AN EXTENSION OF OPIAL'S INEQUALITY TO THE COMPLEX INTEGRAL

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Abstract. In this paper we establish an extension of Opial inequality to the case of complex integral of analytic functions.

1. Introduction

We recall the following Opial type inequalities:

Theorem 1. Assume that \( u : [a, b] \subset \mathbb{R} \to \mathbb{R} \) is an absolutely continuous function on the interval \([a, b]\) and such that \( u' \in L^2[a, b] \).

(i) If \( u(a) = u(b) = 0 \), then

\[
\int_a^b |u(t)u'(t)| \, dt \leq \frac{1}{4} (b - a) \int_a^b |u'(t)|^2 \, dt,
\]

with equality if and only if

\[
u(t) = \begin{cases} 
  c(t - a) & \text{if } a \leq t \leq \frac{a + b}{2}, \\
  c(b - t) & \text{if } \frac{a + b}{2} < t \leq b,
\end{cases}
\]

where \( c \) is an arbitrary constant;

(ii) If \( u(a) = 0 \), then

\[
\int_a^b |u(t)u'(t)| \, dt \leq \frac{1}{2} (b - a) \int_a^b |u'(t)|^2 \, dt,
\]

with equality if and only if \( u(t) = c(t - a) \) for some constant \( c \).

The inequality (1.1) was obtained by Olech in [9] in which he gave a simplified proof of an inequality originally due to Zdzislaw Opial [10].

Embedded in Olech’s proof is the half-interval form of Opial’s inequality, also discovered by Beesack [1], which is satisfied by those \( u \) vanishing only at \( a \).

For various proofs of the above inequalities, see [5]-[8] and [12].

In order to extend this result for the complex integral, we need some preparations as follows.

Suppose \( \gamma \) is a smooth path parametrized by \( z(t), t \in [a, b] \) and \( f \) is a complex function which is continuous on \( \gamma \). Put \( z(a) = u \) and \( z(b) = w \) with \( u, w \in \mathbb{C} \). We define the integral of \( f \) on \( \gamma_{u,w} = \gamma \) as

\[
\int_{\gamma} f(z) \, dz = \int_{\gamma_{u,w}} f(z) \, dz := \int_a^b f(z(t)) z'(t) \, dt.
\]
We observe that that the actual choice of parametrization of \( \gamma \) does not matter. This definition immediately extends to paths that are piecewise smooth. Suppose \( \gamma \) is parametrized by \( z(t), t \in [a, b] \), which is differentiable on the intervals \([a, c]\) and \([c, b]\), then assuming that \( f \) is continuous on \( \gamma \) we define

\[
\int_{\gamma_{u,w}} f(z) \, dz := \int_{\gamma_{u,v}} f(z) \, dz + \int_{\gamma_{v,w}} f(z) \, dz
\]

where \( v := z(c) \). This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

\[
\int_{\gamma_{u,w}} f(z) |dz| := \int_{a}^{b} |f(z(t))| |z'(t)| \, dt
\]

and the length of the curve \( \gamma \) is then

\[
\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_{a}^{b} |z'(t)| \, dt.
\]

Let \( f \) and \( g \) be holomorphic in \( G \), an open domain and suppose \( \gamma \subset G \) is a piecewise smooth path from \( z(a) = u \) to \( z(b) = w \). Then we have the integration by parts formula

\[
(1.3) \quad \int_{\gamma_{u,w}} f(z) g'(z) \, dz = f(w) g(w) - f(u) g(u) - \int_{\gamma_{u,w}} f'(z) g(z) \, dz.
\]

We recall also the triangle inequality for the complex integral, namely

\[
(1.4) \quad \left| \int_{\gamma} f(z) \, dz \right| \leq \int_{\gamma} |f(z)| \, |dz| \leq \|f\|_{\gamma,\infty} \ell(\gamma)
\]

where \( \|f\|_{\gamma,\infty} := \sup_{z \in \gamma} |f(z)| \).

We also define the \( p \)-norm with \( p \geq 1 \) by

\[
\|f\|_{\gamma,p} := \left( \int_{\gamma} |f(z)|^p \, |dz| \right)^{1/p}.
\]

For \( p = 1 \) we have

\[
\|f\|_{\gamma,1} := \int_{\gamma} |f(z)| \, |dz|.
\]

If \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then by Hölder’s inequality we have

\[
\|f\|_{\gamma,1} \leq [\ell(\gamma)]^{1/q} \|f\|_{\gamma,p}.
\]

2. Some Preliminary Results

We have the following refinement and generalization for complex valued function of the Opial inequality:

**Theorem 2.** Assume that \( f : [a, b] \to \mathbb{C} \) are absolutely continuous on \([a, b]\) and \( f' \in L^2[a, b] \).
(i) If either \( f(a) = 0 \) or \( f(b) = 0 \), then

\[
\int_a^b |f'(t) f(t)| \, dt \leq \left( \int_a^b (t-a) |f'(t)|^2 \, dt \right)^{1/2} \left( \int_a^b (b-t) |f'(t)|^2 \, dt \right)^{1/2} \leq \frac{1}{2} (b-a) \int_a^b |f'(t)|^2 \, dt.
\]

(ii) If \( f(a) = f(b) = 0 \), then

\[
\int_a^b |f'(t) f(t)| \, dt \leq \left[ \int_a^b K(t) |f'(t)|^2 \, dt \right]^{1/2} \left[ \int_a^b \frac{|a+b|}{2} - t \left| f'(t) \right|^2 \, dt \right]^{1/2} \leq \frac{1}{4} (b-a) \int_a^b |f'(t)|^2 \, dt,
\]

where

\[
K(t) := \begin{cases} 
    t - a & \text{if } a \leq t \leq \frac{a+b}{2}, \\
    b - t & \text{if } \frac{a+b}{2} < t \leq b.
\end{cases}
\]

**Proof.** (i) Since \( f(a) = 0 \), then \( f(t) = \int_a^t f'(s) \, ds \) for \( t \in [a, b] \). We have

\[
\int_a^b |f'(t) f(t)| \, dt = \int_a^b |f'(t)||f(t)| \, dt = \int_a^b (t-a)^{1/2} |f'(t)| (t-a)^{-1/2} |f(t)| \, dt = \int_a^b (t-a)^{1/2} |f'(t)| (t-a)^{-1/2} \left| \int_a^t f'(s) \, ds \right| \, dt = A.
\]

Using Cauchy-Bunyakovsky-Schwarz (CBS) inequality, we have

\[
A \leq \left( \int_a^b \left[ (t-a)^{1/2} |f'(t)| \right]^2 \, dt \right)^{1/2} \times \left( \int_a^b \left[ (t-a)^{-1/2} \left| \int_a^t f'(s) \, ds \right| \right]^2 \, dt \right)^{1/2} = \left( \int_a^b (t-a) |f'(t)|^2 \, dt \right)^{1/2} \left( \int_a^b (t-a)^{-1} \left| \int_a^t f'(s) \, ds \right|^2 \, dt \right)^{1/2} =: B.
\]

By (CBS) inequality we also have

\[
(t-a)^{-1} \left| \int_a^t f'(s) \, ds \right|^2 \leq \int_a^t |f'(s)|^2 \, ds,
\]

which gives

\[
B \leq \left( \int_a^b (t-a) |f'(t)|^2 \, dt \right)^{1/2} \left( \int_a^b \left( \int_a^t |f'(s)|^2 \, ds \right) \, dt \right)^{1/2}.
\]
Using integration by parts, we have

$$\int_a^b \left( \int_a^t |f' (s)|^2 \, ds \right) \, dt = b \int_a^b |f' (s)|^2 \, ds - \int_a^b t |f' (t)|^2 \, dt = \int_a^b (b - t) |f' (t)|^2 \, dt$$

and by (2.4) we get the first inequality in (2.1).

The last part follows by the elementary inequality

$$(2.6) \quad \sqrt{\alpha \beta} \leq \frac{1}{2} (\alpha + \beta), \quad \alpha, \beta \geq 0.$$  

(ii) If we write the inequality (2.1) on the interval $[a, \frac{a+b}{2}]$, we have

$$(2.7) \quad \int_a^{a+b} |f' (t) f (t)| \, dt \leq \left( \int_a^{a+b} (t - a) |f' (t)|^2 \, dt \right)^{1/2} \left( \int_a^{a+b} \left( \frac{a + b}{2} - t \right) |f' (t)|^2 \, dt \right)^{1/2}$$

and if we write the inequality (2.1) on the interval $[a+b, b]$, we have

$$(2.8) \quad \int_{a+b}^b |f' (t) f (t)| \, dt \leq \left( \int_{a+b}^b (b - t) |f' (t)|^2 \, dt \right)^{1/2} \left( \int_{a+b}^b \left( t - \frac{a + b}{2} \right) |f' (t)|^2 \, dt \right)^{1/2}.$$  

If we add the inequalities (2.7) and (2.8) we get

$$\int_a^b |f' (t) f (t)| \, dt \leq \left( \int_a^{a+b} (t - a) |f' (t)|^2 \, dt \right)^{1/2} \left( \int_a^{a+b} \left( \frac{a + b}{2} - t \right) |f' (t)|^2 \, dt \right)^{1/2} + \left( \int_{a+b}^b (b - t) |f' (t)|^2 \, dt \right)^{1/2} \left( \int_{a+b}^b \left( t - \frac{a + b}{2} \right) |f' (t)|^2 \, dt \right)^{1/2}$$

$$\leq \left[ \int_a^{a+b} (a+b) |f' (t)|^2 \, dt + \int_{a+b}^b (b-t) |f' (t)|^2 \, dt \right]^{1/2}$$

$$\times \left[ \int_a^{a+b} \left( \frac{a+b}{2} - t \right) |f' (t)|^2 \, dt + \int_{a+b}^b \left( t - \frac{a+b}{2} \right) |f' (t)|^2 \, dt \right]^{1/2}$$

$$= \left[ \int_a^b K(t) |f' (t)|^2 \, dt \right]^{1/2} \left[ \int_a^b \frac{a+b}{2} - t \right]^{1/2} |f' (t)|^2 \, dt \right]^{1/2},$$

where for the last inequality we used the elementary (CBS) inequality

$$\alpha \beta + \gamma \delta \leq (\alpha^2 + \gamma^2)^{1/2} (\beta^2 + \delta^2)^{1/2}, \quad \alpha, \beta, \gamma, \delta \geq 0.$$
The last part follows by (2.6), namely
\[
\left[ \int_a^b K(t) |f'(t)|^2 \, dt \right]^{1/2} \left[ \int_a^b \left| \frac{a + b}{2} - t \right| |f'(t)|^2 \, dt \right]^{1/2} \\
\leq \frac{1}{2} \left( \int_a^b K(t) |f'(t)|^2 \, dt + \int_a^b \left| \frac{a + b}{2} - t \right| |f'(t)|^2 \, dt \right) \\
= \frac{1}{2} \int_a^b \left[ K(t) + \left| \frac{a + b}{2} - t \right| \right] |f'(t)|^2 \, dt = \frac{1}{4} \int_a^b |f'(t)|^2 \, dt,
\]

since
\[
K(t) + \left| \frac{a + b}{2} - t \right| = \frac{1}{2} (b - a) \text{ for } t \in [a, b].
\]

\[\square\]

3. Weighted Inequalities

We also have the following composite inequality:

**Theorem 3.** Let \( g : [a, b] \to [g(a), g(b)] \) be a continuous strictly increasing function that is of class \( C^1 \) on \((a, b)\). Assume that \( f : [a, b] \subset \mathbb{R} \to \mathbb{C} \) is an absolutely continuous complex valued function on the interval \([a, b]\) and such that \( \frac{f'}{|g'|} \in L^2[a, b] \).

(i) If \( f(a) = 0 \) or then \( f(b) = 0 \), then

\[(3.1) \quad \int_a^b |f(t) f'(t)| \, dt \leq \left( \int_a^b (g(t) - g(a)) \left| \frac{f'(t)}{g'(t)} \right|^2 \, dt \right)^{1/2} \left( \int_a^b (g(b) - g(t)) \left| \frac{f'(t)}{g'(t)} \right|^2 \, dt \right)^{1/2} \]

\[ \leq \frac{1}{2} |g(b) - g(a)| \int_a^b |f'(t)|^2 \, dt. \]

(ii) If \( f(a) = f(b) = 0 \), then

\[(3.2) \quad \int_a^b |f(t) f'(t)| \, dt \leq \left[ \int_a^b \left( \frac{1}{2} (g(b) - g(a)) - \frac{g(a) + g(b)}{2} - g(t) \right) \left| \frac{f'(t)}{g'(t)} \right|^2 \, dt \right]^{1/2} \]

\[ \times \left[ \int_a^b \left| \frac{g(a) + g(b)}{2} - g(t) \right| \left| \frac{f'(t)}{g'(t)} \right|^2 \, dt \right]^{1/2} \]

\[ \leq \frac{1}{4} |g(b) - g(a)| \int_a^b |f'(t)|^2 \, dt. \]

**Proof.** (i) Consider the function \( u := f \circ g^{-1} : [g(a), g(b)] \to \mathbb{R} \). The function \( u \) is absolutely continuous on \([g(a), g(b)]\), \( u(g(a)) = f \circ g^{-1}(g(a)) = f(a) = 0 \) or \( u(g(b)) = f \circ g^{-1}(g(b)) = f(b) = 0 \).
Using the chain rule and the derivative of inverse functions we have

\[(f \circ g^{-1})' (z) = (f' \circ g^{-1}) (z) (g^{-1})' (z) = \frac{(f' \circ g^{-1}) (z)}{(g' \circ g^{-1}) (z)}\]

for almost every (a.e.) \( z \in [g(a), g(b)] \).

If we apply the inequality (2.1) for the function \( u = f \circ g^{-1} \) on the interval \([g(a), g(b)]\), then we get

\[(3.4) \quad \int_{g(a)}^{g(b)} f \circ g^{-1} (z) \left( \frac{f' \circ g^{-1}}{g' \circ g^{-1}} \right) (z) dz \]

\[\leq \left( \int_{g(a)}^{g(b)} (z - g(a)) \left( \frac{f' \circ g^{-1}}{g' \circ g^{-1}} \right) (z)^2 dz \right)^{1/2} \]
\[\times \left( \int_{g(a)}^{g(b)} (g(b) - z) \left( \frac{f' \circ g^{-1}}{g' \circ g^{-1}} \right) (z)^2 dz \right)^{1/2} \]
\[\leq \frac{1}{2} [g(b) - g(a)] \int_{g(a)}^{g(b)} \left( \frac{f' \circ g^{-1}}{g' \circ g^{-1}} \right) (z)^2 dz.\]

If we make the change of variable \( t = g^{-1}(z), z \in [g(a), g(b)] \), then \( z = g(t), dz = g'(t) \, dt \),

\[\int_{g(a)}^{g(b)} \left| f \circ g^{-1} (z) \left( \frac{f' \circ g^{-1}}{g' \circ g^{-1}} \right) (z) \right| dz = \int_{a}^{b} \left| f (t) \frac{f'(t)}{g'(t)} \right| g'(t) \, dt \]
\[= \int_{a}^{b} |f (t) f'(t)| \, dt, \]
\[\int_{g(a)}^{g(b)} (z - g(a)) \left( \frac{f' \circ g^{-1}}{g' \circ g^{-1}} \right) (z) dz = \int_{a}^{b} (g(t) - g(a)) \left| \frac{f'(t)}{g'(t)} \right|^2 g'(t) \, dt \]
\[= \int_{a}^{b} (g(t) - g(a)) \frac{|f'(t)|^2}{g'(t)} \, dt \]
\[\int_{g(a)}^{g(b)} (g(b) - z) \left( \frac{f' \circ g^{-1}}{g' \circ g^{-1}} \right) (z) dz = \int_{g(a)}^{g(b)} (g(b) - g(t)) \left| \frac{f'(t)}{g'(t)} \right|^2 g'(t) \, dt \]
\[= \int_{a}^{b} (g(b) - g(t)) \frac{|f'(t)|^2}{g'(t)} \, dt \]

and

\[\int_{g(a)}^{g(b)} \left( \frac{f' \circ g^{-1}}{g' \circ g^{-1}} \right) (z) dz = \int_{a}^{b} \left| \frac{f'(t)}{g'(t)} \right|^2 g'(t) \, dt = \int_{a}^{b} \frac{|f'(t)|^2}{g'(t)} \, dt.\]

By utilising (3.4), we then get the desired inequality (3.1).
If we make the change of variable \( g^{-1} \), then we get

\[
\int_{g(a)}^{g(b)} \left| f \circ g^{-1}(z) \frac{(f' \circ g^{-1})(z)}{(g' \circ g^{-1})(z)} \right| dz
\]

\[
\leq \int_{g(a)}^{g(b)} \left( \frac{1}{2} (g(b) - g(a)) - \left| \frac{g(a) + g(b)}{2} - z \right| \right) \frac{(f' \circ g^{-1})(z)}{(g' \circ g^{-1})(z)} dz \]

\[
\times \left[ \int_{a}^{b} \left| \frac{g(a) + g(b)}{2} - z \right| \left| \frac{(f' \circ g^{-1})(z)}{(g' \circ g^{-1})(z)} \right|^2 dz \right]^{1/2}
\]

\[
\leq \frac{1}{4} |g(b) - g(a)| \int_{g(a)}^{g(b)} \left| \frac{(f' \circ g^{-1})(z)}{(g' \circ g^{-1})(z)} \right|^2 dz.
\]

If we make the change of variable \( t = g^{-1}(z) \), \( z \in [g(a), g(b)] \), then by (3.5) we get the desired result (3.2).

If \( w : [a, b] \to \mathbb{R} \) is continuous and positive on the interval \([a, b]\), then the function \( W : [a, b] \to [0, \infty) \), \( W(x) := \int_{a}^{x} w(s) ds \) is strictly increasing and differentiable on \((a, b)\). We have \( W'(x) = w(x) \) for any \( x \in (a, b) \).

**Corollary 1.** Assume that \( w : [a, b] \to (0, \infty) \) is continuous on \([a, b]\) and that \( f : [a, b] \subset \mathbb{R} \to \mathbb{C} \) is an absolutely continuous complex valued function on the interval \([a, b]\) and such that \( \frac{f'}{w^{1/2}} \in L^2[a, b] \).

(i) If \( f(a) = 0 \) or \( f(b) = 0 \), then

\[
\int_{a}^{b} |f(t) f'(t)| dt
\]

\[
\leq \left( \int_{a}^{b} \left( \int_{a}^{t} w(s) ds \right) \left| f'(t) \right|^2 dt \right)^{1/2} \left( \int_{a}^{b} \left( \int_{t}^{b} w(s) ds \right) \left| f'(t) \right|^2 dt \right)^{1/2}
\]

\[
\leq \frac{1}{2} \int_{a}^{b} w(s) ds \int_{a}^{b} \left| f'(t) \right|^2 dt.
\]

(ii) If \( f(a) = f(b) = 0 \), then

\[
\int_{a}^{b} |f(t) f'(t)| dt
\]

\[
\leq \frac{1}{2} \left[ \int_{a}^{b} \left( \int_{a}^{b} w(s) ds - \left| \int_{a}^{t} w(s) ds - \int_{a}^{t} w(s) ds \right| \right) \left| f'(t) \right|^2 dt \right]^{1/2}
\]

\[
\times \left[ \int_{a}^{b} \left| \int_{t}^{b} w(s) ds - \int_{t}^{a} w(s) ds \right| \left| f'(t) \right|^2 dt \right]^{1/2}
\]

\[
\leq \frac{1}{4} \int_{a}^{b} w(s) ds \int_{a}^{b} \left| f'(t) \right|^2 dt.
\]
4. OPIAL TYPE INEQUALITIES FOR COMPLEX INTEGRAL

We have the following Wirtinger type inequality for complex functions:

**Theorem 4.** Let $f$ be analytic in $G$, a domain of complex numbers and suppose $\gamma \subset G$ is a smooth path parametrized by $z(t)$, $t \in [a, b]$ from $z(a) = u$ to $z(b) = w$ and $z'(t) \neq 0$ for $t \in (a, b)$.

(i) If $f(u) = 0$ or $f(w) = 0$, then

$$
\int_{\gamma} |f(z) f'(z)| |dz| 
\leq \left( \int_{\gamma} \ell (\gamma_{u,z}) |f'(z)|^2 |dz| \right)^{1/2} \left( \int_{\gamma} \ell (\gamma_{z,w}) |f'(z)|^2 |dz| \right)^{1/2} 
\leq \frac{1}{2} \ell (\gamma_{u,w}) \int_{\gamma} |f'(z)|^2 |dz|.
$$

(ii) If $f(u) = f(w) = 0$, then

$$
\int_{\gamma} |f(z) f'(z)| |dz| 
\leq \frac{1}{2} \left[ \int_{\gamma} (\ell (\gamma_{u,w}) - |\ell (\gamma_{u,z}) - \ell (\gamma_{z,w})|) |f'(z)|^2 |dz| \right]^{1/2} 
\times \left[ \int_{\gamma} |\ell (\gamma_{u,z}) - \ell (\gamma_{z,w})| |f'(z)|^2 |dz| \right]^{1/2} 
\leq \frac{1}{4} \ell (\gamma_{u,w}) \int_{\gamma} |f'(z)|^2 |dz|.
$$

**Proof.** (i) Consider the function $h(t) = f(z(t))$ and $w(t) = |z'(t)|$, $t \in [a, b]$. Then $h'(t) = (f(z(t)))' = f'(z(t)) z'(t)$ for $t \in (a, b)$. Also $h(a) = f(z(a)) = f(u) = 0$ or $h(b) = f(z(b)) = f(w) = 0$. By utilising the inequality (3.6) we get

$$
\int_{a}^{b} |f(z(t)) f'(z(t)) z'(t)| dt
\leq \left( \int_{a}^{b} \left( \int_{a}^{t} |z'(s)| ds \right) \frac{|f'(z(t)) z'(t)|^2}{|z'(t)|} dt \right)^{1/2} 
\times \left( \int_{a}^{b} \left( \int_{t}^{b} |z'(s)| ds \right) \frac{|f'(z(t)) z'(t)|^2}{|z'(t)|} dt \right)^{1/2} 
\leq \frac{1}{2} \int_{a}^{b} |z'(s)| ds \int_{a}^{b} \frac{|f'(z(t)) z'(t)|^2}{|z'(t)|} dt.
$$

Since

$$
\int_{a}^{b} |f(z(t)) f'(z(t)) z'(t)| dt = \int_{\gamma} |f(z) f'(z)| |dz|,
$$

$$
\int_{a}^{b} \left( \int_{a}^{t} |z'(s)| ds \right) \frac{|f'(z(t)) z'(t)|^2}{|z'(t)|} dt = \int_{\gamma} \ell (\gamma_{u,z}) |f'(z)|^2 |dz|,
$$
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\[
\int_a^b \left( \int_t^b \left| z'(s) \right| ds \right) \frac{\left| f'(z(t)) z'(t) \right|^2}{|z'(t)|} dt = \int_\gamma \ell'(z,w) |f'(z)|^2 |dz|
\]

and

\[
\int_a^b \frac{\left| f'(z(t)) z'(t) \right|^2}{|z'(t)|} dt = \int_\gamma |f'(z)|^2 |dz|
\]

hence by (4.3) we get the desired result (4.1).

(ii) Follows in a similar way from (4.2).

References


[4] G. Grüss, Über das Maximum des absoluten Betrages von \[\frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx,\] Math. Z., 39(1935), 215-226.


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