EXTENSIONS OF STEKLOFF AND ALMANSI INEQUALITIES TO THE COMPLEX INTEGRAL

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Abstract. In this paper we establish some extensions of Stekloff and Almansi inequalities to the complex integral. Applications for bounding the complex *Čebyšev functional* are also given.

1. Introduction

It is well known that, see for instance [5], or [9], if $u \in C^1([a,b],\mathbb{R})$, namely u is continuous on $[a,b]$ and has a derivative that is continuous on (a,b) and satisfies $u(a)=u(b)=0$, then the following *Wirtinger type inequality* is valid

$$\int_a^b u^2(t)\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.1}$$

with the equality holding if and only if $u(t)=K\sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant $K\in\mathbb{R}$.

If $u \in C^1([a,b],\mathbb{R})$ satisfies the condition $u(a)=0$, then also

$$\int_a^b u^2(t)\,dt \le \frac{4(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.2}$$

and the equality holds if and only if $u(t)=L\sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant $L\in\mathbb{R}$.

For some related Wirtinger type integral inequalities see [1], [3], [5] and [8]-[11].

In 1901, W. Stekloff, [13], proved that, if $u\in C^1([a,b],\mathbb{R})$ and $\int_a^b u(t)\,dt=0$, then

$$\int_a^b u^2(x)\,dx \le \frac{(b-a)^2}{\pi^2}\int_a^b [u'(x)]^2\,dx. \tag{1.3}$$

In addition, if $u(a)=u(b)$, then, as proved by E. Almansi in 1905, [1], the inequality (1.3) can be improved as follows

$$\int_a^b u^2(x)\,dx \le \frac{(b-a)^2}{4\pi^2}\int_a^b [u'(x)]^2\,dx. \tag{1.4}$$

We can state the following result for complex functions $h:[a,b]\to\mathbb{C}$.

Theorem 1. *If $h\in C^1([a,b],\mathbb{C})$ and $\int_a^b h(t)\,dt=0$, then*

$$\int_a^b |h(x)|^2\,dx \le \frac{(b-a)^2}{\pi^2}\int_a^b |h'(x)|^2\,dx. \tag{1.5}$$

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In addition, if $h(a) = h(b)$, *then*

$$\int_a^b |h(x)|^2 \, dx \le \frac{(b-a)^2}{4\pi^2} \int_a^b |h'(x)|^2 \, dx. \tag{1.6}$$

The proof follows by (1.3) and (1.4) applied for $u = \operatorname{Re} h$ and $u = \operatorname{Im} h$ and by adding the corresponding inequalities.

In order to extend this result for the complex integral, we need some preparations as follows.

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a, b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z)\, dz = \int_{\gamma_{u,w}} f(z)\, dz := \int_a^b f(z(t))\, z'(t)\, dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a, b]$, which is differentiable on the intervals $[a, c]$ and $[c, b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\, dz := \int_{\gamma_{u,v}} f(z)\, dz + \int_{\gamma_{v,w}} f(z)\, dz$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\, |dz| := \int_a^b f(z(t))\, |z'(t)|\, dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\, dt.$$

Let f and g be holomorphic in G, and open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\, g'(z)\, dz = f(w)\, g(w) - f(u)\, g(u) - \int_{\gamma_{u,w}} f'(z)\, g(z)\, dz. \tag{1.7}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left| \int_\gamma f(z)\, dz \right| \le \int_\gamma |f(z)|\, |dz| \le \|f\|_{\gamma,\infty}\, \ell(\gamma) \tag{1.8}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \ge 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p\, |dz| \right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)|\, |dz|.$$

If $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1}\leq\left[\ell\left(\gamma\right)\right]^{1/q}\|f\|_{\gamma,p}.$$

In this paper we establish some extensions of Stekloff and Almansi inequalities to the complex integral. Applications for bounding the complex *Čebyšev functional* are also given.

2. Some Preliminary Facts

We have:

Theorem 2. *Let $g:[a,b]\rightarrow[g(a),g(b)]$ be a continuous strictly increasing function that is of class C^1 on (a,b).*

(i) *If $f\in C^1([a,b],\mathbb{C})$ with $\frac{f'}{\sqrt{g'(t)}}\in L_2[a,b]$ and $\int_a^b f(t)g'(t)\,dt=0$, then*

$$\int_a^b |f(t)|^2 g'(t)\,dt\leq\frac{[g(b)-g(a)]^2}{\pi^2}\int_a^b\frac{|f'(t)|^2}{g'(t)}dt. \tag{2.1}$$

(ii) *In addition, if $f(a)=f(b)$, then we have the better inequality*

$$\int_a^b |f(t)|^2 g'(t)\,dt\leq\frac{[g(b)-g(a)]^2}{4\pi^2}\int_a^b\frac{|f'(t)|^2}{g'(t)}dt. \tag{2.2}$$

Proof. (i) We write the inequality (1.5) for the function $h=f\circ g^{-1}$ on the interval $[g(a),g(b)]$ to get

$$\int_{g(a)}^{g(b)}\left|\left(f\circ g^{-1}\right)(z)\right|^2dz\leq\frac{(g(b)-g(a))^2}{\pi^2}\int_{g(a)}^{g(b)}\left|\left(f\circ g^{-1}\right)'(z)\right|^2dz, \tag{2.3}$$

provided

$$\int_{g(a)}^{g(b)}f\circ g^{-1}(z)\,dz=0.$$

If $f:[c,d]\rightarrow\mathbb{C}$ is absolutely continuous on $[c,d]$, then $f\circ g^{-1}:[g(c),g(d)]\rightarrow\mathbb{C}$ is absolutely continuous on $[g(c),g(d)]$ and using the chain rule and the derivative of inverse functions we have

$$\left(f\circ g^{-1}\right)'(z)=\left(f'\circ g^{-1}\right)(z)\left(g^{-1}\right)'(z)=\frac{\left(f'\circ g^{-1}\right)(z)}{\left(g'\circ g^{-1}\right)(z)} \tag{2.4}$$

for almost every (a.e.) $z\in[g(c),g(d)]$.

Using the inequality (2.3) we then get

$$\int_{g(a)}^{g(b)}\left|\left(f\circ g^{-1}\right)(z)\right|^2dz\leq\frac{(g(b)-g(a))^2}{\pi^2}\int_{g(a)}^{g(b)}\left|\frac{\left(f'\circ g^{-1}\right)(z)}{\left(g'\circ g^{-1}\right)(z)}\right|^2dz, \tag{2.5}$$

provided $\int_{g(a)}^{g(b)}f\circ g^{-1}(z)\,dz=0$.

Observe also that, by the change of variable $t=g^{-1}(z)$, $z\in[g(a),g(b)]$, we have $z=g(t)$ that gives $dz=g'(t)\,dt$,

$$\int_{g(a)}^{g(b)}f\circ g^{-1}(z)\,dx=\int_a^b f(t)\,h'(t)\,dt,$$

and

$$\int_{g(a)}^{g(b)} \left|\left(f\circ g^{-1}\right)(z)\right|^2 dz = \int_a^b |f(t)|^2 g'(t)\,dt. \tag{2.6}$$

We also have

$$\int_{g(a)}^{g(b)} \left|\frac{\left(f'\circ g^{-1}\right)(z)}{\left(g'\circ g^{-1}\right)(z)}\right|^2 dz = \int_a^b \left|\frac{f'(t)}{g'(t)}\right|^2 g'(t)\,dt = \int_a^b \frac{|f'(t)|^2}{g'(t)}dt.$$

By making use of (2.5) we get (2.1).

(ii) The inequality (2.2) follows by (2.2) in a similar way. □

If $w:[a,b]\to\mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W:[a,b]\to[0,\infty)$, $W(x):=\int_a^x w(s)\,ds$ is strictly increasing and differentiable on (a,b). We have $W'(x)=w(x)$ for any $x\in(a,b)$.

Corollary 1. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ and $f\in C^1([a,b],\mathbb{C})$.*

(i) *If $\frac{f'}{\sqrt{w}}\in L_2[a,b]$ and $\int_a^b f(t)\,w(t)\,dt=0$, then*

$$\int_a^b |f(t)|^2 w(t)\,dt \le \frac{1}{\pi^2}\left(\int_a^b w(s)\,ds\right)^2 \int_a^b \frac{|f'(t)|^2}{w(t)}dt. \tag{2.7}$$

(ii) *In addition, if $f(a)=f(b)$, then we have the better inequality*

$$\int_a^b |f(t)|^2 w(t)\,dt \le \frac{1}{4\pi^2}\left(\int_a^b w(s)\,ds\right)^2 \int_a^b \frac{|f'(t)|^2}{w(t)}dt. \tag{2.8}$$

3. Inequalities for Complex Integral

We have the following extensions of Stekloff and Almansi inequalities to the complex integral:

Theorem 3. *Let f be analytic in G, a domain of complex numbers and suppose $\gamma\subset G$ is a smooth path parametrized by $z(t)$, $t\in[a,b]$ from $z(a)=u$ to $z(b)=w$ and $z'(t)\neq 0$ for $t\in(a,b)$.*

(i) *If $\int_\gamma f(z)\,|dz|=0$, then*

$$\int_\gamma |f(z)|^2 |dz| \le \frac{1}{\pi^2}\ell^2(\gamma)\int_\gamma |f'(z)|^2 |dz|. \tag{3.1}$$

(ii) *In addition, if $f(u)=f(w)=0$, then*

$$\int_\gamma |f(z)|^2 |dz| \le \frac{1}{4\pi^2}\ell^2(\gamma)\int_\gamma |f'(z)|^2 |dz|. \tag{3.2}$$

Proof. (i) Consider the function $h(t)=f(z(t))$ and $w(t)=|z'(t)|$, $t\in[a,b]$. Then $h'(t)=(f(z(t)))'=f'(z(t))\,z'(t)$ for $t\in(a,b)$. Also

$$\int_a^b f'(z(t))\,|z'(t)|\,dt = \int_\gamma f(z)\,|dz| = 0.$$

By utilising the inequality (2.7) for these choices, we get

$$\begin{aligned}\int_a^b |f(z(t))|^2 |z'(t)|\,dt &\le \frac{1}{\pi^2}\left(\int_a^b |z'(s)|\,ds\right)^2 \int_a^b \frac{|f'(z(t))\,z'(t)|^2}{|z'(t)|}dt\\ &= \frac{1}{\pi^2}\left(\int_a^b |z'(s)|\,ds\right)^2 \int_a^b \frac{|f'(z(t))|^2\,|z'(t)|^2}{|z'(t)|}dt\\ &= \frac{1}{\pi^2}\left(\int_a^b |z'(s)|\,ds\right)^2 \int_a^b |f'(z(t))|^2\,|z'(t)|\,dt,\end{aligned}$$

which is equivalent to (3.1).

(ii) Follows by the corresponding result from Corollary 1. □

We have the following reverses of Schwarz inequality:

Corollary 2. *Let h be analytic in G, a domain of complex numbers and suppose $\gamma \subset G$ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ from $z(a) = u$ to $z(b) = w$ and $z'(t) \neq 0$ for $t \in (a,b)$. Then*

$$0 \le \frac{1}{\ell(\gamma)}\int_\gamma |h(z)|^2 |dz| - \left|\frac{1}{\ell(\gamma)}\int_\gamma h(z)\,|dz|\right|^2 \le \frac{1}{\pi^2}\ell(\gamma)\int_\gamma |h'(z)|^2 |dz|. \tag{3.3}$$

In addition, if $h(u) = h(w) = 0$, then

$$0 \le \frac{1}{\ell(\gamma)}\int_\gamma |h(z)|^2 |dz| - \left|\frac{1}{\ell(\gamma)}\int_\gamma h(z)\,|dz|\right|^2 \le \frac{1}{4\pi^2}\ell(\gamma)\int_\gamma |h'(z)|^2 |dz|. \tag{3.4}$$

Proof. First, observe that

$$\begin{aligned}&\frac{1}{\ell(\gamma)}\int_\gamma \left|h(z) - \frac{1}{\ell(\gamma)}\int_\gamma h(y)\,|dy|\right|^2 |dz| \qquad (3.5)\\ &= \frac{1}{\ell(\gamma)}\int_\gamma \left[|h(z)|^2 - 2\operatorname{Re}\left(h(z)\,\overline{\frac{1}{\ell(\gamma)}\int_\gamma h(y)\,|dy|}\right) + \left|\frac{1}{\ell(\gamma)}\int_\gamma h(y)\,|dy|\right|^2\right]|dz|\\ &= \frac{1}{\ell(\gamma)}\int_\gamma |h(z)|^2 |dz| - 2\operatorname{Re}\left(\frac{1}{\ell(\gamma)}\int_\gamma h(z)\,|dz|\,\overline{\frac{1}{\ell(\gamma)}\int_\gamma h(y)\,|dy|}\right)\\ &\quad + \left|\frac{1}{\ell(\gamma)}\int_\gamma h(y)\,|dy|\right|^2\\ &= \frac{1}{\ell(\gamma)}\int_\gamma |h(z)|^2 |dz| - 2\left|\frac{1}{\ell(\gamma)}\int_\gamma h(y)\,|dy|\right|^2 + \left|\frac{1}{\ell(\gamma)}\int_\gamma h(y)\,|dy|\right|^2\\ &= \frac{1}{\ell(\gamma)}\int_\gamma |h(z)|^2 |dz| - \left|\frac{1}{\ell(\gamma)}\int_\gamma h(y)\,|dy|\right|^2.\end{aligned}$$

Now, consider $f(z) := h(z) - \frac{1}{\ell(\gamma)}\int_\gamma h(y)\,|dy|$, $z \in G$. Then

$$\int_\gamma f(z)\,|dz| = \int_\gamma \left(h(z) - \frac{1}{\ell(\gamma)}\int_\gamma h(y)\,|dy|\right)|dz| = 0,$$

$f'(z) = h'(z)$ and by (3.1) we get

$$\int_\gamma \left| h(z) - \frac{1}{\ell(\gamma)} \int_\gamma h(y) |dy| \right|^2 |dz| \le \frac{1}{\pi^2} \ell^2(\gamma) \int_\gamma |h'(z)|^2 |dz|,$$

and by (3.5) we get the desired result (3.3).

The second part follows by (3.2). □

4. Complex Čebyšev Functional

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If f and g are continuous on γ, we consider the *complex Čebyšev functional* defined by

$$\mathcal{D}_\gamma(f,g) := \frac{1}{w-u} \int_\gamma f(z) g(z) dz - \frac{1}{w-u} \int_\gamma f(z) dz \frac{1}{w-u} \int_\gamma g(z) dz.$$

We start with the following identity of interest:

Lemma 1. *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If f and g are continuous on γ, then*

$$\begin{aligned} \mathcal{D}_\gamma(f,g) &= \frac{1}{2(w-u)^2} \int_\gamma \left(\int_\gamma (f(z) - f(w))(g(z) - g(w)) dw \right) dz \\ &= \frac{1}{2(w-u)^2} \int_\gamma \left(\int_\gamma (f(z) - f(w))(g(z) - g(w)) dz \right) dw \\ &= \frac{1}{2(w-u)^2} \int_\gamma \int_\gamma (f(z) - f(w))(g(z) - g(w)) dz dw. \end{aligned} \tag{4.1}$$

Proof. For any $z \in \gamma$ the integral $\int_\gamma (f(z) - f(w))(g(z) - g(w)) dw$ exists and

$$\begin{aligned} I(z) &:= \int_\gamma (f(z) - f(w))(g(z) - g(w)) dw \\ &= \int_\gamma (f(z) g(z) + f(w) g(w) - g(z) f(w) - f(z) g(w)) dw \\ &= f(z) g(z) \int_\gamma dw + \int_\gamma f(w) g(w) dw - g(z) \int_\gamma f(w) dw - f(z) \int_\gamma g(w) dw \\ &= (w-u) f(z) g(z) + \int_\gamma f(w) g(w) dw - g(z) \int_\gamma f(w) dw - f(z) \int_\gamma g(w) dw. \end{aligned}$$

The function $I(z)$ is also continuous on γ, then the integral $\int_\gamma I(z)\,dz$ exists and

$$\begin{aligned}\int_\gamma I(z)\,dz &= \int_\gamma \Big[(w-u) f(z) g(z) + \int_\gamma f(w) g(w)\,dw \\ &\qquad - g(z)\int_\gamma f(w)\,dw - f(z)\int_\gamma g(w)\,dw\Big]\,dz \\ &= (w-u)\int_\gamma f(z) g(z)\,dz + (w-u)\int_\gamma f(w) g(w)\,dw \\ &\qquad - \int_\gamma f(w)\,dw \int_\gamma g(z)\,dz - \int_\gamma g(w)\,dw\int_\gamma f(z)\,dz \\ &= 2(w-u)\int_\gamma f(z) g(z)\,dz - 2\int_\gamma f(z)\,dz\int_\gamma g(z)\,dz = 2(w-u)^2\,\mathcal{P}_\gamma(f,g),\end{aligned}$$

which proves the first equality in (4.1).

The rest follows in a similar manner and we omit the details. □

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $f : \gamma \to \mathbb{C}$ a continuous function on γ. Define the quantity:

$$\begin{aligned}\mathcal{P}_\gamma\left(f,\overline{f}\right) &= \frac{1}{\ell(\gamma)}\int_\gamma |f(z)|^2\,|dz| - \left|\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right|^2 \\ &= \frac{1}{\ell(\gamma)}\int_\gamma \left|f(v) - \frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right|^2 |dv| \geq 0.\end{aligned} \tag{4.2}$$

We have:

Theorem 4. *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If f and g are continuous on γ, then*

$$|\mathcal{D}_\gamma(f,g)| \leq \frac{\ell^2(\gamma)}{|w-u|^2}\left[\mathcal{P}_\gamma\left(f,\overline{f}\right)\right]^{1/2}\left[\mathcal{P}_\gamma(g,\overline{g})\right]^{1/2}. \tag{4.3}$$

Proof. Taking the modulus in the first equality in (4.1), we get

$$\begin{aligned}|\mathcal{D}_\gamma(f,g)| &= \frac{1}{2|w-u|^2}\left|\int_\gamma\left(\int_\gamma (f(z)-f(w))(g(z)-g(w))\,dw\right)dz\right| \\ &\leq \frac{1}{2|w-u|^2}\int_\gamma\left|\int_\gamma (f(z)-f(w))(g(z)-g(w))\,dw\right||dz| =: A.\end{aligned}$$

Using the Cauchy-Bunyakovsky-Schwarz integral inequality, we have

$$\begin{aligned}&\left|\int_\gamma (f(z)-f(w))(g(z)-g(w))\,dw\right| \\ &\qquad \leq \left(\int_\gamma |f(z)-f(w)|^2\,|dw|\right)^{1/2}\left(\int_\gamma |g(z)-g(w)|^2\,|dw|\right)^{1/2},\end{aligned}$$

which implies that

$$
\begin{aligned}
&A \\
&\leq \frac{1}{2|w-u|^2} \int_\gamma \left(\int_\gamma |f(z) - f(w)|^2 |dw| \right)^{1/2} \left(\int_\gamma |g(z) - g(w)|^2 |dw| \right)^{1/2} |dz| \\
&\qquad =: B.
\end{aligned}
$$

By the Cauchy-Bunyakovsky-Schwarz integral inequality, we also have

$$
\begin{aligned}
&\int_\gamma \left(\int_\gamma |f(z) - f(w)|^2 |dw| \right)^{1/2} \left(\int_\gamma |g(z) - g(w)|^2 |dw| \right)^{1/2} |dz| \\
&\leq \left(\int_\gamma \left[\left(\int_\gamma |f(z) - f(w)|^2 |dw| \right)^{1/2} \right]^2 |dz| \right)^{1/2} \\
&\quad \times \left(\int_\gamma \left[\left(\int_\gamma |g(z) - g(w)|^2 |dw| \right)^{1/2} \right]^2 |dz| \right)^{1/2} \\
&= \left(\int_\gamma \left(\int_\gamma |f(z) - f(w)|^2 |dw| \right) |dz| \right)^{1/2} \left(\int_\gamma \left(\int_\gamma |g(z) - g(w)|^2 |dw| \right) |dz| \right)^{1/2},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
B \leq \frac{1}{2|w-u|^2} &\left(\int_\gamma \left(\int_\gamma |f(z) - f(w)|^2 |dw| \right) |dz| \right)^{1/2} \\
&\times \left(\int_\gamma \left(\int_\gamma |g(z) - g(w)|^2 |dw| \right) |dz| \right)^{1/2}.
\end{aligned} \tag{4.4}
$$

Now, observe that

$$
\begin{aligned}
&\int_\gamma \left(\int_\gamma |f(z) - f(w)|^2 |dw| \right) |dz| \\
&= \int_\gamma \left(\int_\gamma \left(|f(z)|^2 - 2\operatorname{Re}\left(f(z)\overline{f(w)} \right) + |f(w)|^2 \right) |dw| \right) |dz| \\
&= \int_\gamma \left(\ell(\gamma) |f(z)|^2 - 2\operatorname{Re}\left(f(z) \int_\gamma \overline{f(w)} |dw| \right) + \int_\gamma |f(w)|^2 |dw| \right) |dz| \\
&= \ell(\gamma) \int_\gamma |f(z)|^2 |dz| - 2\operatorname{Re}\left(\int_\gamma f(z) |dz| \int_\gamma \overline{f(w)} |dw| \right) + \ell(\gamma) \int_\gamma |f(w)|^2 |dw| \\
&= 2\ell(\gamma) \int_\gamma |f(z)|^2 |dz| - 2\operatorname{Re}\left(\int_\gamma f(z) |dz| \overline{\left(\int_\gamma f(w) |dw| \right)} \right) \\
&= 2\left[\ell(\gamma) \int_\gamma |f(z)|^2 |dz| - \left| \int_\gamma f(z) |dz| \right|^2 \right] = 2\ell^2(\gamma) \mathcal{P}_\gamma\left(f, \overline{f}\right)
\end{aligned} \tag{4.5}
$$

and, similarly

$$
\int_\gamma \left(\int_\gamma |g(z) - g(w)|^2 |dw| \right) |dz| = 2\ell^2(\gamma) \mathcal{P}_\gamma(g, \overline{g}). \tag{4.6}
$$

Making use of (4.5) and (4.6), we get

$$B \leq \frac{1}{2\left|w-u\right|^{2}}\left[2\ell^{2}\left(\gamma\right)\mathcal{P}_{\gamma}\left(f,\overline{f}\right)\right]^{1/2}\left[2\ell^{2}\left(\gamma\right)\mathcal{P}_{\gamma}\left(g,\overline{g}\right)\right]^{1/2} = \frac{\ell^{2}\left(\gamma\right)}{\left|w-u\right|^{2}}\left[\mathcal{P}_{\gamma}\left(f,\overline{f}\right)\right]^{1/2}\left[\mathcal{P}_{\gamma}\left(g,\overline{g}\right)\right]^{1/2},$$

which proves the desired result (4.3). □

Remark 1. *For $g=f$ we have*

$$\mathcal{D}_{\gamma}\left(f,f\right)=\frac{1}{w-u}\int_{\gamma}f^{2}\left(z\right)dz-\left(\frac{1}{w-u}\int_{\gamma}f\left(z\right)dz\right)^{2} \tag{4.7}$$

and by (4.3) we get

$$\left|\mathcal{D}_{\gamma}\left(f,f\right)\right| \leq \frac{\ell^{2}\left(\gamma\right)}{\left|w-u\right|^{2}}\mathcal{P}_{\gamma}\left(f,\overline{f}\right). \tag{4.8}$$

For $g=\bar{f}$ we have

$$\mathcal{D}_{\gamma}\left(f,\bar{f}\right)=\frac{1}{w-u}\int_{\gamma}\left|f\left(z\right)\right|^{2}dz-\frac{1}{w-u}\int_{\gamma}f\left(z\right)dz\frac{1}{w-u}\int_{\gamma}\overline{f\left(z\right)}dz \tag{4.9}$$

and by (4.3) we get

$$\left|\mathcal{D}_{\gamma}\left(f,\bar{f}\right)\right| \leq \frac{\ell^{2}\left(\gamma\right)}{\left|w-u\right|^{2}}\mathcal{P}_{\gamma}\left(f,\overline{f}\right). \tag{4.10}$$

We have

Theorem 5. *Let f and g be analytic in G, a domain of complex numbers and suppose $\gamma \subset G$ is a smooth path parametrized by $z\left(t\right)$, $t\in\left[a,b\right]$ from $z\left(a\right)=u$ to $z\left(b\right)=w$ and $z'\left(t\right)\neq 0$ for $t\in\left(a,b\right)$. Then we have*

$$\left|\mathcal{D}_{\gamma}\left(f,g\right)\right| \leq \frac{1}{\pi^{2}}\epsilon^{2}\left(\gamma\right)\ell\left(\gamma\right)\begin{cases}\left(\int_{\gamma}\left|f'\left(z\right)\right|^{2}\left|dz\right|\right)^{1/2}\left(\int_{\gamma}\left|g'\left(z\right)\right|^{2}\left|dz\right|\right)^{1/2}\\[2ex] \frac{1}{2}\left(\int_{\gamma}\left|f'\left(z\right)\right|^{2}\left|dz\right|\right)^{1/2}\left(\int_{\gamma}\left|g'\left(z\right)\right|^{2}\left|dz\right|\right)^{1/2}\\ \text{if } f\left(u\right)=f\left(w\right),\\[2ex] \frac{1}{4}\left(\int_{\gamma}\left|f'\left(z\right)\right|^{2}\left|dz\right|\right)^{1/2}\left(\int_{\gamma}\left|g'\left(z\right)\right|^{2}\left|dz\right|\right)^{1/2}\\ \text{if } f\left(u\right)=f\left(w\right) \text{ and } g\left(u\right)=g\left(w\right),\end{cases} \tag{4.11}$$

where $\epsilon\left(\gamma\right):=\frac{\ell(\gamma)}{|w-u|}\geq 1$.

Proof. From (3.3) we have

$$0\leq \mathcal{P}_{\gamma}\left(f,\overline{f}\right)\leq \frac{1}{\pi^{2}}\ell\left(\gamma\right)\int_{\gamma}\left|f'\left(z\right)\right|^{2}\left|dz\right|$$

and

$$0\leq \mathcal{P}_{\gamma}\left(g,\overline{g}\right)\leq \frac{1}{\pi^{2}}\ell\left(\gamma\right)\int_{\gamma}\left|g'\left(z\right)\right|^{2}\left|dz\right|,$$

which together with the inequality (4.3), produce the first inequality in (4.11).

The rest follows in a similar way and we omit the details. □

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