Two-dimensional Hermite-Hadamard-Type integral Inequalities for coordinated ϕ_h-convex functions on time scales

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Key words: Time scales, ϕ_h-convex function, diamond-ϕ_h, Hermite-Hadamard inequality.

2000 Mathematics Subject Classification: 35A23, 52A41, 34N05.

Abstract

In this paper, double integral calculus via the diamond-ϕ_h dynamic integral for two-variable functions on time scales is introduced to prove Hermite-Hadamard type integral inequalities for the generalized class of ϕ_h-convex functions. Also, a two-dimensional Hermite-Hadamard-type integral inequality for this class of convex functions on time scales is established. Our work generalizes and refines proofs of corresponding results for some known classes of functions.

1 Introduction

The inequality

$$(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx \leq (b-a)\frac{f(a)+f(b)}{2}, \quad a, b \in \mathbb{R},\ a < b, \quad (1.1)$$

holds for any convex function f defined on $\mathbb{R}$. It was first suggested by Hermite in 1881. But this result was nowhere mentioned in literature and was not widely known as Hermite's result. A leading expert on the history and theory of convex functions, Beckenbach [1], wrote that the inequality (1.1) was proven by Hadamard in 1893. In general, (1.1) is now known as the Hermite-Hadamard inequality. It has several extensions and generalizations for univariate, bivariate and multivariate convex functions and its classes on classical intervals(see Dragomir [5]) with recent extensions to time scales(see [4, 10, 13]).

The concept of the theory of time scales was initiated by Stefen Hilger [9] in order to unify and extend the theory of difference and differential calculus consistently. In this theory, the delta and nabla calculus for single and two-variable functions are introduced (see [2, 3, 8]). A linear combination of these delta and nabla dynamics, the diamond-α calculus on time scales was developed by Sheng et al. [12]. Since the advent of this notion, several authors have extended many

RGMIA Res. Rep. Coll. 22 (2019), Art. 101, 12 pp. Received 14/10/19

classical mathematical inequalities to time scales via the diamond-alpha dynamic calculus for univariate, bivariate and multivariate functions (see [4, 10, 11, 13]).

Nwaeze [10], employed Theorem 3.9 of Dinu [4] for a univariate function on time scales to prove the following Hadamard's type result, via the combined diamond-α dynamics, extending (1.1), for functions defined on a rectangle, that are convex on the coordinates.

Theorem 1.1.[10] Let $a, b, x \in \mathbb{T}_1, c, d, y \in \mathbb{T}_2$, with $a < b, c < d$ and $f : [a, b] \times [c, d] \to \mathbb{R}$ be such that the partial mappings $f_y : [a, b] \to \mathbb{R}$, $f_y(u) := f(u, y)$ and $f_x : [c, d] \to \mathbb{R} f_x(v) := f(x, v)$ defined for all $y \in [c, d]$ and $x \in [a, b]$, are continuous and convex. Then the following inequalities hold

$$\begin{aligned}
&\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x,\, s_\alpha) \diamond_\alpha x + \frac{1}{d-c}\int_c^d f(t_\alpha, y) \diamond_\alpha y\right] \\
&\leq \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x, y) \diamond_\alpha x \diamond_\alpha y \\
&\leq \frac{1}{2(b-a)(d-c)}\int_a^b [(d - s_\alpha) f(x,\, c) + \ (s_\alpha - c) f(x,\, d)] \diamond_\alpha x \\
&+ \frac{1}{2(b-a)(d-c)}\int_c^d [(b - t_\alpha) f(a,\, y) \ + (t_\alpha - a) f(b,\, y)] \diamond_\alpha y.
\end{aligned} \tag{1.2}$$

where $t_\alpha = \frac{1}{b-a}\int_a^b t \diamond_\alpha t$, and $s_\alpha = \frac{1}{d-c}\int_c^d s \diamond_\alpha s$.

Recently, the authors [6] introduced the time-scaled version of some classes of convex functions, including a more generalized class of ϕ_h-convex function on time scales thus;

Definition 1.1. [6] Let $h : \mathbb{J}_\mathbb{T} \subset \mathbb{T} \to \mathbb{R}$ be a nonzero non negative function with the property that $h(t) > 0$ for all $t \geq 0$. A mapping $f : I_\mathbb{T} \to \mathbb{R}$ is said to be ϕ_h-convex on time scales if

$$f(\lambda x + (1-\lambda) y) \leq \left(\frac{\lambda}{h(\lambda)}\right)^s f(x) + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f(y), \tag{1.3}$$

for $s \in [0, 1], 0 \leq \lambda \leq 1$ and $x, y \in I_\mathbb{T}$.

Remark 1.1.

(i) If $s = 1$ and $h(\lambda) = 1$, then $f \in SX(I_\mathbb{T})$, i.e, f is convex on time scales (see [5, 12]).

(ii) If $s = 1$, $h(\lambda) = 1$, where $\lambda = \frac{1}{2}$, then $f \in J(I_\mathbb{T})$ is mid-point convex on time scales (see [6]).

(iii) If $s = 0$, then $f \in P(I_\mathbb{T})$ is P-convex on time scales (see [6]).

(iv) If $h(\lambda) = \lambda^{\frac{s}{s+1}}$ for $\lambda > 0$, then $f \in SX(h, I_{\mathbb{T}})$ is h-convex on time scales (see [6]).

(v) If $s = 1$ and $h(\lambda) = 2\sqrt{\lambda(1-\lambda)}$ for $\lambda \geq 0$, then $f \in MT(I_{\mathbb{T}})$ is MT-convex on time scales (see [6]).

More recently, Fagbemigun et al.[7] proved the following Hadamard's type result for the new class of ϕ_h-convex functions earlier introduced by the authors [6], for a univariate function to obtain several generalizations of the Hermite-Hadamard inequality (1.1) on time scales.

Theorem 1.2. **[7]** Let $f : I_{\mathbb{T}} \to \mathbb{R}$ be a continuous, nondecreasing ϕ_h-convex function on $I_{\mathbb{T}}$, $a, b, t \in I_{\mathbb{T}}$, with $a < b$. Then

$$f(x_{\phi_h}) \leq \frac{1}{b-a} \int_a^b f(t) \diamond_{\phi_h} t \leq \frac{b - x_{\phi_h}}{b-a} f(a) + \frac{x_{\phi_h} - a}{b-a} f(b), \tag{1.4}$$

where $x_{\phi_h} = \frac{1}{b-a} \int_a^b t \diamond_{\phi_h} t$.

Remark 1.2. (i) When $\phi_h = \alpha$ in (1.4), Theorem 3.9 of Dinu [4] is obtained.

(ii) Setting $\phi_h = \frac{1}{2}$ and using the relation (Q) of [7] in Theorem 1.2 gives inequality (5.1) of Dinu [4], which is the middle point Hermite-Hadamard inequality on time scales.

(iii) The nabla integral version of Theorem 1.2 is obtained if we choose $\phi_h = 0$.

It is the purpose of this paper to extend inequality (1.1) to time scales via the combined diamond-ϕ_h dynamics, for a function of two variables.

2 Preliminaries

In the sequel, we shall need the following new definitions recently introduced in [8].

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales with $\mathbb{T}_1 \times \mathbb{T}_2 = \{(x, y) : x \in \mathbb{T}_1, y \in \mathbb{T}_2\}$ which is a complete metric space with the metric d defined by

$$d((x, y), (x^{'}, y^{'})) = ((x - x^{'})^2 + (y - y^{'})^2)^{\frac{1}{2}}, \qquad \forall \quad (x, y), (x^{'}, y^{'}) \in \mathbb{T}_1 \times \mathbb{T}_2.$$

Let σ_i, ρ_i, $(i = 1, 2)$ denote respectively the forward jump operator, backward jump operator, and the diamond-ϕ_h dynamic differentiation operator on $\mathbb{T}_i$.

Definition 2.1. Let f be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$, $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ a nonzero non negative function with the property that $h(t) > 0$ for all $t \geq 0$. f is said to have a partial $\diamond_{(\phi_h)_1}$ derivative $\frac{\partial f(t_1,t_2)}{\diamond_{(\phi_h)_1} t_1}$ (wrt t_1), at $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$, if for each $\epsilon > 0$, there exists a neighbourhood Ut_1 of t_1 such that

$$\Bigg| \left(\frac{\lambda}{h(\lambda)}\right)_1^s [f(\sigma_1(t_1), t_2) - f(m, t_2,)]\mu t_1 m \\
+ \left(\frac{1-\lambda}{h(1-\lambda)}\right)_1^s [f(\rho_1(t_1), t_2) - f(m, t_2)]\nu t_1 m - f^{\diamond_{(\phi_h)_1}}(t_1,\, t_2)\mu t_1 m \nu t_1 m \Bigg|$$

$$< \epsilon|\mu t_1 m \nu t_1 m|, \tag{2.1}$$
for $s \in [0,1], 0 \leq \lambda \leq 1$ and for all $m \in Ut_1$, where $Ut_1m = \sigma_1(t_1) - m$, $\nu t_1 m = \rho_1(t_1) - m$.

Definition 2.2. Let f be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$ and $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ an increasing function with the property that $h(t) > 0$ for all $t \geq 0$. f is said to have a "partial $\diamond_{(\phi_h)_2}$ derivative" $\frac{\partial f(t_1,t_2)}{\diamond_{(\phi_h)_2} t_2}$(wrt t_2), at $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$, if for each $\epsilon > 0$, there exists a neighbourhood Ut_2 of t_2 such that

$$\Bigg|\left(\frac{\lambda}{h(\lambda)}\right)_2^s [f(t_1,\, \sigma_2(t_2) - f(t_1,\, m)]\mu t_2 m$$

$$+\left(\frac{1-\lambda}{h(1-\lambda)}\right)_2^s [f(t_1,\, \rho_2(t_2) - f(t_1,\, m)]\nu t_2 m - f^{\diamond_{(\phi_h)_2}}(t_1,\, t_2)\mu t_2 m \nu t_2 m\Bigg|$$

$$< \epsilon|\mu t_2 m \nu t_2 m|, \tag{2.2}$$
for $s \in [0,1], 0 \leq \lambda \leq 1$ and for all $n \in Ut_2$, where $Ut_2m = \sigma_2(t_2) - m$, $\nu t_2 m = \rho_2(t_2) - m$.

These derivatives are also denoted by $f^{\diamond_{(\phi_h)_1}}(t_1,\, t_2)$ and $f^{\diamond_{(\phi_h)_2}}(t_1,\, t_2)$ respectively.

Before we define the double diamond-ϕ_h dynamic integral, we shall employ the following remark of [2].

Remark 2.1.[2] Let f be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$. If the delta (Δ) and nabla (∇) integrals of f exist on $\mathbb{T}_1 \times \mathbb{T}_2$, then the following types of integrals can be defined:

(i) $\Delta\Delta$-integral over $R^0 = [a,b) \times [c,d)$, which is introduced by using partitions consisting of subrectangles of the form $[\alpha,\beta) \times [\gamma,\partial)$;

(ii) $\nabla\nabla$-integral over $R^1 = (a,b] \times (c,d]$, which is introduced by using partitions consisting of subrectangles of the form $(\alpha,\beta] \times (\gamma,\partial]$;

(iii) $\Delta\nabla$-integral over $R^2 = [a,b) \times (c,d]$, which is introduced by using partitions consisting of subrectangles of the form $[\alpha,\beta) \times (\gamma,\partial]$;

(iv) $\nabla\Delta$-integral over $R^3 = (a,b] \times [c,d)$, which is introduced by using partitions consisting of subrectangles of the form $(\alpha,\beta] \times [\gamma,\partial)$.

Now let $\bar{U}(f)$ and $\bar{L}(f)$ denote the upper and lower Darboux Δ-integral of f from a to b ; $\underline{\mathrm{U}}(f)$ and $\underline{\mathrm{L}}(f)$ denote the upper and lower Darboux ∇-integral of f from a to b respectively. Given the construction of $U(f)$ and $L(f)$, which follows from the properties of supremum and infimum, we give the following definition.

Definition 2.3. Let f be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$, $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ a nonzero non negative function with the property that $h(t) > 0$ for all $t \geq 0$. If f is Δ-integrable on $R^0 = [a,b) \times [c,d)$ and ∇-integrable on $R^1 = (a,b] \times (c,d]$, then it is $\diamond_{\phi_h}$-integrable on $R = [a, b] \times [c, d]$ and

$$\int_R f(t,\, k) \diamond_{(\phi_h)_1} t \diamond_{(\phi_h)_2} k = \left(\frac{\lambda}{h(\lambda)}\right)^s \int\int_{R^0} f(t,\, k)\Delta_1 t\Delta_2 k + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \int\int_{R^1} f(t,\, k)\nabla_1 t\nabla_2 k, \quad (2.3)$$

for all $s \in [0, 1]$, $0 \le \lambda \le 1$ and $t, k \in J_{\mathbb{T}}$.

Since $\bar{U}(f) \ge \bar{L}(f)$ and $\underline{\mathrm{U}}(f) \ge \underline{\mathrm{L}}(f)$, we obtain the following result.

Theorem 2.1. Let f be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$, $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ a nonzero non negative function with the property that $h(t) > 0$ for all $t \ge 0$. If f be $\diamond_{\phi_h}$-integrable on $R = [a, b] \times [c, d]$, provided its delta (Δ) and nabla (∇) integrals exist, then

(i) If $\phi_h = 1$, f is $\Delta\Delta$-integrable on $R^0 = [a, b) \times [c, d)$;

(ii) If $\phi_h = 0$, f is $\nabla\nabla$-integrable on $R^1 = (a, b] \times (c, d]$;

(iii) If $\phi_h = \frac{1}{2}$, f is $\Delta\Delta$-integrable and $\nabla\nabla$-integrable on R^0 and R^1

(iv) If $\phi_h = \alpha$, f is double $\diamond_\alpha$-integrable on $R = [a, b] \times [c, d]$.

3 Two-dimensional Hermite-Hadamard type inequalities for ϕ_h-convex functions on the co-ordinates

Consider the bi-dimensional time scale interval $I^2_{\mathbb{T}} : [a, b]_{I_{\mathbb{T}}} \times [c, d]_{I_{\mathbb{T}}}$ in $\mathbb{T}^2$ with $a < b$, $c < d$.

Definition 3.1. Let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero non negative function with the property that $h(t) > 0$ for all $t \ge 0$. A monotonically increasing function $f : I^2_{\mathbb{T}} \to \mathbb{R}$ on $I^2_{\mathbb{T}}$ is ϕ_h-convex on time scale co-ordinates if the partial mappings

$$f_y : [a, b]_{I_{\mathbb{T}}} \to \mathbb{R}, \quad f_y(u) := f(u, y), \qquad \forall\ y \in [c, d]_{I_{\mathbb{T}}}$$

and

$$f_x : [c, d]_{I_{\mathbb{T}}} \to \mathbb{R}, \quad f_x(v) := f(x, v), \qquad \forall\ x \in [a, b]_{I_{\mathbb{T}}}$$

are continuous and ϕ_h-convex.

Definition 3.2. Let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero non negative function with the property that $h(t) > 0$ for all $t \ge 0$. A monotonically increasing function $f : I^2_{\mathbb{T}} \to \mathbb{R}$ is ϕ_h-convex on time scale co-ordinates if the inequality

$$f(\lambda x + (1-\lambda)y,\, tu + (1-t)v) \le \left(\frac{t}{h(t)}\right)^s \left(\frac{\lambda}{h(\lambda)}\right)^s f(x, u) + \left(\frac{\lambda}{h(\lambda)}\right)^s \left(\frac{1-t}{h(1-t)}\right)^s f(x, v)$$

$$+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s\left(\frac{t}{h(t)}\right)^s f(y,u)+\left(\frac{1-t}{h(1-t)}\right)^s\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f(y,v),$$

holds for $s \in [0,1]$, $0 \le \lambda,\, t \le 1$ and $x, y \in I_{\mathbb{T}}$ and $(x,u),(x,v),(y,u),(y,v) \in I_{\mathbb{T}}^2$.
Thus the mapping $f : I_{\mathbb{T}}^2 \to \mathbb{R}$ is ϕ_h-convex in $I_{\mathbb{T}}^2$ if the following inequality:

$$f(\lambda x+(1-\lambda)u,\, \lambda y+(1-\lambda)v) \le \left(\frac{\lambda}{h(\lambda)}\right)^s f(x,\, y)+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f(u,\, v) \quad (3.1)$$

holds for all $(x,\, y),\, (u,\, v) \in I_{\mathbb{T}}^2$, $s \in [0,1]$ and $0 \le \lambda \le 1$.

We state and prove the following Lemma

Lemma 3.1. Every ϕ_h-convex mapping $f : I_{\mathbb{T}}^2 \to \mathbb{R}$ on $I_{\mathbb{T}}^2$ is ϕ_h-convex on the co-ordinates.

Proof. Suppose that the mapping $f : I_{\mathbb{T}}^2 \to \mathbb{R}$ is ϕ_h-convex in $I_{\mathbb{T}}^2$ by (4.4). Consider the partial mapping
$f_x : [c,\, d]_{I_{\mathbb{T}}} \to \mathbb{R}, \quad f_x(v) := f(x,v).$
Then for all $s \in [0,1]$, $0 \le \lambda \le 1$ and $f(u,v)$ monotonically increasing functions on $I_{\mathbb{T}}$, we have

$$\begin{aligned}
&f_x(\lambda u+(1-\lambda)v) = f\left(x, \left(\frac{\lambda}{h(\lambda)}\right)^s u + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s v\right)\\
=\;& f\left(\left(\frac{\lambda}{h(\lambda)}\right)^s x + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s x, \left(\frac{\lambda}{h(\lambda)}\right)^s u + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s v\right)\\
\le\;& \left(\frac{\lambda}{h(\lambda)}\right)^s f(x,u) + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f(x,v)\\
=\;& \left(\frac{\lambda}{h(\lambda)}\right)^s f_x u + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f_x v,
\end{aligned}$$

which shows ϕ_h-convexity of f_x.
By a similar argument, the partial mappings
$f_y : [a,b]_{I_{\mathbb{T}}} \to \mathbb{R}, \quad f_y(u) := f(u,y)$, is also ϕ_h-convex for all
$s \in [0,1]$, $0 \le \lambda \le 1$ and $f(v,r)$ monotonically increasing functions on $I_{\mathbb{T}}$ goes likewise and the proof is omitted.

Note that in some special cases, some co-ordinated ϕ_h-convex functions may not necessarily be ϕ_h-convex on time scales.

With the aid of Lemma 3.1, we first discuss and establish a double integral inequality of Hermite-Hadamard type for a ϕ_h-convex function on time scale co-ordinates.

Theorem 3.1. Let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero non negative function with the property that $h(t) > 0$ for all $t \ge 0$. Let $f : I_{\mathbb{T}}^2 \to \mathbb{R}$ be a continuous and an integrable ϕ_h-convex function with respect to the function ϕ_h on the co-ordinates on $I_{\mathbb{T}}^2$. Then for any $a, b, c, d \geq 0$, with $b > a, d > c$ and $s \in [0,1]$,

$$\begin{aligned} f(M_{\phi_h}, N_{\phi_h}) &\leq \frac{I_{\lambda,t}(a,b;c,d)}{(b-a)(d-c)} \\ &\leq \left(\frac{t}{h(t)}\right)^s \frac{I_{M,y}(a,b;c,d)}{(b-a)(d-c)} + \left(\frac{1-t}{h(1-t)}\right)^s \frac{I_{M,N}(a,b;c,d)}{(b-a)(d-c)} \\ &\leq \frac{4I_{x,y}(a,b;c,d)}{(b-a)(d-c)}, \end{aligned} \tag{3.2}$$

where $M_{\phi_h} = \int_a^b u \diamond_{\phi_h} u$ and $N_{\phi_h} = \int_a^b v \diamond_{\phi_h} v$,

$I_{\lambda,t}(a,b;c,d)$

$$= \int_a^b \int_c^d f(\lambda x + (1-\lambda)M_{\phi_h}, ty + (1-t)N_{\phi_h}) \diamond_{\phi_h} y \diamond_{\phi_h} x,$$

$I_{M,y}(a,b;c,d)$

$$= \int_a^b \int_c^d f\left(\left(\frac{\lambda}{h(\lambda)}\right)^s x + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s M_{\phi_h}, y\right) \diamond_{\phi_h} x \diamond_{\phi_h} y,$$

$I_{M,N}(a,b;c,d)$

$$= \int_a^b \int_c^d f\left(\left(\frac{\lambda}{h(\lambda)}\right)^s x + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s M_{\phi_h}, N_{\phi_h})\right) \diamond_{\phi_h} x \diamond_{\phi_h} y,$$

and

$$I_{x,y}(a,b;c,d) = \int_a^b \int_c^d f(\phi(x), y) \diamond_{\phi_h} x \diamond_{\phi_h} y.$$

Proof. (A) To show the first inequality in (3.2).
We have that,

$$\begin{aligned} f(M_{\phi_h}), N_{\phi_h})) &\leq f\left(\frac{1}{b-a}\int_a^b [\lambda x + (1-\lambda)M_{\phi_h}], N_{\phi_h})\right) \diamond_{\phi_h} x \\ &= \frac{1}{b-a}\int_a^b f\left(\lambda x + (1-\lambda)M_{\phi_h}, N_{\phi_h})\right) \diamond_{\phi_h} x \\ &\leq \frac{1}{b-a}\int_a^b f\left(\lambda x + (1-\lambda)M_{\phi_h}, \frac{1}{d-c}\int_c^d [ty + (1-t)N_{\phi_h}] \diamond_{\phi_h} y\right) \diamond_{\phi_h} x \\ &\leq \frac{1}{b-a}\int_a^b \left[\frac{1}{d-c}\int_c^d f\left(\lambda x + (1-\lambda)M_{\phi_h}, ty + (1-t)N_{\phi_h}\right) \diamond_{\phi_h} y\right] \diamond_{\phi_h} x. \end{aligned}$$

This proves the first inequality in (3.2).
Then by Definition 3.2, we have that

$$\frac{1}{b-a}\int_a^b \left[\frac{1}{d-c}\int_c^d f\left(\lambda x + (1-\lambda)M_{\phi_h}, ty + (1-t)N_{\phi_h}\right) \diamond_{\phi_h} y\right] \diamond_{\phi_h} x$$

$$
\begin{aligned}
\leq\ & \left(\frac{t}{h(t)}\right)^s \frac{1}{b-a}\int_a^b \left(\frac{1}{d-c}\int_c^d f\left(\left(\frac{\lambda}{h(\lambda)}\right)^s x+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s M_{\phi_h}, y\right)\diamond_{\phi_h} y\right)\\
+\ & \left(\frac{1-t}{h(1-t)}\right)^s\\
\times\ & \frac{1}{d-c}\int_c^d f\left(\left(\frac{\lambda}{h(\lambda)}\right)^s x+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s M_{\phi_h}, M_{\phi_h})\right)\diamond_{\phi_h} y\diamond_{\phi_h} x, \quad (*)
\end{aligned}
$$

satisfying the second inequality in (3.2).
Thus from the right hand side of (*), we have

$$
\begin{aligned}
& \left(\frac{t}{h(t)}\right)^s \frac{1}{b-a}\int_a^b \left(\frac{1}{d-c}\int_c^d f\left(\left(\frac{\lambda}{h(\lambda)}\right)^s x+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s M_{\phi_h}, y\right)\diamond_{\phi_h} y\right)\\
+\ & \left(\frac{1-t}{h(1-t)}\right)^s\\
\times\ & \frac{1}{d-c}\int_c^d f\left(\left(\frac{\lambda}{h(\lambda)}\right)^s x+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s M_{\phi_h}, M_{\phi_h})\right)\diamond_{\phi_h} y\diamond_{\phi_h} x\\
\leq\ & \left(\frac{t}{h(t)}\right)^s \times \frac{1}{d-c}\int_c^d [\left(\frac{\lambda}{h(\lambda)}\right)^s \frac{1}{b-a}\int_a^b f(x,y)\diamond_{\phi_h} y\diamond_{\phi_h} x\\
+\ & \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \frac{1}{b-a}\int_a^b f\left(M_{\phi_h}, y\right)\diamond_{\phi_h} x]\diamond_{\phi_h} y\\
+\ & \left(\frac{1-t}{h(1-t)}\right)^s \times \frac{1}{d-c}\int_c^d [\left(\frac{\lambda}{h(\lambda)}\right)^s \cdot \frac{1}{b-a}\int_a^b f\left(\phi(x), N_{\phi_h}\right)\diamond_{\phi_h} x\\
+\ & \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f\left(M_{\phi_h}, N_{\phi_h})\right]\diamond_{\phi_h} y\\
\leq\ & \left(\frac{t}{h(t)}\right)^s \left(\frac{\lambda}{h(\lambda)}\right)^s \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\diamond_{\phi_h} x\diamond_{\phi_h} y\\
+\ & \left(\frac{t}{h(t)}\right)^s \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \frac{1}{d-c}\int_c^d f(M_{\phi_h}, y)\diamond_{\phi_h} y\\
+\ & \left(\frac{1-t}{h(1-t)}\right)^s \left(\frac{\lambda}{h(\lambda)}\right)^s \frac{1}{b-a}\int_a^b f\left(x, N_{\phi_h}\right)\diamond_{\phi_h} x\\
+\ & \left(\frac{1-t}{h(1-t)}\right)^s \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f\left(M_{\phi_h}), N_{\phi_h}\right). \qquad (3.3)
\end{aligned}
$$

Also, from the first inequality in Theorem 1.3, ϕ_h-convexity of y on the co-ordinates and using Lemma 3.1, we have

$$
f(M_{\phi_h}, y) \leq \frac{1}{b-a}\int_a^b f(x,y)\diamond_{\phi_h} x \qquad (3.4)
$$

and

$$
f(x, N_{\phi_h})) \leq \frac{1}{d-c}\int_c^d f(x,y)\diamond_{\phi_h} y. \qquad (3.5)
$$

Integrating (3.4) and (3.5), we have

$$\frac{1}{d-c}\int_c^d f(M_{\phi_h},y)\diamond_{\phi_h} y \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\diamond_{\phi_h} x. \tag{3.6}$$

And,

$$\frac{1}{b-a}\int_a^b f(x,N_{\phi_h})\diamond_{\phi_h} x \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\diamond_{\phi_h} y. \tag{3.7}$$

Using (3.4), (3.5),(3.6) and (3.7), we deduce that (3.3) becomes

$$\begin{aligned}
& \left(\frac{t}{h(t)}\right)^s\left(\frac{\lambda}{h(\lambda)}\right)^s\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\diamond_{\phi_h} x\diamond_{\phi_h} y \\
+ & \left(\frac{t}{h(t)}\right)^s\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s\frac{1}{d-c}\int_c^d f(M_{\phi_h},y)\diamond_{\phi_h} y \\
+ & \left(\frac{1-t}{h(1-t)}\right)^s\left(\frac{\lambda}{h(\lambda)}\right)^s\frac{1}{b-a}\int_a^b f\left(x,N_{\phi_h}\right)\diamond_{\phi_h} x \\
+ & \left(\frac{1-t}{h(1-t)}\right)^s\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f\left(M_{\phi_h},N_{\phi_h}\right) \\
\le & \left[\left(\frac{t}{h(t)}\right)^s\left(\frac{\lambda}{h(\lambda)}\right)^s+\left(\frac{t}{h(t)}\right)^s\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s\right. \\
+ & \left.\left(\frac{1-t}{h(1-t)}\right)^s\left(\frac{\lambda}{h(\lambda)}\right)^s+\left(\frac{1-t}{h(1-t)}\right)^s\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s\right] \\
\times & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\diamond_{\phi_h} x\diamond_{\phi_h} y \\
\le & \frac{4}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\diamond_{\phi_h} x\diamond_{\phi_h} y.
\end{aligned}$$

This proves the third inequality in (3.2).

Theorem 3.2. Let $h:\mathbb{J}_{\mathbb{T}}\subset\mathbb{T}\to\mathbb{R}$ be a non zero non negative function with the property that $h(t)>0$ for all $t\ge 0$. Let $f: I_{\mathbb{T}}^2=[a,b]_{I_{\mathbb{T}}}\times[c,d]_{I_{\mathbb{T}}}\to\mathbb{R}$ be continuous, integrable and co-ordinated ϕ_h-convex on $I_{\mathbb{T}}^2$. Then for any $a,b,c,d\ge 0$, with $b>a, d>c$, the following inequalities hold

$$\begin{aligned}
& \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x,\,M_{\phi_h})\diamond_{\phi_h} x+\frac{1}{d-c}\int_c^d f(N_{\phi_h},y)\diamond_{\phi_h} y\right] \\
\le & \frac{I_{x,y}(a,b;c,d)}{(b-a)(d-c)} \\
\le & \frac{1}{2(b-a)(d-c)}\int_a^b[(d-M_{\phi_h})f(x,\,c)+\ (M_{\phi_h}-c)f(x,\,t_4)]\diamond_{\phi_h} x \\
+ & \frac{1}{2(b-a)(d-c)}
\end{aligned}$$

$$\times \quad \int_c^d [(y-N_{\phi_h})f(x,\, y) \ + (N_{\phi_h}-x)f(x,\, y)] \diamond_{\phi_h} y, (3.8)$$

where $\quad M_{\phi_h} = \int_a^b u \diamond_{\phi_h} u,\ N_{\phi_h} = \int_a^b v \diamond_{\phi_h} v,$
and

$$I_{x,y}(a,b;c,d) = \int_a^b \int_c^d f(x,y) \diamond_{\phi_h} x \diamond_{\phi_h} y.$$

Proof. By Definition 3.1, we have

$$f_x(M_{\phi_h}) \leq \frac{1}{d-c}\int_c^d f_x(y) \diamond_{\phi_h} y \leq \frac{d-M_{\phi_h}}{d-c} f_x c \ + \ \frac{M_{\phi_h - c}}{d-c} f_x d.$$

That is,

$$\begin{aligned} f(x, M_{\phi_h}) &\leq \frac{1}{d-c}\int_c^d f(x,y) \diamond_{\phi_h} y \\ &\leq \frac{d-M_{\phi_h}}{d-c} f(x,c) + \frac{M_{\phi_h}-c}{d-c} f(x,d). \end{aligned} \tag{3.9}$$

Integrating both sides of (3.9) over $\diamond_{\phi_h} x$ on $[a,b]_{I_{\mathbb{T}}}$, we obtain

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x, M_{\phi_h}) \diamond_{\phi_h} x &\leq \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x,y) \diamond_{\phi_h} x \diamond_{\phi_h} y \\ &\leq \frac{d-M_{\phi_h}}{(b-a)(d-c)}\int_a^b f(x,c) \diamond_{\phi_h} x \\ &+ \frac{M_{\phi_h}-c}{(b-a)(d-c)}\int_a^b f(x,d) \diamond_{\phi_h} x. \end{aligned} \tag{3.10}$$

By a similar argument, for the partial mapping $f_y : [a,b] \to \mathbb{R}, \quad f_y(u) := f(u,y)$, we obtain

$$\begin{aligned} f(N_{\phi_h}, y) &\leq \frac{1}{b-a}\int_a^b f(x,y) \diamond_{\phi_h} x \\ &\leq \frac{y-N_{\phi_h}}{b-a} f(x,y) \ + \ \frac{N_{\phi_h - x}}{b-a} f(x,y). \end{aligned} \tag{3.11}$$

Integrating both sides of (3.11) over $\diamond_{\phi_h} y$ on $[a,b]_{I_{\mathbb{T}}}$, we get

$$\begin{aligned} &\frac{1}{d-c}\int_c^d f(N_{\phi_h}, y) \diamond_{\phi_h} y \\ \leq \quad &\frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x,y) \diamond_{\phi_h} x \diamond_{\phi_h} y \\ \leq \quad &\frac{y-N_{\phi_h}}{(b-a)(d-c)}\int_c^d f(x,y) \diamond_{\phi_h} y \\ + \quad &\frac{N_{\phi_h - x}}{(b-a)(d-c)}\int_c^d f(x,y) \diamond_{\phi_h} y. \end{aligned} \tag{3.12}$$

Adding (3.10) and (3.12),we get the desired result (3.8).

Remark 3.1. If $\phi_h = \alpha$ Theorem 3.1 of Nwaeze [10] is recovered.

Remark 3.2. The case $\phi_h = 0$ gives corollary 3.2 of Nwaeze [10].

Remark 3.3. If we choose $\phi_h = \frac{1}{2}$ and substitute the relation Q of Fagbemigun et al. [7] in Theorem 3.2, we obtain corollary 3.3 of Nwaeze [10].

Remark 3.4. Corollary 3.4 of [10] is obtained if $\phi_h = 1$ in Theorem 3.2.

Remark 3.5. If we take $I_{\mathbb{T}_1} = I_{\mathbb{T}_2} = \mathbb{R}$ in Theorem 3.2, we get the second and third inequalities of Theorem 1.1 due to Dragomir [5].

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