(i) *If $f(u) = 0$ or $f(w) = 0$, then*

$$\int_\gamma |f(z) f'(z)| |dz| \leq \left(\int_\gamma \ell(\gamma_{u,z}) |f'(z)|^2 |dz| \right)^{1/2} \left(\int_\gamma \ell(\gamma_{z,w}) |f'(z)|^2 |dz| \right)^{1/2} \leq \frac{1}{2} \ell(\gamma_{u,w}) \int_\gamma |f'(z)|^2 |dz| . \tag{4.1}$$

(ii) *If $f(u) = f(w) = 0$, then*

$$\int_\gamma |f(z) f'(z)| |dz| \leq \frac{1}{2} \left[\int_\gamma \left(\ell(\gamma_{u,w}) - \left| \ell(\gamma_{u,z}) - \ell(\gamma_{z,w}) \right| \right) |f'(z)|^2 |dz| \right]^{1/2} \times \left[\int_\gamma \left| \ell(\gamma_{u,z}) - \ell(\gamma_{z,w}) \right| |f'(z)|^2 |dz| \right]^{1/2} \leq \frac{1}{4} \ell(\gamma_{u,w}) \int_\gamma |f'(z)|^2 |dz| . \tag{4.2}$$

Proof. (i) Consider the function $h(t) = f(z(t))$ and $w(t) = |z'(t)|$, $t \in [a,b]$. Then $h'(t) = (f(z(t)))' = f'(z(t)) z'(t)$ for $t \in (a,b)$. Also $h(a) = f(z(a)) = f(u) = 0$ or $h(b) = f(z(b)) = f(w) = 0$. By utilising the inequality (3.6) we get

$$\int_a^b |f(z(t)) f'(z(t)) z'(t)| dt \leq \left(\int_a^b \left(\int_a^t |z'(s)| ds \right) \frac{|f'(z(t)) z'(t)|^2}{|z'(t)|} dt \right)^{1/2} \times \left(\int_a^b \left(\int_t^b |z'(s)| ds \right) \frac{|f'(z(t)) z'(t)|^2}{|z'(t)|} dt \right)^{1/2} \leq \frac{1}{2} \int_a^b |z'(s)| ds \int_a^b \frac{|f'(z(t)) z'(t)|^2}{|z'(t)|} dt. \tag{4.3}$$

Since

$$\int_a^b |f(z(t)) f'(z(t)) z'(t)| dt = \int_\gamma |f(z) f'(z)| |dz| ,$$

$$\int_a^b \left(\int_a^t |z'(s)| ds \right) \frac{|f'(z(t)) z'(t)|^2}{|z'(t)|} dt = \int_\gamma \ell(\gamma_{u,z}) |f'(z)|^2 |dz, |$$

$$\int_a^b \left(\int_t^b |z'(s)|\, ds \right) \frac{|f'(z(t))\, z'(t)|^2}{|z'(t)|} dt = \int_\gamma \ell(\gamma_{z,w})\, |f'(z)|^2\, |dz|$$

and

$$\int_a^b \frac{|f'(z(t))\, z'(t)|^2}{|z'(t)|} dt = \int_\gamma |f'(z)|^2\, |dz|$$

hence by (4.3) we get the desired result (4.1).

(ii) Follows in a similar way from (4.2). □

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