ON SOME ψ CAPUTO FRACTIONAL ČEBYŠEV LIKE INEQUALITIES FOR FUNCTIONS OF TWO AND THREE VARIABLES

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Abstract. In this paper we obtain some ψ Caputo fractional Čebyšev like inequalities. Some new Čebyšev type inequalities involving functions of two and three variables using ψ Caputo fractional derivatives definition are obtained.

1. Introduction

P.L Čebyšev in the year 1882 has proved the following interesting inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)\right| \leq \frac{1}{12}(b-a)^2\left\|f'\right\|_\infty\left\|g'\right\|_\infty.$$

where f, g are absolutely continuous functions defined on $[a, b]$ and $f', g' \in L_\infty[a, b]$.

In last few decades many researchers have obtained various extensions and generalizations of above inequalities using various techniques see [9, 10]. Study of inequalities have attracted the attention of researchers from various fields due to its wide applications in various fields [5, 8].

During last few years the subject of Fractional Calculus has been developed rapidly due to the applications in various fields of science and engineering. Various new definitions of fractional derivatives and integrals have been obtained by various researchers depending on the applications such as Riemann liouville, Caputo, Saigo, Hilfer, Hadmard, Katugampola and other See [2, 3, 6, 12]. Many results on study of mathematical inequalities using various new fractional definitions such as Conformable and generalized fractional integral were obtained in [4, 13]. Recently in [7, 11, 14] the authors have obtained the results on Cebysev inequalities using various fractional integral and derivatives definitions.

In [6] fractional derivative and integrals of a functions with respect to another functions are defined. Recently in [1, 16] authors have studied the ψ Caputo and ψ Hilfer fractional derivative of a function with respect to another functions and its applications.

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Motivated from the above mentioned literature the aim of this paper is to obtain ψ Caputo fractional Čebyšev inequalities involving functions of two and three variables.

2. Preliminaries

Now in this section we give some basic definitions and properties which are used in our subsequent discussions. In [6, 12] the authors have defined the fractional integrals and fractional derivative of a function with respect to another function as follows.

Definition 2.1. [1, 6] Let $I = [a, b]$ be an interval, $\alpha > 0$, f is an integrable function defined on I and $\psi \in C^1(I)$ an increasing function such that $\psi'(x) \neq 0$ for all $x \in I$ then fractional derivative and integral of f is given by

$$I_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} f(t)\, dt$$

and

$$\begin{aligned} D_{a+}^{\alpha,\psi} f(x) &= \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^n I_{a+}^{n-\alpha,\psi} f(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^n \int_a^x \psi'(t) (\psi(x) - \psi(t))^{n-\alpha-1} f(t)\, dt, \end{aligned}$$

respectively. Similarly right fractional integral and right fractional derivative are given by

$$I_{b-}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(t) - \psi(x))^{\alpha-1} f(t)\, dt$$

and

$$\begin{aligned} D_{b-}^{\alpha,\psi} f(x) &= \left(-\frac{1}{\psi'(x)} \frac{d}{dx}\right)^n I_{b-}^{n-\alpha,\psi} f(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^n \int_a^x \psi'(t) (\psi(t) - \psi(x))^{n-\alpha-1} f(t)\, dt. \end{aligned}$$

In [1] Almedia has considered a Caputo type fractional derivative with respect to another function

Definition 2.2. [1] Let $\alpha > 0$, $n \in \mathbb{N}$, I is the interval $-\infty \leq a < b \leq \infty$, $f, \psi \in C^n(I)$ two functions such that ψ is increasing and $\psi'(x) \neq 0$ for all $x \in I$. The left ψ-Caputo fractional derivative of f of order α is given by

$${}^C D_{a+}^{\alpha,\psi} f(x) = I_{a+}^{n-\alpha,\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^n f(x),$$

and the right ψ-Caputo fractional derivative of f is given by

$$ {}^{C}D_{b-}^{\alpha,\psi} f(x) = I_{b-}^{n-\alpha,\psi} \left(-\frac{1}{\psi'(x)} \frac{d}{dx} \right)^{n} f(x). $$

For given $\alpha \notin \mathbb{N}$

$$ {}^{C}D_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{x} \psi'(t) (\psi(x) - \psi(t))^{n-\alpha-1} f_{\psi}^{[n]}(t)\, dt $$

and

$$ {}^{C}D_{b-}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{x} \psi'(t) (\psi(t) - \psi(x))^{n-\alpha-1} (-1)^{n} f_{\psi}^{[n]}(t)\, dt. $$

In particular when $\alpha \in (0,1)$ then

$$ {}^{C}D_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_{a}^{x} (\psi(x) - \psi(t))^{-\alpha} f'(t)\, dt $$

and

$$ {}^{C}D_{b-}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_{a}^{x} (\psi(t) - \psi(x))^{-\alpha} f'(t)\, dt. $$

In [15] the author has defined the ψ fractional partial integral with respect to another functions as

Definition 2.3. Let $\theta = (a,b)$ and $\alpha = (\alpha_1, \alpha_2)$ where $0 \leq \alpha_1, \alpha_2 \leq 1$. Also put $I = [a,k] \times [b,m]$ where a, b and k, m are positive constants. Also let $\psi(.)$ be an increasing positive monotone function on $(a,k] \times (b,m]$ having continuous derivative $\psi'(.)$ on $(a,k] \times (b,m]$. Then the fractional partial integral is

$$ I_{\theta}^{\alpha;\psi} u(x,y) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{a}^{x} \int_{b}^{y} \psi'(s)\psi'(t) (\psi(x) - \psi(s))^{\alpha_1 - 1} (\psi(y) - \psi(t))^{\alpha_2 - 1}\, dtds. $$

The Caputo fractional partial derivative is defined as follows

Definition 2.4. Let $\theta = (a,b)$ and $\alpha = (\alpha_1, \alpha_2)$ where $0 \leq \alpha_1, \alpha_2 \leq 1$. Also put $I = [a,k] \times [b,m]$ where a, b and a, b are positive constants. Also let $\psi(.)$ be an increasing function on $(a,k] \times (b,m]$ and $\psi'(.) \neq 0$ on $(a,k] \times (b,m]$. The ψ Caputo fractional partial derivative of functions of two variables of order α is given by

$$ {}^{C}D_{\theta}^{\alpha;\psi} u(x,y) = I_{\theta}^{2-\alpha;\psi} \left(\frac{1}{\psi'(s)\psi'(t)} \frac{\partial^2 \alpha}{\partial y \partial x} \right) u(x,y). $$

We use the following notation:

$$ {}^{C}D_{\theta}^{\alpha;\psi}u\left(x,y\right)=\frac{\partial_{\psi}^{2\alpha}u}{\partial_{\psi}y^{\alpha}\partial_{\psi}x^{\alpha}}\left(x,y\right) $$

. We define the norm for a function of two variables as follows

$$ \left\|{}^{C}D_{\theta}^{\alpha;\psi}f\right\|_{\infty}=\sup\left|{}^{C}D_{\theta}^{\alpha;\psi}f\left(x,y\right)\right| $$

Similarly as in Definition (2.3) and (2.4) we define the ψ fractional partial integral with respect to another functions and ψ Caputo fractional partial derivative of functions of three variables as follows

Definition 2.5. Let $\Theta=(a,b,c)$ and $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ where $0\leq\alpha_1,\alpha_2,\alpha_3\leq 1$. Also put $I=[a,k]\times[b,m]\times[c,n]$ where a,b,c and k,m,n are positive constants. Also let $\psi(.)$ be an increasing positive monotone function on $(a,k]\times(b,m]\times[c,n]$ having continuous derivative $\psi'(.)$ on $(a,k]\times(b,m]\times(c,n]$. Then the fractional partial integral is

$$ I_{\Theta}^{\alpha;\psi}u\left(x,y,z\right)=\frac{1}{\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)}\int_{a}^{x}\int_{b}^{y}\int_{c}^{z}\psi'(s)\psi'(t)\psi'(r) $$
$$ \left(\psi\left(x\right)-\psi\left(s\right)\right)^{\alpha_1-1}\left(\psi\left(y\right)-\psi\left(t\right)\right)^{\alpha_2-1}\left(\psi\left(z\right)-\psi\left(r\right)\right)^{\alpha_3-1}drdtds. $$

Definition 2.6. Let $\theta=(a,b,c)$ and $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ where $0\leq\alpha_1,\alpha_2,\alpha_3\leq 1$. Also put $I=[a,k]\times[b,m]\times[c,n]$ where a,b,c and k,m,n are positive constants. Also let $\psi(.)$ be an increasing function on $(a,k]\times(b,m]\times(c,n]$ and $\psi'(.)\neq 0$ on $(a,k]\times(b,m]\times(c,n]$. The ψ Caputo fractional partial derivative of functions of two variables of order α is given by

$$ {}^{C}D_{\Theta}^{\alpha;\psi}u\left(x,y,z\right)=I_{\Theta}^{3-\alpha;\psi}\left(\frac{1}{\psi'(s)\psi'(t)\psi'(r)}\frac{\partial^3}{\partial z\partial y\partial x}\right)u\left(x,y,z\right). $$

We use the following notation:

$$ {}^{C}D_{\Theta}^{\alpha;\psi}u\left(x,y,z\right)=\frac{\partial_{\psi}^{3\alpha}u}{\partial_{\psi}z^{\alpha}\partial_{\psi}y^{\alpha}x^{\alpha}}\left(x,y,z\right). $$

We define the norm for a function of three variables as follows

$$ \left\|{}^{C}D_{\Theta}^{\alpha;\psi}f\right\|_{\infty}=\sup\left|{}^{C}D_{\Theta}^{\alpha;\psi}f\left(x,y,z\right)\right|. $$

3. Čebyšev inequality involving functions of two variables

Now we give the ψ Caputo fractional Čebyšev inequality involving functions of two variables as follows:

Theorem 3.1. Let $f, g : [a, l]\times[b, m] \to R$ be a continuous function on $[a, l]\times[b, m]$ and $\frac{\partial^{2\alpha} f}{\partial_\psi y^\alpha \partial_\psi x^\alpha}$, $\frac{\partial^{2\alpha} g}{\partial_\psi y^\alpha \partial_\psi x^\alpha}$ exists continuous and bounded on $[a, l] \times [b, m]$ and $\alpha = (\alpha_1, \alpha_2)$

$$\begin{aligned}
&\left| \int_a^l \int_b^m \left[f(x,y)g(x,y) - \frac{1}{2}\left[G(f(x,y))g(x,y) + G(g(x,y))f(x,y)\right] dydx \right] \right| \\
&\leq \frac{1}{8}\left(\psi(l) - \psi(a)\right)\left(\psi(m) - \psi(b)\right) \\
&\int_a^l \int_b^m \left[|g(x,y)| \left\| D_\theta^{\alpha;\psi} f \right\|_\infty + g(x,y) \left\| D_\theta^{\alpha;\psi} g \right\|_\infty \right] dydx,
\end{aligned}$$

where

$$\begin{aligned}
G(f(x,y)) = &\frac{1}{2}\left[f(a,y) + f(x,m) + f(x,b) + f(l,y)\right] \\
&- \frac{1}{4}\left[f(a,b) + f(a,m) + f(l,b) + f(l,m)\right]
\end{aligned}$$

and

$$\begin{aligned}
&H\left(\frac{\partial^{2\alpha} f}{\partial_\psi y^\alpha \partial_\psi x^\alpha}(x,y)\right) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \times \\
&\times \left[\int_a^x \int_b^y \psi'(t)\psi'(s)\left(\psi(x) - \psi(t)\right)^{\alpha_1 - 1}\left(\psi(y) - \psi(s)\right)^{\alpha_2 - 1} \frac{\partial^{2\alpha} f}{\partial_\psi s^\alpha \partial_\psi t^\alpha}(t,s)\, dsdt \right. \\
&- \int_a^x \int_y^m \psi'(t)\psi'(s)\left(\psi(x) - \psi(t)\right)^{\alpha_1 - 1}\left(\psi(m) - \psi(s)\right)^{\alpha_2 - 1} \frac{\partial^{2\alpha} f}{\partial_\psi s^\alpha \partial_\psi t^\alpha}(t,s)\, dsdt \\
&- \int_x^l \int_b^y \psi'(t)\psi'(s)\left(\psi(l) - \psi(t)\right)^{\alpha_1 - 1}\left(\psi(y) - \psi(s)\right)^{\alpha_2 - 1} \frac{\partial^{2\alpha} f}{\partial_\psi s^\alpha \partial_\psi t^\alpha}(t,s)\, dsdt \\
&\left. + \int_x^l \int_y^m \psi'(t)\psi'(s)\left(\psi(l) - \psi(t)\right)^{\alpha_1 - 1}\left(\psi(m) - \psi(s)\right)^{\alpha_2 - 1} \frac{\partial^{2\alpha} f}{\partial_\psi s^\alpha \partial_\psi t^\alpha}(t,s)\, dsdt \right].
\end{aligned}$$

Proof. From the given hypotheses for $(x, y) \in [a, l] \times [b, m]$ we have

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_a^x \int_b^y \psi'(t)\psi'(s) \\
&\times \left(\psi(x) - \psi(t)\right)^{\alpha_1 - 1}\left(\psi(y) - \psi(s)\right)^{\alpha_2 - 1} \frac{\partial^{2\alpha} f}{\partial_\psi s^\alpha \partial_\psi t^\alpha}(t,s)\, dsdt
\end{aligned}$$

$$= \frac{1}{\Gamma(\alpha_1)} \int_a^x \psi'(s) (\psi(x) - \psi(t))^{\alpha_1 - 1} \left[\frac{\partial^\alpha f}{\partial_\psi s^\alpha}(s,t) \Big|_c^y \right]$$

$$= \frac{1}{\Gamma(\alpha_1)} \int_a^x \psi'(s) (\psi(y) - \psi(t))^{\alpha_1 - 1} \left[\frac{\partial^\alpha f}{\partial_\psi s^\alpha}(t,y) - \frac{\partial^\alpha f}{\partial_\psi s^\alpha}(t,b) \right]$$

$$= f(t,y)|_a^x - f(t,b)|_a^x$$

$$= f(x,y) - f(a,y) - f(x,b) + f(a,b). \tag{3.2}$$

Similarly we have

$$\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_a^x \int_y^m \psi'(t)\psi'(s)$$

$$\times (\psi(x) - \psi(t))^{\alpha_1 - 1} (\psi(m) - \psi(s))^{\alpha_2 - 1} \frac{\partial^{2\alpha} f}{\partial_\psi s^\alpha \partial_\psi t^\alpha}(t,s)\, dsdt$$

$$= -f(x,y) - f(a,m) + f(x,m) + f(a,y), \tag{3.3}$$

$$\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_x^l \int_b^y \psi'(t)\psi'(s)$$

$$\times (\psi(l) - \psi(t))^{\alpha_1 - 1} (\psi(y) - \psi(s))^{\alpha_2 - 1} \frac{\partial^{2\alpha} f}{\partial_\psi s^\alpha \partial_\psi t^\alpha}(t,s)\, dsdt$$

$$= -f(x,y) - f(l,b) + f(x,b) + f(l,y), \tag{3.4}$$

$$\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_x^l \int_y^m \psi'(t)\psi'(s)$$

$$\times (\psi(l) - \psi(t))^{\alpha_1 - 1} (\psi(m) - \psi(s))^{\alpha_2 - 1} \frac{\partial^{2\alpha} f}{\partial_\psi s^\alpha \partial_\psi t^\alpha}(s,t)\, dsdt$$

$$= f(x,y) + f(l,b) - f(x,b) - f(l,y). \tag{3.5}$$

Adding the above identities we have

$$4f(x,y) - 2[f(a,y) + f(x,m) + f(x,b) + f(l,y)]$$
$$+ [f(a,b) + f(a,m) + f(l,b) + f(l,m)]$$
$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}$$
$$\left[\int_a^x \int_b^y \psi'(t)\psi'(s) (\psi(x) - \psi(t))^{\alpha_1 - 1} (\psi(y) - \psi(s))^{\alpha_2 - 1} \frac{\partial^{2\alpha} f}{\partial_\psi s^\alpha \partial_\psi t^\alpha}(t,s) dsdt \right.$$

$$
\begin{aligned}
&-\int_a^x\int_y^d \psi'(t)\,\psi'(s)\,(\psi(x)-\psi(t))^{\alpha_1-1}\,(\psi(m)-\psi(s))^{\alpha_2-1}\,\frac{\partial^{2\alpha}f}{\partial_\psi s^\alpha \partial_\psi t^\alpha}(t,s)dsdt\\
&-\int_x^l\int_b^y \psi'(t)\,\psi'(s)\,(\psi(l)-\psi(t))^{\alpha_1-1}\,(\psi(y)-\psi(s))^{\alpha_2-1}\,\frac{\partial^{2\alpha}f}{\partial_\psi s^\alpha \partial_\psi t^\alpha}(t,s)dsdt\\
&+\int_x^l\int_y^m \psi'(t)\,\psi'(s)\,(\psi(l)-\psi(t))^{\alpha_1-1}\,(\psi(m)-\psi(s))^{\alpha_2-1}\,\frac{\partial^{2\alpha}f}{\partial_\psi s^\alpha \partial_\psi t^\alpha}(t,s)dsdt\Bigg].
\end{aligned}
\tag{3.6}
$$

Similarly we have

$$
\begin{aligned}
&4g(x,y)-2[g(a,y)+g(x,m)+g(x,b)+g(l,y)]\\
&+[g(a,b)+g(a,m)+g(l,b)+g(l,m)]\\
&=\frac{1}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)}\\
&\Bigg[\int_a^x\int_b^y \psi'(t)\,\psi'(s)\,(\psi(x)-\psi(t))^{\alpha_1-1}\,(\psi(y)-\psi(s))^{\alpha_2-1}\,\frac{\partial^{2\alpha}g}{\partial_\psi s^\alpha \partial_\psi t^\alpha}(t,s)dsdt\\
&-\int_a^x\int_y^d \psi'(t)\,\psi'(s)\,(\psi(x)-\psi(t))^{\alpha_1-1}\,(\psi(m)-\psi(s))^{\alpha_2-1}\,\frac{\partial^{2\alpha}g}{\partial_\psi s^\alpha \partial_\psi t^\alpha}(t,s)dsdt\\
&-\int_x^l\int_b^y \psi'(t)\,\psi'(s)\,(\psi(l)-\psi(t))^{\alpha_1-1}\,(\psi(y)-\psi(s))^{\alpha_2-1}\,\frac{\partial^{2\alpha}g}{\partial_\psi s^\alpha \partial_\psi t^\alpha}(t,s)dsdt\\
&+\int_x^l\int_y^m \psi'(t)\,\psi'(s)\,(\psi(l)-\psi(t))^{\alpha_1-1}\,(\psi(m)-\psi(s))^{\alpha_2-1}\,\frac{\partial^{2\alpha}g}{\partial_\psi s^\alpha \partial_\psi t^\alpha}(t,s)dsdt\Bigg].
\end{aligned}
\tag{3.7}
$$

From (3.6) and (3.7) we have

$$
f(x,y)-G(f(x,y))=\frac{1}{4}H\left(\frac{\partial^{2\alpha}f}{\partial_\psi y^\alpha \partial_\psi x^\alpha}(x,y)\right), \tag{3.8}
$$

for $(x,y)\in[a,l]\times[b,m]$. Similarly we have

$$
g(x,y)-G(g(x,y))=\frac{1}{4}H\left(\frac{\partial^{2\alpha}g}{\partial_\psi y^\alpha \partial_\psi x^\alpha}(x,y)\right), \tag{3.9}
$$

for $(x,y)\in[a,l]\times[b,m]$.

Multiplying (3.8) by $g(x,y)$ and (3.9) by $f(x,y)$ and adding them

$$\begin{aligned}&2f(x,y)g(x,y)-g(x,y)G(f(x,y))-f(x,y)G(g(x,y))\\&=\frac{1}{4}g(x,y)H\left(\frac{\partial^{2\alpha}f}{\partial_\psi y^\alpha\partial_\psi x^\alpha}(t,s)\right)+\frac{1}{4}f(x,y)H\left(\frac{\partial^{2\alpha}g}{\partial_\psi y^\alpha\partial_\psi x^\alpha}(t,s)\right).\end{aligned}\tag{3.10}$$

Integrating (3.11) over $(x,y)\in[a,l]\times[b,m]$ we get

$$\begin{aligned}&\int_a^l\int_b^m[2f(x,y)g(x,y)-g(x,y)G(f(x,y))-f(x,y)G(g(x,y))]dydx\\&=\frac{1}{8}\int_a^l\int_b^m\left[H\left(\frac{\partial^{2\alpha}f}{\partial_\psi y^\alpha\partial_\psi x^\alpha}(x,y)\right)g(x,y)+\frac{1}{4}f(x,y)H\left(\frac{\partial^{2\alpha}g}{\partial_\psi y^\alpha\partial_\psi x^\alpha}(x,y)\right)\right].\end{aligned}\tag{3.11}$$

From the properties of modulus we have

$$\begin{aligned}&\left|H\left(\frac{\partial^{2\alpha}f}{\partial_\psi y^\alpha\partial_\psi x^\alpha}(x,y)\right)\right|\leq\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\\&\int_a^l\int_b^m\psi'(t)\psi'(s)(\psi(l)-\psi(t))^{\alpha_1-1}(\psi(m)-\psi(s))^{\alpha_2-1}\left|\frac{\partial^{2\alpha}f}{\partial_\psi s^\alpha\partial_\psi t^\alpha}(t,s)\right|dsdt\\&\leq(\psi(l)-\psi(a))^{\alpha_1}(\psi(m)-\psi(b))^{\alpha_2}\left\|{}^cD_\theta^{\alpha;\psi}f\right\|_\infty,\end{aligned}\tag{3.12}$$

and we have

$$\begin{aligned}&\left|H\left(\frac{\partial^{2\alpha}g}{\partial_\psi y^\alpha\partial_\psi x^\alpha}(x,y)\right)\right|\leq\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\\&\int_a^l\int_b^m\psi'(t)\psi'(s)(\psi(l)-\psi(t))^{\alpha_1-1}(\psi(m)-\psi(s))^{\alpha_2-1}\left|\frac{\partial^{2\alpha}g}{\partial_\psi s^\alpha\partial_\psi t^\alpha}(t,s)\right|dsdt\\&\leq(\psi(l)-\psi(a))^{\alpha_1}(\psi(m)-\psi(b))^{\alpha_2}\left\|{}^cD_\theta^{\alpha;\psi}g\right\|_\infty.\end{aligned}\tag{3.13}$$

From (3.11), (3.12) and (3.13) we have

$$\begin{aligned}&\left|\int_a^l\int_b^m\left[f(x,y)g(x,y)-\frac{1}{2}[G(f(x,y))g(x,y)+G(g(x,y))f(x,y)]\right]dydx\right|\\&\leq\frac{1}{8}\int_a^l\int_b^m\left[\left|H\left(\frac{\partial^{2\alpha}f}{\partial_\psi y^\alpha\partial_\psi x^\alpha}(x,y)\right)\right||g(x,y)|+\left|H\left(\frac{\partial^{2\alpha}g}{\partial_\psi y^\alpha\partial_\psi x^\alpha}(x,y)\right)\right||f(x,y)|\right]\end{aligned}$$

$$\leq \frac{1}{8}\int_a^l\int_b^m \{|g(x,y)| \left[\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\right.$$
$$\times \left[\int_a^l\int_b^m \psi'(t)\psi'(s)(\psi(l)-\psi(t))^{\alpha_1-1}(\psi(m)-\psi(s))^{\alpha_2-1}\left|\frac{\partial^{2\alpha}f}{\partial_\psi s^\alpha \partial_\psi t^\alpha}(t,s)\right| dsdt\right]$$
$$+|f(x,y)|$$
$$\times \left[\int_a^l\int_b^m \psi'(t)\psi'(s)(\psi(l)-\psi(t))^{\alpha_1-1}(\psi(m)-\psi(s))^{\alpha_2-1}\right.$$
$$\left.\left.\left|\frac{\partial^{2\alpha}g}{\partial_\psi s^\alpha \partial_\psi t^\alpha}(t,s)\right| dsdt\right]\right\} dydx$$
$$\leq \frac{1}{8}(\psi(l)-\psi(a))^{\alpha_1}(\psi(m)-\psi(b))^{\alpha_2}$$
$$\times \int_a^l\int_b^m \left[|g(x,y)|\left\|{}^cD_\theta^{\alpha;\psi}f\right\|_\infty + |f(x,y)|\left\|{}^cD_\theta^{\alpha;\psi}g\right\|_\infty\right] dydx, \tag{3.14}$$

which is required inequality.

Theorem 3.2. Let $f, g, G(f(x,y)), G(g(f(x,y)), \frac{\partial^{2\alpha}f}{\partial_\psi y^\alpha \partial_\psi x^\alpha}, \frac{\partial^{2\alpha}g}{\partial_\psi y^\alpha \partial_\psi x^\alpha}$ be as in Theorem 3.1 then

$$\left|\int_a^l\int_b^m \{f(x,y)g(x,y) - [G(f(x,y))g(x,y) + G(g(x,y))f(x,y)\right.$$
$$\left.-G(f(x,y))G(g(x,y))]\} dydx\right|$$
$$\leq \frac{1}{16}\{(\psi(l)-\psi(a))^{\alpha_1}(\psi(m)-\psi(b))^{\alpha_2}\}^2 \left\|{}^cD_\theta^{\alpha;\psi}f\right\|_\infty \left\|{}^cD_\theta^{\alpha;\psi}g\right\|_\infty, \tag{3.15}$$

for $(x,y)\in[a,l]\times[b,m]$.

Proof. From (3.9) and (3.10) we have

$$f(x,y) - G(f(x,y)) = \frac{1}{4}H\left(\frac{\partial^{2\alpha}f}{\partial_\psi y^\alpha \partial_\psi x^\alpha}(x,y)\right) \tag{3.16}$$

and

$$g(x,y) - G(g(x,y)) = \frac{1}{4}H\left(\frac{\partial^{2\alpha}g}{\partial_\psi y^\alpha \partial_\psi x^\alpha}(x,y)\right). \tag{3.17}$$

for $(x,y)\in[a,l]\times[b,m]$.

Multiplying left hand side and right hand side of (3.16) and (3.17) we have

$$f(x,y)g(x,y) - [f(x,y)G(g(x,y)) + g(x,y)G(f(x,y))]$$
$$= \frac{1}{16}H\left(\frac{\partial^{2\alpha}f}{\partial_\psi y^\alpha \partial_\psi x^\alpha}(x,y)\right)H\left(\frac{\partial^{2\alpha}g}{\partial_\psi y^\alpha \partial_\psi x^\alpha}(x,y)\right). \tag{3.18}$$

Integrating (3.18) over $[a,l]\times[b,m]$ and from the properties of modulus we get

$$\begin{aligned}&\left|\int_a^l\int_b^m \{f(x,y)g(x,y) - [G(g(x,y))f(x,y)+G(f(x,y))g(x,y)]\right.\\&\left.-G(f(x,y))G(g(x,y))\}\,dydx\right|\\&\leq \frac{1}{16}\int_a^l\int_b^m \left|H\left(\frac{\partial^{2\alpha}f}{\partial_\psi y^\alpha \partial_\psi x^\alpha}(x,y)\right)\right|\left|H\left(\frac{\partial^{2\alpha}g}{\partial_\psi y^\alpha \partial_\psi x^\alpha}(x,y)\right)\right| dydx. \qquad (3.19)\end{aligned}$$

Now using (3.13), (3.14) in (3.19) we have

$$\begin{aligned}&\left|\int_a^l\int_b^m \{f(x,y)g(x,y) - [G(f(x,y))g(x,y)+G(g(x,y))f(x,y)\right.\\&\left.-G(f(x,y))G(g(x,y))]\}\,dydx\right|\\&\leq \frac{1}{16}\left\{(\psi(l)-\psi(a))^{\alpha_1}(\psi(m)-\psi(b))^{\alpha_2}\right\}^2 \left\|{}^cD_\theta^{\alpha;\psi}f\right\|_\infty \left\|{}^cD_\theta^{\alpha;\psi}g\right\|_\infty, \qquad (3.20)\end{aligned}$$

which is required inequality.

4. Čebyšev inequality involving functions of three variables

Now in our result we give the ψ Caputo fractional Čebyšev inequality involving functions of three variables. We use some notations as follows

$$\begin{aligned}A(p(u,v,w)) =&\frac{1}{8}[p(a,b,c)+p(k,m,n)]\\&-\frac{1}{4}[p(u,b,c)+p(u,m,n)+p(u,m,c)+p(u,b,n)]\\&-\frac{1}{4}[p(a,v,c)+p(k,v,n)+p(a,v,n)+p(k,v,c)]\\&-\frac{1}{4}[p(a,b,w)+p(k,m,w)+p(k,b,w)+p(a,m,w)]\\&+\frac{1}{2}[p(a,v,w)+p(k,v,w)]\\&+\frac{1}{2}[p(u,b,w)+p(u,m,w)]\\&+\frac{1}{2}[p(u,v,c)+p(u,v,n)] \qquad (4.1)\end{aligned}$$

and

$$B\left(\frac{\partial^{3\alpha}p}{\partial_\psi w^\alpha \partial_\psi v^\alpha \partial_\psi u^\alpha}(u,v,w)\right)$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_a^u \int_b^v \int_c^w \psi'(r)\psi'(s)\psi'(t)(\psi(u)-\psi(r))^{\alpha_1-1}$$

$$\times (\psi(v)-\psi(s))^{\alpha_2-1}(\psi(w)-\psi(t))^{\alpha_3-1} \frac{\partial^{3\alpha} p}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,t)\,dtdsdr$$

$$- \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_a^u \int_b^v \int_c^n \psi'(r)\psi'(s)\psi'(t)(\psi(u)-\psi(r))^{\alpha_1-1}$$

$$\times (\psi(v)-\psi(s))^{\alpha_2-1}(\psi(n)-\psi(t))^{\alpha_3-1} \frac{\partial^{3\alpha} p}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,t)\,dtdsdr$$

$$\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_a^u \int_v^m \int_c^w \psi'(r)\psi'(s)\psi'(t)(\psi(u)-\psi(r))^{\alpha_1-1}$$

$$\times (\psi(m)-\psi(s))^{\alpha_2-1}(\psi(w)-\psi(t))^{\alpha_3-1} \frac{\partial^{3\alpha} p}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,t)\,dtdsdr$$

$$- \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_u^k \int_b^v \int_c^w \psi'(r)\psi'(s)\psi'(t)(\psi(k)-\psi(r))^{\alpha_1-1}$$

$$\times (\psi(u)-\psi(s))^{\alpha_2-1}(\psi(w)-\psi(t))^{\alpha_3-1} \frac{\partial^{3\alpha} p}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,t)\,dtdsdr$$

$$\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_a^u \int_r^m \int_w^n \psi'(r)\psi'(s)\psi'(t)(\psi(u)-\psi(r))^{\alpha_1-1}$$

$$\times (\psi(m)-\psi(s))^{\alpha_2-1}(\psi(n)-\psi(t))^{\alpha_3-1} \frac{\partial^{3\alpha} p}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,t)\,dtdsdr$$

$$- \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_u^k \int_v^m \int_c^w \psi'(r)\psi'(s)\psi'(t)(\psi(k)-\psi(r))^{\alpha_1-1}$$

$$\times (\psi(m)-\psi(s))^{\alpha_2-1}(\psi(w)-\psi(t))^{\alpha_3-1} \frac{\partial^{3\alpha} p}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,t)\,dtdsdr$$

$$\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_u^k \int_b^v \int_w^n \psi'(r)\psi'(s)\psi'(t)(\psi(k)-\psi(r))^{\alpha_1-1}$$

$$\times (\psi(v)-\psi(s))^{\alpha_2-1}(\psi(n)-\psi(t))^{\alpha_3-1} \frac{\partial^{3\alpha} p}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,t)\,dtdsdr$$

$$- \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_u^k \int_v^m \int_w^n \psi'(r)\psi'(s)\psi'(t)(\psi(k)-\psi(r))^{\alpha_1-1}$$

$$\times \left(\psi\left(m\right)-\psi\left(s\right)\right)^{\alpha_2-1}\left(\psi\left(n\right)-\psi\left(t\right)\right)^{\alpha_3-1}\frac{\partial^{3\alpha}p}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}\left(r,s,t\right)dtdsdr. \tag{4.2}$$

Now we give our next result as

Theorem 4.1. Let $f,g:[a,k]\times[b,m]\times[c,n]\to R$ be a continuous function on $[a,l]\times[b,m]$ and $\frac{\partial^{3\alpha}f}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}$, $\frac{\partial^{3\alpha}g}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}$ exists and continuous and bounded on $[a,k]\times[b,m]\times[c,n]$. Then

$$\begin{aligned}
&\int_a^k\int_b^m\int_c^n [f(u,v,w)g(u,v,w)\\
&-\frac{1}{2}\left[f(u,v,w)A\left(g(u,v,w)\right)+g(u,v,w)A\left(f(u,v,w)\right)\right]\Big] dwdvdu\\
&\leq \frac{1}{16}\left(\psi\left(k\right)-\psi\left(a\right)\right)^{\alpha_1}\left(\psi\left(m\right)-\psi\left(b\right)\right)^{\alpha_2}\left(\psi\left(n\right)-\psi\left(c\right)\right)^{\alpha_3}\\
&\times\int_a^k\int_b^m\int_c^n\Big[|g(u,v,w)|\left\|{}^cD_\Theta^{\alpha;\psi}f\right\|_\infty+|f(u,v,w)|\left\|{}^cD_\Theta^{\alpha;\psi}g\right\|_\infty\Big]dwdvdu,
\end{aligned} \tag{4.3}$$

where A,B are as given in (4.1), (4.2).

Proof. From the hypotheses we have for $u,v,w\in[a,k]\times[b,m]\times[c,n]$

$$\begin{aligned}
&\frac{1}{\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)\Gamma\left(\alpha_3\right)}\int_a^u\int_b^v\int_c^w\psi'\left(r\right)\psi'\left(s\right)\psi'\left(t\right)\left(\psi\left(u\right)-\psi\left(r\right)\right)^{\alpha_1-1}\\
&\left(\psi\left(v\right)-\psi\left(s\right)\right)^{\alpha_2-1}\left(\psi\left(w\right)-\psi\left(t\right)\right)^{\alpha_3-1}\frac{\partial^{3\alpha}f}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,t)dtdsdr\\
&=\frac{1}{\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)}\int_a^u\int_b^v\psi'\left(r\right)\psi'\left(s\right)\left(\psi\left(u\right)-\psi\left(r\right)\right)^{\alpha_1-1}\\
&\left(\psi\left(v\right)-\psi\left(s\right)\right)^{\alpha_2-1}\frac{\partial^{2\alpha}f}{\partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,t)\Big|_c^w dsdr\\
&=\frac{1}{\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)}\int_a^u\int_b^v\psi'\left(r\right)\psi'\left(s\right)\left(\psi\left(u\right)-\psi\left(r\right)\right)^{\alpha_1-1}\\
&\left(\psi\left(v\right)-\psi\left(s\right)\right)^{\alpha_2-1}\frac{\partial^{2\alpha}f}{\partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,w)dsdr\\
&-\frac{1}{\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)}\int_a^u\int_b^v\psi'\left(r\right)\psi'\left(s\right)\left(\psi\left(u\right)-\psi\left(r\right)\right)^{\alpha_1-1}
\end{aligned}$$

$$
\begin{aligned}
&(\psi(v)-\psi(s))^{\alpha_2-1}\frac{\partial^{2\alpha}f}{\partial_\psi s^\alpha\partial_\psi r^\alpha}(r,s,c)dsdr\\
&=\frac{1}{\Gamma(\alpha_1)}\int_a^u\psi'(r)(\psi(u)-\psi(r))^{\alpha_1-1}\frac{\partial^\alpha f}{\partial_\psi r^\alpha}(r,s,w)\Big|_b^v dr\\
&-\frac{1}{\Gamma(\alpha_1)}\int_a^u\psi'(r)(\psi(u)-\psi(r))^{\alpha_1-1}\frac{\partial^\alpha f}{\partial_\psi r^\alpha}(r,s,c)\Big|_b^v dr\\
&=\frac{1}{\Gamma(\alpha_1)}\int_a^u\psi'(r)(\psi(u)-\psi(r))^{\alpha_1-1}\frac{\partial^\alpha f}{\partial_\psi r^\alpha}(r,v,w)dr\\
&-\frac{1}{\Gamma(\alpha_1)}\int_a^u\psi'(r)(\psi(u)-\psi(r))^{\alpha_1-1}\frac{\partial^\alpha f}{\partial_\psi r^\alpha}(r,b,w)dr\\
&-\frac{1}{\Gamma(\alpha_1)}\int_a^u\psi'(r)(\psi(u)-\psi(r))^{\alpha_1-1}\frac{\partial^\alpha f}{\partial_\psi r^\alpha}(r,v,c)dr\\
&+\frac{1}{\Gamma(\alpha_1)}\int_a^u\psi'(r)(\psi(u)-\psi(r))^{\alpha_1-1}\frac{\partial^\alpha f}{\partial_\psi r^\alpha}(r,b,c)dr\\
&=f(r,v,w)|_a^u-f(r,b,w)|_a^u-f(r,v,c)|_a^u+f(r,b,c)|_a^u\\
&=f(u,v,w)-f(a,v,w)-f(u,b,w)+f(a,b,w)\\
&-f(u,v,c)+f(a,v,c)+f(u,b,c)+f(a,b,c).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&f(u,v,w)=f(a,v,w)+f(u,b,w)-f(a,b,w)\\
&+f(u,v,c)-f(a,v,c)-f(u,b,c)-f(a,b,c)\\
&\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\int_a^u\int_b^v\int_c^w\psi'(r)\psi'(s)\psi'(t)(\psi(u)-\psi(r))^{\alpha_1-1}\\
&(\psi(v)-\psi(s))^{\alpha_2-1}(\psi(w)-\psi(t))^{\alpha_3-1}\frac{\partial^{3\alpha}f}{\partial_\psi t^\alpha\partial_\psi s^\alpha\partial_\psi r^\alpha}(r,s,t)dtdsdr, \qquad (4.4)
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
&f(u,v,w)=f(u,v,n)+f(a,v,w)+f(u,b,w)\\
&+f(a,b,n)-f(a,b,w)-f(a,v,n)-f(v,b,n)\\
&-\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\int_a^u\int_b^v\int_w^n\psi'(r)\psi'(s)\psi'(t)(\psi(u)-\psi(r))^{\alpha_1-1}
\end{aligned}
$$

$$(\psi(v)-\psi(s))^{\alpha_2-1}(\psi(n)-\psi(t))^{\alpha_3-1}\frac{\partial^{3\alpha}f}{\partial_\psi t^\alpha\partial_\psi s^\alpha\partial_\psi r^\alpha}(r,s,t)dtdsdr, \tag{4.5}$$

$$\begin{aligned}
&f(u,v,w)=f(u,m,w)+f(u,v,c)+f(a,m,c)\\
&+f(a,v,w)-f(u,m,c)-f(a,m,w)-f(a,v,c)\\
&-\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\int_a^u\int_v^m\int_c^w\psi'(r)\psi'(s)\psi'(t)(\psi(u)-\psi(r))^{\alpha_1-1}\\
&(\psi(m)-\psi(s))^{\alpha_2-1}(\psi(w)-\psi(t))^{\alpha_3-1}\frac{\partial^{3\alpha}f}{\partial_\psi t^\alpha\partial_\psi s^\alpha\partial_\psi r^\alpha}(r,s,t)dtdsdr,
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
&f(u,v,w)=f(k,s,t)+f(k,b,c)+f(u,v,c)\\
&+f(u,b,w)-f(k,v,c)-f(k,b,w)-f(u,b,c)\\
&-\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\int_u^k\int_b^v\int_c^w\psi'(r)\psi'(s)\psi'(t)(\psi(k)-\psi(r))^{\alpha_1-1}\\
&(\psi(v)-\psi(s))^{\alpha_2-1}(\psi(w)-\psi(t))^{\alpha_3-1}\frac{\partial^{3\alpha}f}{\partial_\psi t^\alpha\partial_\psi s^\alpha\partial_\psi r^\alpha}(r,s,t)dtdsdr,
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
&f(u,v,w)=f(u,m,w)+f(u,v,n)+f(a,m,n)\\
&+f(a,v,w)-f(u,m,n)-f(a,m,w)-f(a,v,n)\\
&+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\int_a^u\int_v^m\int_w^n\psi'(r)\psi'(s)\psi'(t)(\psi(u)-\psi(r))^{\alpha_1-1}\\
&(\psi(m)-\psi(s))^{\alpha_2-1}(\psi(n)-\psi(t))^{\alpha_3-1}\frac{\partial^{3\alpha}f}{\partial_\psi t^\alpha\partial_\psi s^\alpha\partial_\psi r^\alpha}(r,s,t)dtdsdr,
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
&f(u,v,w)=f(r,m,t)+f(u,v,c)+f(k,s,t)\\
&+f(k,m,c)-f(k,m,w)-f(k,v,c)-f(u,m,c)\\
&+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\int_u^k\int_v^m\int_c^w\psi'(r)\psi'(s)\psi'(t)(\psi(k)-\psi(r))^{\alpha_1-1}\\
&(\psi(m)-\psi(s))^{\alpha_2-1}(\psi(w)-\psi(t))^{\alpha_3-1}\frac{\partial^{3\alpha}f}{\partial_\psi t^\alpha\partial_\psi s^\alpha\partial_\psi r^\alpha}(r,s,t)dtdsdr,
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
&f(u,v,w)=f(k,v,w)+f(k,b,n)+f(u,v,n)\\
&+f(u,b,t)-f(k,v,n)-f(k,b,w)-f(u,b,n)
\end{aligned}$$

$$+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\int_u^k\int_b^v\int_w^n \psi'(r)\psi'(s)\psi'(t)(\psi(k)-\psi(r))^{\alpha_1-1}$$

$$(\psi(v)-\psi(s))^{\alpha_2-1}(\psi(n)-\psi(t))^{\alpha_3-1}\frac{\partial^{3\alpha}f}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,t)dtdsdr \tag{4.10}$$

and

$$\begin{aligned} &f(u,v,w)=f(k,m,n)+f(k,v,w)+f(u,m,w)\\ &+f(u,v,n)-f(k,m,w)-f(k,v,n)-f(u,m,n)\\ &+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\int_u^k\int_v^m\int_w^n \psi'(r)\psi'(s)\psi'(t)(\psi(k)-\psi(r))^{\alpha_1-1}\\ &(\psi(m)-\psi(s))^{\alpha_2-1}(\psi(n)-\psi(t))^{\alpha_3-1}\frac{\partial^{3\alpha}f}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,t)dtdsdr. \end{aligned} \tag{4.11}$$

Adding the above identities we have

$$f(u,v,w)-A(f(u,v,w))=\frac{1}{8}B\left(\frac{\partial^{3\alpha}f}{\partial_\psi w^\alpha \partial_\psi v^\alpha \partial_\psi u^\alpha}(u,v,w)\right), \tag{4.12}$$

for $(u,v,w)\in[a,k]\times[b,m]\times[c,n]$.
Similarly we have

$$g(u,v,w)-A(g(u,v,w))=\frac{1}{8}B\left(\frac{\partial^{3\alpha}g}{\partial_\psi w^\alpha \partial_\psi v^\alpha \partial_\psi u^\alpha}(u,v,w)\right), \tag{4.13}$$

for $(u,v,w)\in[a,k]\times[b,m]\times[c,n]$.
Now multiplying (4.12) and (4.13) by $g(u,v,w)$ and $f(u,v,w)$ respectively and adding them we get

$$\begin{aligned} &2f(u,v,w)g(u,v,w)-g(u,v,w)A(f(u,v,w))-f(u,v,w)A(g(u,v,w))\\ &=\frac{1}{8}g(u,v,w)B\left(\frac{\partial^{3\alpha}f}{\partial_\psi w^\alpha \partial_\psi v^\alpha \partial_\psi u^\alpha}(u,v,w)\right)\\ &+\frac{1}{8}f(u,v,w)B\left(\frac{\partial^{3\alpha}g}{\partial_\psi w^\alpha \partial_\psi v^\alpha \partial_\psi u^\alpha}(u,v,w)\right). \end{aligned} \tag{4.14}$$

Integrating (4.14) over $[a,k]\times[b,m]\times[c,n]$ we have

$$\begin{aligned} &\int_a^k\int_b^m\int_c^n [f(u,v,w)g(u,v,w)-\frac{1}{2}[g(u,v,w)A(f(u,v,w))\\ &g(u,v,w)A(f(u,v,w))]]dwdvdu\\ &=\frac{1}{16}\int_a^k\int_b^m\int_c^n\left[g(u,v,w)B\left(\frac{\partial^{3\alpha}f}{\partial_\psi w^\alpha \partial_\psi v^\alpha \partial_\psi u^\alpha}(u,v,w)\right)\right. \end{aligned}$$

$$+f(u,v,w)\,B\left(\frac{\partial^{3\alpha}g}{\partial_\psi w^\alpha \partial_\psi v^\alpha \partial_\psi u^\alpha}(u,v,w)\right)\Bigg]. \tag{4.15}$$

From the properties of modulus we have

$$\begin{aligned}
&\left|B\left(\frac{\partial^{3\alpha}f}{\partial_\psi w^\alpha \partial_\psi v^\alpha \partial_\psi u^\alpha}(u,v,w)\right)\right| \\
&\leq \int_a^k \int_b^m \int_c^n \psi'(r)\psi'(s)\psi'(t)\left(\psi(k)-\psi(r)\right)^{\alpha_1-1}\left(\psi(m)-\psi(s)\right)^{\alpha_2-1} \\
&\times \left(\psi(n)-\psi(t)\right)^{\alpha_3-1}\frac{\partial^{3\alpha}f}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,t)\,dtdsdr \\
&\leq \left(\psi(k)-\psi(a)\right)^{\alpha_1}\left(\psi(m)-\psi(b)\right)^{\alpha_2}\left(\psi(n)-\psi(c)\right)^{\alpha_3}\left\|{}^{C}D_{\Theta}^{\alpha;\psi}f\right\|_\infty
\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}
&\left|B\left(\frac{\partial^{3\alpha}g}{\partial_\psi w^\alpha \partial_\psi v^\alpha \partial_\psi u^\alpha}(u,v,w)\right)\right| \\
&\leq \int_a^k \int_b^m \int_c^n \psi'(r)\psi'(s)\psi'(t)\left(\psi(k)-\psi(r)\right)^{\alpha_1-1}\left(\psi(m)-\psi(s)\right)^{\alpha_2-1} \\
&\times \left(\psi(n)-\psi(t)\right)^{\alpha_3-1}\frac{\partial^{3\alpha}g}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}(r,s,t)\,dtdsdr \\
&\leq \left(\psi(k)-\psi(a)\right)^{\alpha_1}\left(\psi(m)-\psi(b)\right)^{\alpha_2}\left(\psi(n)-\psi(c)\right)^{\alpha_3}\left\|{}^{C}D_{\Theta}^{\alpha;\psi}g\right\|_\infty.
\end{aligned} \tag{4.17}$$

Now by substituting the values from equation (4.16) and (4.17) in (4.15) we get the required inequality (4.3)

Theorem 4.2. Let f, g, $\frac{\partial^{3\alpha}f}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}$ and $\frac{\partial^{3\alpha}g}{\partial_\psi t^\alpha \partial_\psi s^\alpha \partial_\psi r^\alpha}$ be as in Theorem 4.1. Then

$$\begin{aligned}
&\Bigg|\int_a^k \int_b^m \int_c^n [f(u,v,w)\,g(u,v,w) - [A(f(u,v,w))\,g(u,v,w) \\
&A(g(u,v,w))\,f(u,v,w) - A(f(u,v,w))\,A(g(u,v,w))|\,dwdvdu \\
&\leq \frac{1}{64}\left\{\left(\psi(k)-\psi(a)\right)^{\alpha_1}\left(\psi(m)-\psi(b)\right)^{\alpha_2}\left(\psi(n)-\psi(c)\right)^{\alpha_3}\right\}^2 \\
&\left\|{}^{C}D_{\Theta}^{\alpha;\psi}f\right\|_\infty \left\|{}^{C}D_{\Theta}^{\alpha;\psi}g\right\|_\infty,
\end{aligned} \tag{4.18}$$

for $(r,s,t) \in [a,k]\times[b,m]\times[c,n]$ and A, B are as given in (4.1), (4.2).

Proof. From (4.12) and (4.13) we have

$$f(u,v,w) - A(f(u,v,w)) = \frac{1}{8}B\left(\frac{\partial^{3\alpha}f}{\partial_\psi w^\alpha \partial_\psi v^\alpha \partial_\psi u^\alpha}(u,v,w)\right) \tag{4.19}$$

and

$$g(u,v,w) - A(g(u,v,w)) = \frac{1}{8} B\left(\frac{\partial^{3\alpha} g}{\partial_\psi w^\alpha \partial_\psi v^\alpha \partial_\psi u^\alpha}(u,v,w)\right). \tag{4.20}$$

for $(u,v,w) \in [a,k] \times [b,m] \times [c,n]$.

Multiplying left hand and right hand side of equation (4.19) and (4.20) we have

$$\begin{aligned} & f(u,v,w)\, g(u,v,w) - [f(u,v,w)\, A(g(u,v,w)) \\ & + g(u,v,w)\, A(f(u,v,w)) - A(f(u,v,w))\, A(g(u,v,w))] \\ & = \frac{1}{64} B\left(\frac{\partial^{3\alpha} f}{\partial_\psi w^\alpha \partial_\psi v^\alpha \partial_\psi u^\alpha}(u,v,w)\right) B\left(\frac{\partial^{3\alpha} g}{\partial_\psi w^\alpha \partial_\psi v^\alpha \partial_\psi u^\alpha}(u,v,w)\right). \end{aligned} \tag{4.21}$$

Integrating over $[a,k] \times [b,m] \times [c,n]$ and from the properties of modulus we have

$$\begin{aligned} & \left| \int_a^k \int_b^m \int_c^n [f(u,v,w)\ g(u,v,w) - [f(u,v,w)\, A(g(u,v,w)) \right. \\ & \left. + g(u,v,w)\, A(f(u,v,w)) - A(f(u,v,w))\, A(g(u,v,w))]] \right| dwdvdu \\ & \leq \frac{1}{64} \int_a^k \int_b^m \int_c^n \left| B\left(\frac{\partial^{3\alpha} f}{\partial_\psi w^\alpha \partial_\psi v^\alpha \partial_\psi u^\alpha}(u,v,w)\right) \right. \\ & \left. B\left(\frac{\partial^{3\alpha} f}{\partial_\psi w^\alpha \partial_\psi v^\alpha \partial_\psi u^\alpha}(u,v,w)\right) \right| dwdvdu. \end{aligned} \tag{4.22}$$

Using (4.16) and (4.17) in (4.22) we get the required inequality (4.18).

Remark: In this paper we have obtained the Čebyšev inequality using Caputo fractional deriative of a function with respect to another function for funtions of two and three variables. If we put different values for $\psi(x)$ then it reduces to various types of fractional Čebyšev inequalities such as Riemann Liouville fractional, Hadmard Fractional and Erdelyi-Kober fractional inequalities.

If we put $\psi(x) = x$ then the above inequalities given in the theorems reduces to Riemann-Liouville type fractional Chebysev inequality.

If we put $\psi(x) = lnx$ then the above inequalities given in the theorems reduces to Hadmard fractional type fractional Chebysev inequality.

If we put $\psi(x) = x^\sigma$ then the above inequalities given in the theorems reduces to Erdelyi-Kober type fractional Chebysev inequality.

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